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On Some Identities And Generating Function of Both K-jacobsthal Numbers and Symmetric Functions in Several Variables

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Abstract

In this paper, we derive new generating functions for the products of k-Fibonacci numbers at negative indices, k-Pell numbers at negative indices, k-Jacobsthal numbers at negative indices, the product of k-Fibonacci numbers and Tribonacci numbers, Jacobsthal polynomials and symmetric functions in several variables by making use of useful properties of the symmetric functions.

Keywords: *k-Fibonacci and k- Jacobsthal numbers at negative indices; Generating functions; Symmetric functions.*
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1. Introduction and Preliminaries

The Fibonacci numbers and their generalizations have many interesting properties and applications to almost every field of science and art (for e.g, see [15]). Fibonacci numbers F_n are defined by the recurrence relation

$$\begin{cases} F_0 = 1, F_1 = 1 \\ F_{n+1} = F_n + F_{n-1},\ n \geq 1, \end{cases}$$

so, there exist many properties for Fibonacci numbers. In particular, there is a beautiful combinatorial identity to Fibonacci numbers [16]

$$F_n = \sum_{i=0}^{\left[\frac{n-1}{2}\right]} \left(\begin{array}{c} n-i-1 \\ i \end{array} \right) \tag{1.1}$$

From (1.1), Falcon [17] introduced the incomplete Fibonacci numbers $F_n(s)$**,** which are defined by

$$F_n(s) = \sum_{j=0}^{s} \left(\begin{array}{c} n-j-1 \\ j \end{array} \right),\ 0 \leq s \leq \left[\frac{n-1}{2}\right];\ n = 0,1,2,..,$$

On the other hand, many kinds of bf Fibonacci numbers generalizations have been presented in the literature, in particular, the k-Fibonacci Numbers. For any positive real number k, the k-Fibonacci sequence, say $(F_{n,k})_{n\in\mathbb{N}}$, is defined recurrently by [17]

$$\begin{cases} F_{k,0} = 1, F_{k,1} = k \\ F_{k,n+1} = kF_{k,n} + F_{k,n-1},\ n \geq 1 \end{cases}$$

In [16], the k-Fibonacci numbers were found by studying the recursive application of two geometrical transformations used in the four-triangle longest-edge (4TLE) partition. These numbers have been studied in several works [16, 17].

Definition 1.1. *For any positive real number k, the k-Lucas numbers, say* $\{L_{k,n}\}_{n\in\mathbb{N}}$ *is defined recurrently by*

$L_{k,n+1} = kL_{k,n} + L_{k,n-1}$ *for* $n \geq 1$,

with initial conditions $L_{k,0} = 2; L_{k,1} = k$.

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Definition 1.2. *[12] For any positive real number k, the k-Pell numbers, say $\{P_{k,n}\}_{n\in\mathbb{N}}$ is defined recurrently by*

$P_{k,n+1} = 2P_{k,n} + kP_{k,n-1}$ *for* $n \geq 1$,

with initial conditions $P_{k,0} = 0; P_{k,1} = 1$.

Definition 1.3. *For any positive real number k, the k-Pell Lucas numbers, say $\{Q_{k,n}\}_{n\in\mathbb{N}}$ is defined recurrently by*

$Q_{k,n+1} = 2Q_{k,n} + kQ_{k,n-1}$ *for* $n \geq 1$,

with initial conditions $Q_{k,0} = 2; Q_{k,1} = 2$.

Definition 1.4. *[14] For any positive real number k, the k-Jacobsthal numbers, say $(J_{n,k})_{n\in\mathbb{N}}$, is defined recurrently by*

$$\begin{cases} J_{k,0} = 0, J_{k,1} = 1 \\ J_{k,n+1} = kJ_{k,n} + 2J_{k,n-1},\ n \geq 1 \end{cases}.$$

Definition 1.5. *[13]For $n \in \mathbb{N}$, the Jacobsthal polynomials, say $(J_n(x))_{n\in\mathbb{N}}$, is defined recurrently by*

$$\begin{cases} J_0(x) = 0, J_1(x) = 1 \\ J_{n+1}(x) = J_n(x) + 2xJ_{n-1}(x),\ n \geq 1 \end{cases}.$$

