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A New Modular Space Derived by Euler Totient Function

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Abstract: In this study, we introduce the Euler Totient sequence spaces in generalized Orlicz space and we examine some topological properties of these spaces by using the Luxemburg norm.

Keywords: Euler Totient function, Modular space, Orlicz sequence space, Luxemburg norm

1 Introduction and background

Lindenstrauss and Tzafriri [1] used the idea of Orlicz function M to construct the sequence space ℓ_M of all sequences of scalars (x_k) such that $\sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) < \infty$ for some $\rho > 0$. The space ℓ_M with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}$$

is a Banach space and it is called as Orlicz sequence space. The space ℓ_M is closely related to the space $\ell_p = \{(x_k) : \sum_{k=1}^{\infty} |x_k|^p < \infty\}$ which is an Orlicz space with $M(x) = x^p$, for $1 \leq p < \infty$.

Definition 1. *[2] A function $M : [0,\infty) \to [0,\infty)$ is called an Orlicz function if it is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for all $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.*

An Orlicz function M is said to satisfy the Δ_2-condition if there exists a constant $K > 0$ such that $M(2x) \leq KM(x)$ for all $x \geq 0$. It is easy to see that always $K > 2$.

Equivalently, an Orlicz function M is said to satisfy the Δ_2-condition if $M(lx) \leq K(l)M(x)$ for all $x \geq 0$, where $l > 1$.

A simple example of an Orlicz function which satisfies the Δ_2-condition is given by $M(x) = \alpha|x|^{\alpha}$ $(\alpha > 1)$, since we have $M(2x) = \alpha 2^{\alpha}|x|^{\alpha} = 2^{\alpha}M(x)$.

Definition 2. *[2] Let X be a linear space over $\mathbb{R}$. A function $\rho : X \to [0,\infty]$ is called a modular if the following conditions hold:*
(1) $\rho(x) = 0 \Leftrightarrow x = \theta$ (zero vector of X),
(2)$\rho(x) = \rho(-x)$ for all $x \in X$,
(3)$\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all $x, y \in X$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.
If the condition
(3')$\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ for all $x, y \in X$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$
holds instead of (3), then ρ is called a convex modular.
If ρ is a modular on X, then the linear space

$$X_\rho = \{x \in X : \lim_{\delta \to 0} \rho(\delta x) = 0\}$$

is called a modular space.

Definition 3. *[2] A sequence (x_n) in X_ρ is called ρ-convergent to $x_0 \in X_\rho$ if $\rho(\delta(x_n - x_0)) \to 0$ as $n \to \infty$ for some $\delta > 0$.*
A sequence (x_n) in X_ρ is called ρ-Cauchy if $\rho(\delta(x_n - x_m)) \to 0$ as $n, m \to \infty$ for some $\delta > 0$.
The space X_ρ is called ρ-complete if every ρ-Cauchy sequence in this space is ρ-convergent.

Definition 4. *Let E be a Lebesgue measurable subset of $\mathbb{R}$. The generalized Orlicz space is defined as follows:*

$$L_M = \{f : E \to \mathbb{R} : f \text{ is Lebesgue measurable and} \int_E M(\delta\,|f(x)|)dx < \infty \text{ for some } \delta > 0\}.$$

The function $\rho_M : L_M \to [0, \infty)$ defined by

$$\rho_M(f) = \int_E M(|f(x)|)dx$$

is a modular on L_M and the space L_M is ρ_M-complete.

The generalized Orlicz space L_M is a Banach space with the Luxemburg norm given by

$$\|f\|_M = \inf\{\gamma > 0 : \rho_M\left(\frac{f}{\gamma}\right) \le 1\}.$$

Throughout the study, by $\omega(L_M)$, we denote the space of all sequences in L_M.

Let φ denote the Euler function. For every $m \in \mathbb{N}$ with $m > 1$, $\varphi(m)$ is the number of positive integers less than m which are coprime with m and $\varphi(1) = 1$. If $p_1^{a_1} p_2^{a_2} ... p_r^{a_r}$ is the prime factorization of a natural number $m > 1$, then

$$\varphi(m) = m(1 - \frac{1}{p_1})(1 - \frac{1}{p_2})...(1 - \frac{1}{p_r}).$$

Also, the equality

$$m = \sum_{k|m} \varphi(k)$$

holds for every $m \in \mathbb{N}$ and $\varphi(m_1 m_2) = \varphi(m_1)\varphi(m_2)$, where $m_1, m_2 \in \mathbb{N}$ are coprime [4]. One can consult to [5] for more details related to these functions.

