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Weighted Set Sharing and Uniqueness of Meromorphic Functions

Dilip Chandra Pramanik[1*]

[1] *Department of Mathematics, University of North Bengal, Raja Rammohanpur, Darjeeling-734013, West Bengal, India*
[*] *Corresponding author E-mail: dcpramanik.nbu2012@gmail.com*

Abstract

In this paper, we study the uniqueness problem of meromorphic functions sharing a set of small functions and proved that under certain essential conditions $P[f] = tp(f)$ for some t such that $t^m = 1$ (m is a positive number), where $P[f]$ is a differential polynomial in f and $p(z)$ is a polynomial in z of degree at least one such that $p(0) = 0$. Our results generalizes the results due to Zhang and Lü, Banerjee and Majumder, Bhoosnurmath and Kabur, and Charak and Lal.

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1. Introduction and main result

Let $\mathbb{C}$ denote the complex plane and let $f(z)$ be a non-constant meromorphic function defined on $\mathbb{C}$. We assume that the reader is familiar with the standard definitions and notions used in the Nevanlinna value distribution theory, such as $T(r,f), m(r,f), N(r,f)$(see [5, 7, 10, 11]). By $S(r,f)$ we denote any quantity satisfying the condition $S(r,f) = \circ(T(r,f))$ as $r \to \infty$ possibly outside an exceptional set of finite linear measure. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$ if either $a \equiv \infty$ or $T(r,a) = S(r,f)$. We denote by $S(f)$ the collection of all small functions with respect to f. Clearly $\mathbb{C} \cup \{\infty\} \in S(f)$ and $S(f)$ is a field over the set of complex numbers. For $a \in \mathbb{C} \cup \{\infty\}$ the quantities

$$\delta(a,f) = 1 - \limsup_{r\to\infty} \frac{N(r, \frac{1}{f-a})}{T(r,f)}$$

and

$$\Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, \frac{1}{f-a})}{T(r,f)}$$

are respectively called the deficiency and ramification index of a for the function f.

In this paper, we also need the following definitions:

Definition 1.1. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and let $a(z) \in S(f) \cap S(g)$. We write $E(a,f) = \{z \in \mathbb{C} : f(z) - a(z) = 0\}$, where zeros of $f(z) - a(z)$ are counted according to their multiplicities. Also by $\overline{E}(a,f)$, we denote the zeros of $f(z) - a(z)$, where a zero is counted only once. We say that f and g share the function $a(z)$ CM(counting multiplicity) if $E(a,f) = E(a,g)$. Further, if $\overline{E}(a,f) = \overline{E}(a,g)$ we say that f and g share the function $a(z)$ IM(ignoring multiplicity).*

Definition 1.2. *Let k be a nonnegative integer or infinity and $a(z) \in S(f)$. We denote by $E_k(a,f)$ the set of all zeros of $f - a$, where a zero of multiplicity m is counted m times if $m \le k$ and $k+1$ times if $m > k$. If $E_k(a,f) = E_k(a,g)$, we say that f, g share the function $a(z)$ with weight k. We write f and g share (a,k) to mean that f and g share the function $a(z)$ with weight k. Since $E_k(a,f) = E_k(a,g)$ implies that $E_l(a,f) = E_l(a,g)$ for any integer l $(0 \le l < k)$, if f, g share (a,k), then f, g share (a,l), $(0 \le l < k)$. Moreover, we note that f and g share the function $a(z)$ IM (ignoring multilicity) or CM (counting multiplicity) if and only if f and g share $(a,0)$ or (a,∞) respectively.*

Email addresses: **dcpramanik.nbu2012@gmail.com (D. C. Pramanik)**

Definition 1.3. *Let f and g share 1 IM and let z_0 be a zero of $f-1$ with multiplicity m and a zero of $g-1$ with multiplicity n. We denote by $N_E^{1)}(r,\frac{1}{f-1})$ the counting function of the zeros of $f-1$ when $m=n=1$. By $\overline{N}_E^{(2}(r,\frac{1}{f-1})$ we denote the counting function of the zeros of $f-1$ when $m=n\geq 2$ and by $\overline{N}_L(r,\frac{1}{f-1})$ we denote the counting function of the zeros of $f-1$ when $m>n\geq 1$; each point in these counting functions is counted only once. Similarly, we can define the terms $N_E^{1)}(r,\frac{1}{g-1})$, $\overline{N}_E^{(2}(r,\frac{1}{g-1})$ and $\overline{N}_L(r,\frac{1}{g-1})$. In addition, we denote by $\overline{N}_{f>k}(r,\frac{1}{g-1})$ the reduced counting function of those zeros of $f-1$ and $g-1$ such that $m>n=k$ and $\overline{N}_{g>k}(r,\frac{1}{f-1})$ is defined analogously.*