Definition 1.6. *For $n \in \mathbb{N}$, the Tribonacci numbers, say $(T_n)_{n\in\mathbb{N}}$, is defined recurrently by*

$$\begin{cases} T_0 = 1, T_1 = 1, T_2 = 2 \\ T_{n+1} = T_n + T_{n-1} + T_{n-2},\ n \geq 2 \end{cases}.$$

The main purpose of this paper is to present some results involving the k-Fibonacci, k-Pell and k-Jacobsthal numbers at negative indices using define a useful operator denoted by $\delta_{p_1p_2}$. By making use of this operator, we can derive new results based on our previous ones [5, 9, **?**]. In order to determine the generating functions of the product of k-Fibonacci, k-Pell and k-Jacobsthal numbers with negative indices and Jacobsthal polynomials.

In order to render the work self-contained, we give the necessary preliminaries tools and recall some defnitions and results.

Definition 1.7. *[6] Let B and P be any two alphabets. We define $S_n(B-P)$ by the following form*

$$\frac{\Pi_{p\varepsilon P}(1-pt)}{\Pi_{b\varepsilon B}(1-bt)} = \sum_{n=0}^{\infty} S_n(B-P)t^n, \tag{1.2}$$

with the condition $S_n(B-P) = 0$ *for* $n < 0$. *[7]*

Corollary 1.8. *[7] Taking $B = \{0, 0, ..., 0\}$ in (1.2) gives*

$$\Pi_{p\in P}(1-pt) = \sum_{n=0}^{\infty} S_n(-P)t^n.$$

Equation (1.2) can be rewritten in the following form

$$\sum_{n=0}^{\infty} S_n(B-P)t^n = \left(\sum_{n=0}^{\infty} S_n(B)t^n\right) \times \left(\sum_{n=0}^{\infty} S_n(-P)t^n\right),$$

where

$$S_n(B-P) = \sum_{j=0}^{n} S_{n-j}(-P)S_j(B).$$

Definition 1.9. *[10] Given a function f on $\mathbb{R}^n$, the divided difference operator is defined as follows*

$$\partial_{p_ip_{i+1}}(f) = \frac{f(p_1, \cdots, p_i, p_{i+1}, \cdots p_n) - f(p_1, \cdots p_{i-1}, p_{i+1}, p_i, p_{i+2} \cdots p_n)}{p_i - p_{i+1}}.$$

Definition 1.10. *The symmetrizing operator $\delta^k_{e_1e_2}$ is defined by*

$$\delta^k_{p_1p_2}(g(p_1)) = \frac{p_1^k g(p_1) - p_2^k g(p_2)}{p_1 - p_2} \text{ for all } k \in \mathbb{N}.$$

Proposition 1.11. *[**?**] Let $P = \{p_1, p_2\}$ an alphabet, we define the operator $\delta^k_{p_1p_2}$ as follows*

$$\delta^k_{p_1p_2} g(p_1) = S_{k-1}(p_1+p_2)g(p_1) + p_2^k \partial_{p_1p_2} g\ (p_1), \text{ for all } k \in \mathbb{N}.$$

Proposition 1.12. *[5] The relations*

$$\begin{aligned} 1)F_{k,-n} &= (-1)^{n+1}F_{k,n}, \\ 2)P_{k,-n} &= (-1)^{n+1}P_{k,n} \\ 3)L_{k,-n} &= (-1)^{n}L_{k,n}, \\ 4)Q_{k,-n} &= (-1)^{n}Q_{k,n}, \\ 5)J_{k,-n} &= (-1)^{n-1}2^{-n}J_{k,n} \end{aligned}$$

hold for all $n \geq 0$.

Proposition 1.13. *Let $\{P_n\}_{n\geq 0}$ sequence is called symmetric when*

$P_n(-x) = (-1)^n P_n(x),\ n \geq 0.$

2. Theorem and Proof

The following Theorem is one of the key tools of the proof of our main result. It has been proved in [10]. For the completeness of the paper we state its proof here.