The Φ-summability was introduced by Schoenberg [3] for the purpose of studying the Riemann integrability of a generalized Dirichlet function in the range $[0, 1]$. This method is called φ-convergence which is a weaker form of usual convergence. The infinite matrix $\Phi = (\phi_{ij})$ is defined as

$$\phi_{ij} = \begin{cases} \frac{\varphi(j)}{i} & , \quad \text{if } j \mid i \\ 0 & , \quad \text{if } j \nmid i \end{cases}$$

The matrix Φ satisfies the following conditions:

1. $\sup_{i \in \mathbb{N}} (\sum_{j=1}^{\infty} |\phi_{ij}|) < \infty$,
2. $\lim_{i \to \infty} \phi_{ij} = 0$ for each fixed $j \in \mathbb{N}$,
3. $\lim_{i \to \infty} \sum_{j=1}^{\infty} \phi_{ij} = 1$

and so it is a regular matrix.

By using this matrix, İlkhan and Kara [6] have introduced the sequence spaces $\ell_p(\Phi)$ and $\ell_\infty(\Phi)$ as

$$\ell_p(\Phi) = \left\{ u = (u_n) \in \omega : \sum_n \left| \frac{1}{n} \sum_{k|n} \varphi(k) u_k \right|^p < \infty \right\} \quad (1 \le p < \infty)$$

and

$$\ell_\infty(\Phi) = \left\{ u = (u_n) \in \omega : \sup_n \left| \frac{1}{n} \sum_{k|n} \varphi(k) u_k \right| < \infty \right\}.$$

In the literature, there are many papers on sequence spaces using Orlicz function. Later these spaces are generalized by using the Lebesgue integral with Orlicz function. In [7], the authors have generalized the Cesàro sequence spaces in the classical Banach space L_p to the generalized Orlicz space L_M. In this paper, we generalize Euler sequence spaces to the generalized Orlicz space and obtain a modular space. Also, we examine some topological properties of these spaces by using the Luxemburg norm.

2 Main results

Now, we introduce the Euler Totient sequence spaces in generalized Orlicz space as follows:

$$W(M, \Phi) = \{(f_k) \in \omega(L_M) : \lim_{n \to \infty} \frac{1}{n} \sum_{k|n} \varphi(k) \rho_M(\lambda |f_k - f_0|) = 0 \text{ for some } \lambda > 0, f_0 \in L_M\},$$

$$W^\infty(M, \Phi) = \{(f_k) \in \omega(L_M) : \sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{k|n} \varphi(k) \rho_M(\lambda |f_k|) = 0 \text{ for some } \lambda > 0\}.$$

Theorem 1. *If the Orlicz function M satisfies the Δ_2-condition, then the following equalities hold:*

$$W(M,\Phi)=\{(f_k)\in\omega(L_M):\lim_{n\to\infty}\frac{1}{n}\sum_{k|n}\varphi(k)\rho_M(|f_k-f_0|)=0, f_0\in L_M\},$$

$$W^{\infty}(M,\Phi)=\{(f_k)\in\omega(L_M):\sup_{n\in\mathbb{N}}\frac{1}{n}\sum_{k|n}\varphi(k)\rho_M(|f_k|)=0\}.$$

Proof: Denote the right hand side of the first equality by $W_0(\rho_M,\Phi)$. It is clear that $W_0(\rho_M,\Phi)\subset W(M,\Phi)$.

Now, choose $(f_k)\in W(M,\Phi)$. If $\lambda\geq 1$, we have $(f_k)\in W_0(\rho_M,\Phi)$ since M is a non-decreasing function. If $\lambda<1$, there exists $K(\lambda)>0$ such that $M(\frac{x}{\lambda})\leq K(\lambda)M(x)$ for all $x\geq 0$ since M satisfies Δ_2-condition. Hence, we deduce that

$$\begin{aligned}\frac{1}{n}\sum_{k|n}\varphi(k)\rho_M(|f_k-f_0|) &= \frac{1}{n}\sum_{k|n}\varphi(k)\int_E M\left(\frac{\lambda}{\lambda}|f_k(x)-f_0(x)|\right)dx\\ &\leq \frac{K(\lambda)}{n}\sum_{k|n}\varphi(k)\int_E M\left(\lambda|f_k(x)-f_0(x)|\right)dx\to 0\end{aligned}$$

as $n\to\infty$. This proves that $(f_k)\in W_0(\rho_M,\Phi)$. Hence, we conclude that $W(M,\Phi)=W_0(\rho_M,\Phi)$. □

Remark 1. *Note that if the Orlicz function M is defined by $M(x)=|x|^p$ for $1<p<\infty$, then the space is reduced to the following space*

$$W(p,\Phi)=\{(f_k)\in\omega(L_p):\lim_{n\to\infty}\frac{1}{n}\sum_{k|n}\varphi(k)\int_E|f_k(x)-f_0(x)|^p dx=0, f_0\in L_p\},$$

where $L_p=\{f:E\to\mathbb{R}:\int_E|f(x)|^p dx<\infty\}$.