Definition 1.4. *Let $n_{0j}, n_{1j}, n_{2j}, \ldots, n_{kj}$ are nonnegative integers. The expression*

$$M_j[f]=(f)^{n_{0j}}(f^{(1)})^{n_{1j}}(f^{(2)})^{n_{2j}}(f^{(k)})^{n_{kj}}$$

is called a differential monomial generated by f of degree $d(M_j)=\sum_{i=0}^{k} n_{ij}$ and weight $\Gamma_{M_j}=\sum_{i=0}^{k}(i+1)n_{ij}$. Let $a_j\in S(f)$ and $a_j\not\equiv 0 (j=1,2,\ldots,t)$. The sum $P[f]=\sum_{j=1}^{t} a_jM_j[f]$ is called a differential polynomial generated by f of degree $\overline{d}(P)=\max\{d(M_j):1\leq j\leq t\}$ and weight $\Gamma_P=\max\{\Gamma_{M_j}:1\leq j\leq t\}$. The numbers $\underline{d}_P=\min\{d(M_j):1\leq j\leq t\}$ and k (the highest order of the derivative of f in $P[f]$) are called respectively the lower degree and the order of $P[f]$. $P[f]$ is said to be homogeneous differential polynomial of degree d if $\overline{d}_P=\underline{d}_P=d$. $P[f]$ is called a linear differential Polynomial generated by f if $\overline{d}_P=1$. Otherwise, $P[f]$ is called non-linear differential polynomial. Also, we denote by Q the quantity $Q=\max_{1\leq j\leq t}\sum_{i=0}^{k} i.n_{ij}$.

For the last few decades, the value sharing problems related to a meromorphic function f and its derivative $f^{(k)}$ have been a more widely studied subtopic among the researchers(see [6, 8, 9]) of the uniqueness theory of entire and meromorphic functions in the field of complex analysis.

In 2008, Zhang and Lü[13] proved the following result:

Theorem 1.5. *Let k, n be positive integers, f be a nonconstant meromorphic function, and $a(\not\equiv 0,\infty)$ be a meromorphic function satisfying $T(r,a)=\circ(T(r,f))$ as $r\to\infty$. If f^n and $f^{(k)}$ share a IM and*

$$(2k+6)\Theta(\infty,f)+4\Theta(0,f)+2\delta_{2+k}(0,f)>2k+12-n,$$

or f^n and $f^{(k)}$ share a CM and

$$(k+3)\Theta(\infty,f)+2\Theta(0,f)+\delta_{2+k}(0,f)>k+6-n,$$

then $f^n\equiv f^{(k)}$.

Bhoosnurmath and Kabbur [3] considered the uniqueness of f and $P[f]$, which is the more natural extention of $f^{(k)}$ and proved the following result:

Theorem 1.6. *Let f be a nonconstant meromorphic function and $a(\not\equiv 0,\infty)$ be a meromorphic function satisfying $T(r,a)=\circ(T(r,f))$ as $r\to\infty$. Let $P[f]$ be a nonconstant differential polynomial of f. If f and $P[f]$ share a IM and*

$$(2Q+6)\Theta(\infty,f)+(2+3\underline{d}(P))\delta(0,f)>2Q+2\underline{d}(P)+\overline{d}(P)+7,$$

or if f and $P[f]$ share a CM and

$$3\Theta(\infty,f)+(\underline{d}(P)+1)\delta(0,f)>4,$$

then $f\equiv P[f]$.

Banerjee and Majumder [2] considered the weighted sharing of values of f^n and $(f^m)^{(k)}$ and proved the following result:

Theorem 1.7. *Let f be a nonconstant meromorphic function, $k,n,m\in\mathbb{N}$ and l be a non-negative integer. Suppose $a(\not\equiv 0,\infty)$ is a meromorphic function satisfying $T(r,a)=\circ(T(r,f))$ as $r\to\infty$ such that f^n and $(f^m)^{(k)}$ share (a,l). If $l\geq 2$ and*

$$(k+3)\Theta(\infty,f)+(k+4)\Theta(0,f)>2k+7-n,$$

or $l=1$ and

$$(k+\frac{7}{2})\Theta(\infty,f)+(k+\frac{9}{2})\Theta(0,f)>2k+8-n,$$

or $l=0$ and

$$(2k+6)\Theta(\infty,f)+(2k+7)\Theta(0,f)>4k+13-n,$$

then $f^n\equiv(f^m)^{(k)}$.