Theorem 2.1. *Given two alphabets $P = \{p_1, p_2\}$ and $B = \{b_1, b_2, b_3\}$, we have*

$$\sum_{n=0}^{\infty} S_n(B)\partial_{p_1p_2}(p_1^{n+1})t^n = \frac{S_0(-B) - p_1p_2S_2(-B)t^2 - p_1p_2S_3(-B)(p_1+p_2)t^3}{\left(\sum_{n=0}^{\infty} S_n(-B)p_1^nt^n\right)\left(\sum_{n=0}^{\infty} S_n(-B)p_2^nt^n\right)}. \tag{2.1}$$

Proof. Let $\sum_{n=0}^{\infty} S_n(B)t^n$ and $\sum_{n=0}^{\infty} S_n(-B)t^n$ be two sequences such that $\sum_{n=0}^{\infty} S_n(B)t^n = \frac{1}{\sum_{n=0}^{\infty} S_n(-B)t^n}$. On one hand, since $g(p_1) = \sum_{n=0}^{\infty} S_n(B)p_1^nt^n$, we have

$$\begin{aligned}
\delta_{p_1p_2}g(p_1) &= \delta_{p_1p_2}\left(\sum_{n=0}^{\infty} S_n(B)p_1^nt^n\right) \\
&= \frac{p_1\sum_{n=0}^{\infty} S_n(B)p_1^nt^n - p_2\sum_{n=0}^{\infty} S_n(B)p_2^nt^n}{p_1-p_2} \\
&= \sum_{n=0}^{\infty} S_n(B)\left(\frac{p_1^{n+1}-p_2^{n+1}}{p_1-p_2}\right)t^n \\
&= \sum_{n=0}^{\infty} S_n(B)\partial_{p_1p_2}(p_1^{n+1})t^n
\end{aligned}$$

which is the right-hand side of (2.1). On the other part, since

$$g(p_1) = \frac{1}{\sum_{n=0}^{\infty} S_n(-B)p_1^nt^n},$$

we have

$$\delta_{p_1p_2}g(p_1) = \frac{p_1\prod_{b\in B}(1-bp_2)t - p_2\prod_{b\in B}(1-bp_1t)}{(p_1-p_2)\left(\sum_{n=0}^{\infty} S_n(-B)p_1^nt^n\right)\left(\sum_{n=0}^{\infty} S_n(-B)p_2^nt^n\right)}.$$

Using the fact that : $\sum_{n=0}^{\infty} S_n(-B)p_1^nt^n = \prod_{b\in B}(1-bp_1t)$, then

$$\begin{aligned}
\delta_{p_1p_2}g(p_1) &= \frac{\sum_{n=0}^{\infty} S_n(-B)\frac{p_1p_2^n-p_2p_1^n}{p_1-p_2}t^n}{\left(\sum_{n=0}^{\infty} S_n(-B)p_1^nt^n\right)\left(\sum_{n=0}^{\infty} S_n(-B)p_2^nt^n\right)} \\
&= \frac{S_0(-B) - p_1p_2S_2(-B)t^2 - p_1p_2(p_1+p_2)S_3(-B)t^3}{\left(\sum_{n=0}^{\infty} S_n(-B)p_1^nt^n\right)\left(\sum_{n=0}^{\infty} S_n(-B)p_2^nt^n\right)}.
\end{aligned}$$

This completes the proof. □

3. On the Generating Functions of Some Numbers and Polynomails

In this part, we derive the new generating functions of the products of some known numbers. For the applications of generating functions of some known functions, we refer the reader to see the references [?, ?].

Remark 1: Replacing p_2 by $(-p_2)$ and assuming that $p_1p_2 = 1$, $p_1 - p_2 = k$, $S_1(-B) = -1$, $S_2(-B) = -1$ and $S_3(-B) = -1$ in the relationship (2.1), we deduce the following theorems.