Using the fact that ρ_M is a convex modular on L_M, we obtain the following results.

Theorem 2. *The function $\rho:\omega(L_M)\to[0,\infty)$ given by*

$$\rho(f)=\sup_{n\in\mathbb{N}}\frac{1}{n}\sum_{k|n}\rho_M(f_k)$$

is a convex modular, where $f=(f_k)\in\omega(L_M)$.

Theorem 3. *The space*

$$W^{\infty}(M,\Phi)=\{f\in\omega(L_M):\rho(\lambda f)<\infty,\lambda>0\}$$

is a modular space.

Proof: Clearly, the space $W^{\infty}(M,\Phi)$ is linear. Also, $(\omega(L_M))_\rho=\{f\in\omega(L_M):\lim_{\lambda\to 0}\rho(\lambda f)=0\}\subset W^{\infty}(M,\Phi)$ holds. To prove the inverse inclusion, choose $f\in W^{\infty}(M,\Phi)$ which means $\rho(\lambda f)<\infty$ for some $\lambda>0$. By convexity of ρ, for $|\frac{\alpha}{\lambda}|<1$, we have

$$\lim_{\alpha\to 0}\rho(\alpha f)=\lim_{\alpha\to 0}\frac{\alpha}{\lambda}\rho(\lambda f)=0.$$

This implies that $f\in(\omega(L_M))_\rho$. □

Since ρ is a modular, we can define the Luxemburg norm $\|.\|_\rho$ on $W^{\infty}(M,\Phi)$ as

$$\|f\|_\rho=\inf\{\gamma>0:\rho\left(\frac{f}{\gamma}\right)\leq 1, f\in W^{\infty}(M,\Phi)\}.$$

Definition 5. *Let (f^n) be a sequence in $W^{\infty}(M,\Phi)$.*

It is said to be ρ-convergent or modular convergent to $f\in W^{\infty}(M,\Phi)$ if there exists $\lambda>0$ such that $\lim_{n\to\infty}\rho(\lambda(f^n-f))=0$.

It is said to be ρ-Cauchy if there exists $\lambda>0$ such that $\lim_{n,m\to\infty}\rho(\lambda(f^n-f^m))=0$.

Theorem 4. *The space $W^{\infty}(M,\Phi)$ is ρ-complete.*

Theorem 5. *The space $W^{\infty}(M,\Phi)$ is complete with the Luxemburg norm $\|.\|_\rho$.*

Theorem 6. *If the Orlicz function M satisfies Δ_2-condition, then the norm convergence and modular convergence are equivalent.*

Theorem 7. *The space $W(M, \Phi)$ is a closed subspace of $W^{\infty}(M, \Phi)$.*

Proof: It is clear that $W(M, \Phi)$ is a linear subspace of $W^{\infty}(M, \Phi)$. Now, let (f^m) be a convergent sequence in $W(M, \Phi)$. Since $f^m \in W(M, \Phi)$ for each $m \in \mathbb{N}$, then there exists $f_0^m \in L_M$ and $\lambda > 0$ such that $\lim_n \frac{1}{n} \sum_{k|n} \varphi(k) \rho_M(\lambda(f_k^m - f_0^m)) = 0$. Also, since (f^m) is convergent, then $\rho(\lambda(f^m - f)) \to 0$ as $m \to \infty$ for some $f = (f_k) \in W^{\infty}(M, \Phi)$. Hence, we have

$$\frac{1}{n} \sum_{k|n} \varphi(k) \rho_M(\lambda(f_k^m - f_k)) \to 0$$

. It follows that

$$\frac{1}{n} \sum_{k|n} \varphi(k) \rho_M(\lambda(f_k - f_0^m)) \to 0$$

as $n \to \infty$ which implies that $f = (f_k) \in W(M, \Phi)$. Thus the space $W(M, \Phi)$ is closed. □

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