Theorem 3.1. *The new generating function of product of k-Fibonacci numbers and Tribonacci numbers is given by*

$$\sum_{n=0}^{\infty} T_nF_{k,n}t^n = \frac{1-t^2-kt^3}{1-kt-(k^2+3)t^2-(k^3+4k)t^3-(k^2+1)t^4+kt^5-t^6}. \tag{3.1}$$

Theorem 3.2. *The new generating function of product of k-Fibonacci numbers with negative indices and Tribonacci numbers is given by*

$$\sum_{n=0}^{\infty} T_nF_{k,-n}t^n = \frac{-1+t^2-kt^3}{1+kt-(k^2+3)t^2+(k^3+4k)t^3-(k^2+1)t^4-kt^5-t^6}. \tag{3.2}$$

- Put $k=1$ in the relationship (3.2) we have

$$\sum_{n=0}^{\infty} T_n F_{-n} t^n = \frac{-1+t^2-t^3}{1+t+4t^2+5t^3-2t^4-t^5-t^6},$$

which representing a new generating function of Fibonacci numbers at negative indices and Tribonacci numbers.

Setting $b_3=0$ and replacing b_2 by $(-b_2), p_2$ by $(-p_2)$ in the relationship (2.1), and assuming $b_1-b_2=p_1-p_2=k$; $b_1b_2=p_1p_2=1$; we deduce the following theorems.

Theorem 3.3. *For $n\in\mathbb{N}$, the new generating function of the product of k-Fibonacci numbers at negative indices is given by*

$$\sum_{n=0}^{\infty} F_{k,-n}^2 t^n = \frac{1-t^2}{1-k^2t-2(k^2+1)t^2-k^2t^3+t^4}. \tag{3.3}$$

Corollary 3.4. *If $k=1$ in the relationship (3.3) we get*

$$\sum_{n=0}^{\infty} F_{-n}^2 t^n = \frac{1-t^2}{1-t-4t^2-t^3+t^4},$$

which representing a new generating function of the product of Fibonacci numbers with negative indices.

Theorem 3.5. *For $n\in\mathbb{N}$, the new generating function of the product of k-Lucas numbers at negative indices is given by*

$$\sum_{n=0}^{\infty} L_{k,-n}^2 t^n = \frac{4-3k^2t-4(k^2+1)t^2-k^2t^3}{1-k^2t-2(k^2+1)t^2-k^2t^3+t^4}. \tag{3.4}$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} L_{k,-n}^2 t^n &= \sum_{n=0}^{\infty}[(-1)^n((2+k^2)S_n(b_1+[-b_2])-kS_{n+1}(b_1+[-b_2]))\\
&\quad \times(-1)^n((2+k^2)S_n(p_1+[-p_2])-kS_{n+1}(p_1+[-p_2]))]t^n\\
&= (2+k^2)^2\sum_{n=0}^{\infty}(-1)^{2n}S_n(b_1+[-b_2])S_n(p_1+[-p_2])t^n-\\
&\quad k(2+k^2)\sum_{n=0}^{\infty}(-1)^{2n}S_{n+1}(p_1+[-p_2])S_n(b_1+[-b_2])t^n\\
&\quad -k(2+k^2)\sum_{n=0}^{\infty}(-1)^{2n}S_{n+1}(b_1+[-b_2])S_n(p_1+[-p_2])t^n+\\
&\quad k^2\sum_{n=0}^{\infty}(-1)^{2n}S_{n+1}(b_1+[-b_2])S_{n+1}(p_1+[-p_2])t^n\\
&= (2+k^2)^2\sum_{n=0}^{\infty}F_{k,n}t^n-k(2+k^2)\\
&\quad \times[\frac{k+(p_1-p_2)t}{1-k(p_1-p_2)t-[(p_1-p_2)^2+2p_1p_2+k^2p_1p_2]t^2-k(p_1-p_2)p_1p_2t^3+p_1^2p_2^2t^4}]\\
&\quad -k(2+k^2)[\frac{k+(b_1-b_2)t}{1-k(b_1-b_2)t-[(b_1-b_2)^2+2b_1b_2+k_1^2b_1b_2]t^2-k(b_1-b_2)b_1b_2t^3+b_1^2b_2^2t^4}]\\
&\quad +k^2[\frac{k(p_1-p_2)-[(p_1-p_2)^2+p_1p_2+k^2p_1p_2]t-k(p_1-p_2)p_1p_2t^2+p_1^2p_2^2t^3}{1-k(p_1-p_2)t-[(p_1-p_2)^2+2p_1p_2+k^2p_1p_2]t^2-k(p_1-p_2)p_1p_2t^3+p_1^2p_2^2t^4}].
\end{aligned}$$

Since

$$\sum_{n=0}^{\infty} F_{k,n}^2 t^n = \frac{1-t^2}{1-k^2t-2(k^2+1)t^2-k^2t^3+t^4},$$

therefore

$$\sum_{n=0}^{\infty} L_{k,-n}^2 t^n = \frac{4-3k^2t-4(k^2+1)t^2-k^2t^3}{1-k^2t-2(k^2+1)t^2-k^2t^3+t^4}.$$

This completes the proof. □

Theorem 3.6. *For $n\in\mathbb{N}$, the new generating function of the product of k-Pell Lucas numbers at negative indices is given by*

$$\sum_{n=0}^{\infty} Q_{k,-n}^2 t^n = \frac{4-12t-4(4k+k^2)t^2-4k^2t^3}{1+4t-2(k^2+4k)t^2-4k^2t^3+k^4t^4}. \tag{3.5}$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} Q_{k,-n}^{2} t^{n} &= \sum_{n=0}^{\infty} [(-1)^{n}(S_{n+1}(b_1+[-b_2])-(2+k)S_{n-1}(b_1+[-b_2])) \\
&\quad \times(-1)^{n}(S_{n+1}(p_1+[-p_2])-(2+k)S_{n-1}(p_1+[-p_2]))]t^{n} \\
&= \sum_{n=0}^{\infty}(-1)^{2n}S_{n+1}(b_1+[-b_2])S_{n+1}(p_1+[-p_2])t^{n}-(2+k)\sum_{n=0}^{\infty}(-1)^{2n}S_{n+1}(b_1+[-b_2])S_{n-1}(p_1+[-p_2])t^{n} \\
&\quad -(2+k)\sum_{n=0}^{\infty}(-1)^{2n}S_{n+1}(p_1+[-p_2])S_{n-1}(b_1+[-b_2])t^{n}+ \\
&\quad (2+k)^{2}\sum_{n=0}^{\infty}(-1)^{2n}S_{n-1}(b_1+[-b_2])S_{n-1}(p_1+[-p_2])t^{n} \\
&= [\frac{2(p_1-p_2)+[k(p_1-p_2)^{2}+kp_1p_2+4p_1p_2]t+2kp_1p_2(p_1-p_2)^{2}-k^{2}p_1^{2}p_2^{2}t^{3}}{1-2(p_1-p_2)t-[k(p_1-p_2)^{2}+2kp_1p_2+4p_1p_2]t^{2}-2k(p_1-p_2)p_1p_2t^{3}+k^{2}p_1^{2}p_2^{2}t^{4}}] \\
&\quad -(2+k)[\frac{(k+4)t+2k(p_1-p_2)t^{2}-k^{2}p_1p_2t^{3}}{1-2(p_1-p_2)t-[k(p_1-p_2)^{2}+2kp_1p_2+4p_1p_2]t^{2}-2k(p_1-p_2)p_1p_2t^{3}+k^{2}p_1^{2}p_2^{2}t^{4}}] \\
&\quad -(2+k)[\frac{(k+4)t+2k(b_1-b_2)t^{2}-k^{2}b_1b_2t^{3}}{1-2(b_1-b_2)t-[k(b_1-b_2)^{2}+2kb_1b_2+4b_1b_2]t^{2}-2k(b_1-b_2)b_1b_2t^{3}+k^{2}b_1^{2}b_2^{2}t^{4}}] \\
&\quad +(2+k)^{2}\sum_{n=0}^{\infty}P_{k,n}^{2}t^{n}.
\end{aligned}$$

Since

$$\sum_{n=0}^{\infty} P_{k,n}^{2} t^{n} t^{n} = \frac{t-k^{2}t^{3}}{1+4t-2(k^{2}+4k)t^{2}-4k^{2}t^{3}+k^{4}t^{4}},$$

therefore

$$\sum_{n=0}^{\infty} Q_{k,-n}^{2} t^{n} = \frac{4-12t-4(4k+k^{2})t^{2}-4k^{2}t^{3}}{1+4t-2(k^{2}+4k)t^{2}-4k^{2}t^{3}+k^{4}t^{4}}.$$

This completes the proof. □

Remark 2: Replacing p_2 by $(-p_2)$ and assuming that $p_1p_2=1,\ p_1-p_2=x$ in the relationship (2.1), we deduce the following theorems

Theorem 3.7. *The new generating function of both Fibonacci polynomials and symmetric functions in several variables as*

$$\sum_{n=0}^{\infty} S_n(b_1+b_2+b_3)F_n(x)t^{n} = \frac{S_0(-B)+S_2(-B)t^{2}+xS_3(-B)t^{3}}{\prod_{i=1}^{3}\left(1-xb_it-b_i^{2}t^{2}\right)}. \tag{3.6}$$

Theorem 3.8. *The new generating function of both symmetric Fibonacci polynomials and symmetric functions in several variables are defined as*

$$\sum_{n=0}^{\infty} S_n(b_1+b_2+b_3)F_n(-x)t^{n} = \frac{S_0(-B)+S_2(-B)t^{2}-xS_3(-B)t^{3}}{\prod_{i=1}^{3}\left(1+xb_it-b_i^{2}t^{2}\right)}.$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} S_n(b_1+b_2+b_3)F_n(-x)t^{n} &= \sum_{n=0}^{\infty} S_n(b_1+b_2+b_3)(-1)^{n}F_n(x)t^{n} \\
&= \sum_{n=0}^{\infty} S_n(b_1+b_2+b_3)F_n(x)(-t)^{n} \\
&= \frac{S_0(-B)+S_2(-B)t^{2}-xS_3(-B)t^{3}}{\prod_{i=1}^{3}\left(1+xb_it-b_i^{2}t^{2}\right)}.
\end{aligned}$$

This completes the proof. □

Remark 3: Replacing b_2 by $(-b_2)$, p_2 by $(-p_2)$ and assuming that $b_1b_2=p_1p_2=2$, $b_1-b_2=p_1-p_2=k$ in the relationship (2.1), we deduce the following theorems.

Theorem 3.9. *For $n\in\mathbb{N}$, the new generating function of product of k-Jcobsthal numbers is given by*

$$\sum_{n=0}^{\infty} J_{k,n}^2 t^n = \frac{t+2kt^2}{1-k^2t-4(k^2+1)t^2-4kt^3+16t^4}.$$

Theorem 3.10. *For $n \in \mathbb{N}$, the new generating function of the product of k-Jacobsthal numbers and k-Jacobsthal numbers with negative indices is given*

$$\sum_{n=0}^{\infty} J_{k,n} J_{k,-n} t^n = \frac{\frac{t}{2} - \frac{k}{2}t^2}{1+\frac{k^2}{2}t-(k^2+1)t^2+\frac{k}{2}t^3+t^4}. \tag{3.7}$$

Corollary 3.11. *If $k = 1$ in the relationship (3.7) we get*

$$\sum_{n=0}^{\infty} J_n J_{-n} t^n = \frac{\frac{t}{2}t + \frac{t^2}{2}}{1+\frac{t}{2}-2t^2+\frac{t^3}{2}+t^4},$$

which representing a new generating function of the product of Jacobsthal numbers and Jacobsthal numbers with negative indices.

Theorem 3.12. *For $n \in \mathbb{N}$, the new generating function of the product of k-Jacobsthal numbers at negative indices is given by*

$$\sum_{n=0}^{\infty} J_{k,-n}^2 t^n = \frac{\frac{t}{4} + \frac{k}{8}t^2}{1-\frac{k^2}{4}t-\frac{(k^2+1)}{4}t^2-\frac{k}{16}t^3+\frac{t^4}{16}}.$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} J_{k,-n}^2 t^n &= \sum_{n=0}^{\infty} J_{k,-n} J_{k,-n} t^n \\
&= \sum_{n=0}^{\infty} (-1)^{n-1} 2^{-n} J_{k,n} (-1)^{n-1} 2^{-n} J_{k,n} t^n \\
&= \sum_{n=0}^{\infty} (-1)^{2n-2} 2^{-2n} J_{k,n}^2 t^n \\
&= \sum_{n=0}^{\infty} J_{k,n}^2 \left(\frac{t}{4}\right)^n \\
&= \frac{\frac{t}{4} + \frac{k}{8}t^2}{1-\frac{k^2}{4}t-\frac{(k^2+1)}{4}t^2-\frac{k}{16}t^3+\frac{t^4}{16}}.
\end{aligned}$$

This completes the proof. □

Theorem 3.13. *[2]Given two alphabets $P = \{p_1, p_2\}$ and $B = \{b_1, b_2, b_3\}$, we have*

$$\sum_{n=0}^{\infty} S_n(b_1+b_2+b_3) S_{n-1}(p_1+p_2) t^n = \frac{-S_1(-B)t-(p_1+p_2)S_2(-B)t^2-((p_1+p_2)^2-p_1p_2)S_3(-B)t^3}{\left(\sum_{n=0}^{\infty} S_n(-B)p_1^n t^n\right)\left(\sum_{n=0}^{\infty} S_n(-B)p_2^n t^n\right)}. \tag{3.8}$$

Remark 4: Replacing p_2 by $(-p_2)$ and assuming that $p_1p_2 = k,\ p_1 - p_2 = 2$, $S_1(-B) = -1$, $S_2(-B) = -1$ and $S_3(-B) = -1$ in the relationship (3.8), we deduce the following theorems.

Theorem 3.14. *For $n \in \mathbb{N}$, the new generating function of product of k-Pell numbers and Tribonacci numbers as follows*

$$\sum_{n=0}^{\infty} T_n P_{k,n} t^n = \frac{t+2t^2+(4+k)t^3}{1-2t-(3k+4)t^2-8(k+1)t^3-(k^2+4k)t^4+2k^2t^5-k^3t^6}. \tag{3.9}$$

Theorem 3.15. *For $n \in \mathbb{N}$, the new generating function of product of k-Pell numbers with negative indices and Tribonacci numbers as follows*

$$\sum_{n=0}^{\infty} T_n P_{k,-n} t^n = \frac{t-2t^2+(4+k)t^3}{1+2t-(3k+4)t^2+8(k+1)t^3-(k^2+4k)t^4-2k^2t^5-k^3t^6}. \tag{3.10}$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} T_n P_{k,-n} t^n &= \sum_{n=0}^{\infty} T_n\ (-1)^{n+1} P_{k,n} t^n \\
&= -\sum_{n=0}^{\infty} T_n P_{k,n} (-t)^n \\
&= -\left[\frac{-t+2t^2-(4+k)t^3}{1+2t-(3k+4)t^2+8(k+1)t^3-(k^2+4k)t^4-2k^2t^5-k^3t^6}\right] \\
&= \frac{t-2t^2+(4+k)t^3}{1+2t-(3k+4)t^2+8(k+1)t^3-(k^2+4k)t^4-2k^2t^5-k^3t^6}.
\end{aligned}$$

This completes the proof. □

Remark 5: Replacing p_2 by $(-p_2)$ and assuming that $p_1p_2 = 2,\ p_1 - p_2 = k$, in the relationship (3.8), we deduce the following theorems.

Theorem 3.16. *For $n \in \mathbb{N}$, the new generating function of both k-Jacobsthal numbers and symmetric functions in several variables as*

$$\sum_{n=0}^{\infty} S_n(b_1+b_2+b_3)J_{k,n}t^n = \frac{-S_1(-B)t - kS_2(-B)t^2 - (k^2+2)S_3(-B)t^3}{\prod_{i=1}^{3}\left(1 - kb_it - 2b_i^2t^2\right)}. \tag{3.11}$$

Corollary 3.17. *If $k = 1$ in the relationship (3.11) we get*

$$\sum_{n=0}^{\infty} S_n(b_1+b_2+b_3)J_nt^n = \frac{-S_1(-B)t - S_2(-B)t^2 - 3S_3(-B)t^3}{\prod_{i=1}^{3}\left(1 - b_it - 2b_i^2t^2\right)}.$$

which representing a new generating function of both Jacobsthal numbers and symmetric functions in several variables.

Theorem 3.18. *For $n \in \mathbb{N}$, the new generating function of both k-Jcobsthal numbers with negative indices and symmetric functions in several variables as*

$$\sum_{n=0}^{\infty} S_n(b_1+b_2+b_3)J_{k,-n}t^n = \frac{\frac{-S_1(-B)}{2}t + \frac{k}{4}S_2(-B)t^2 - \frac{(k^2-2)}{8}S_3(-B)t^3}{\prod_{i=1}^{3}\left(1 + \frac{k}{2}b_it - \frac{b_i^2}{2}t^2\right)}. \tag{3.12}$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} S_n(b_1+b_2+b_3)J_{k,-n}t^n &= \sum_{n=0}^{\infty} S_n(b_1+b_2+b_3)(-1)^{n-1}2^{-n}J_{k,n}t^n \\
&= -\sum_{n=0}^{\infty} S_n(b_1+b_2+b_3)J_{k,n}(\frac{-t}{2})^n \\
&= -[\frac{\frac{S_1(-B)}{2}t - \frac{k}{4}S_2(-B)t^2 + \frac{(k^2-2)}{8}S_3(-B)t^3}{\prod_{i=1}^{3}\left(1 + \frac{k}{2}b_it - \frac{b_i^2}{2}t^2\right)}] \\
&= \frac{\frac{-S_1(-B)}{2}t + \frac{k}{4}S_2(-B)t^2 - \frac{(k^2-2)}{8}S_3(-B)t^3}{\prod_{i=1}^{3}\left(1 + \frac{k}{2}b_it - \frac{b_i^2}{2}t^2\right)}.
\end{aligned}$$

This completes the proof. □

- Put $k = 1$ in the relationship (3.12) we have

  $$\sum_{n=0}^{\infty} S_n(b_1+b_2+b_3)J_{-n}t^n = \frac{\frac{-S_1(-B)}{2}t + \frac{S_2(-B)}{4}t^2 - \frac{S_3(-B)}{8}t^3}{\prod_{i=1}^{3}\left(1 + \frac{b_i}{2}t - \frac{b_i^2}{2}t^2\right)}, \tag{3.13}$$

  which representing a new generating function of both Jacobsthal numbers at negative indices and symmetric functions in several variables.

Remark 6: Replacing p_2 by $(-p_2)$ and assuming that $p_1p_2 = 2x,\ p_1 - p_2 = 1$, in the relationship (3.8), we deduce the following theorems

Theorem 3.19. *For $n \in \mathbb{N}$, The new generating function of both Jacobsthal polynomials and symmetric functions in several variables as*

$$\sum_{n=0}^{\infty} S_n(b_1+b_2+b_3)J_n(x)t^n = \frac{-S_1(-B)t - S_2(-B)t^2 - (1+2x)S_3(-B)t^3}{\prod_{i=1}^{3}\left(1 - b_it - 2xb_i^2t^2\right)}. \tag{3.14}$$

Theorem 3.20. *For $n \in \mathbb{N}$, The new generating function of both symmetric Jacobsthal polynomials and symmetric functions in several variables as*

$$\sum_{n=0}^{\infty} S_n(b_1+b_2+b_3)J_n(-x)t^n = \frac{S_1(-B)t - S_2(-B)t^2 + (1+2x)S_3(-B)t^3}{\prod_{i=1}^{3}\left(1 + b_it - 2xb_i^2t^2\right)}. \tag{3.15}$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} S_n(b_1+b_2+b_3)J_n(-x)t^n &= \sum_{n=0}^{\infty} S_n(b_1+b_2+b_3)(-1)^nJ(x)t^n \\
&= \sum_{n=0}^{\infty} S_n(b_1+b_2+b_3)J(x)(-t)^n \\
&= \frac{S_1(-B)t - S_2(-B)t^2 + (1+2x)S_3(-B)t^3}{\prod_{i=1}^{3}\left(1 + b_it - 2xb_i^2t^2\right)}.
\end{aligned}$$

This completes the proof. □

4. Conclusion

In this work, a theorem has been proposed in order to determine the generating functions for the products of k-Fibonacci numbers, k-Pell numbers, k-Jacobsthal numbers all with negative indices, the product of k-Fibonacci numbers and Tribonacci numbers, k-Jacobsthal numbers and symmetric functions in several variables and product of Jacobsthal polynomials and symmetric functions in several variables. The proposed theorem is based on the symmetric functions.The obtained results agree with the results obtained in some previous works.

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