Motivated by such uniqueness investigation, Charak and Lal [4] considered the uniqueness of $p(f)$ and $P[f]$ sharing (a,l), where $p(z)$ is a polynomial of degree $n\geq 1$. They have shown by an example that in general this is not true, but under certain essential conditions they proved the following result:

Theorem 1.8. *Let f be a nonconstant meromorphic function, $a(\not\equiv 0,\infty)$ be a meromorphic function satisfying $T(r,a)=\circ(T(r,f))$ as $r\to\infty$, and $p(z)$ be a polynomial of degree $n\geq 1$ with $p(0)=0$. Let $P[f]$ be a nonconstant differential polynomial of f. Suppose $p(f)$ and $P[f]$ share (a,l) with one of the following conditions:*

(i) $l\geq 2$ and

$(Q+3)\Theta(\infty,f)+2n\Theta(0,p(f))+\overline{d}(P)\delta(0,f)>Q+3+2\overline{d}(P)-\underline{d}(P)+n,$

(ii) $l=1$ and

$(Q+\frac{7}{2})\Theta(\infty,f)+\frac{5n}{2}\Theta(0,p(f))+\overline{d}(P)\delta(0,f)>Q+\frac{7}{2}+2\overline{d}(P)-\underline{d}(P)+\frac{3n}{2},$

(iii) $l=0$ and

$(2Q+6)\Theta(\infty,f)+4n\Theta(0,p(f))+2\overline{d}(P)\delta(0,f)>2Q+6+4\overline{d}(P)-2\underline{d}(P)+3n.$

Then $p(f)\equiv P[f]$.

Regarding Theorems $1.1-1.4$, it is natural to ask the following question:

Question 1.9. *What will happen when the small function $a(z)$ is replaced by a set of small functions $S_m=\{a(z),a(z)\omega,...,a(z)\omega^{m-1}\}$ in Theorems $1.1-1.4$, where $\omega=\cos\frac{2\pi}{m}+i\sin\frac{2\pi}{m}$ and m is a positive integer?*

Now, we recall the following definition:

Definition 1.10. *[8] Let S be a subset of $S(f)\cap S(g)$. We denote by $E_f(S)$ the set $\cup_{a\in S}\{z: f(z)-a(z)=0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $\cup_{a\in S}\{z: f(z)-a(z)=0\}$ is denoted by $\overline{E}_f(S)$. Let k be a nonnegative integer or infinity. We denote by $E_f(S,k)$ the set $\cup_{a\in S}E_k(a,f)$. Clearly $E_f(S)=E_f(S,\infty)$ and $\overline{E}_f(S)=\overline{E}_f(S,0)$. If $E_f(S,k)=E_g(S,k)$ we say that f, g share the set S with weight k and we write f, g share (S,k) to mean that f, g share the set S with weight k. Moreover, we note that f and g share the set S IM (ignoring multilicity) or CM (counting multiplicity) if and only if f and g share $(S,0)$ or (S,∞) respectively.*

In this paper, we consider the weighted set sharing of $p(f)$ and $P[f]$ and prove the following result:

Theorem 1.11. *Let f be a nonconstant meromorphic function and $p(z)$ be a polynomial in z of degree n (≥ 1) with $p(0)=0$. Let $a(z)$ $(\not\equiv 0,\infty)$ be an element of $S(f)$. Let $P[f]$ be a nonconstant differential polynomial of f as defined in Definition 1.4. Suppose that $p(f)$ and $P[f]$ share (S_m,l) with one of the following conditions:*

(i) $l\geq 2$ and

$$(mQ+3)\Theta(\infty,f)+2n\Theta(0,p(f))+m\overline{d}(P)\delta(0,f)>(mQ+3)+2m\overline{d}(P)-m\underline{d}(P)-(m-2)n, \tag{1.1}$$

(ii) $l=1$ and

$$(mQ+\frac{7}{2})\Theta(\infty,f)+\frac{5n}{2}\Theta(0,p(f))+\overline{d}(P)\delta(0,f)>mQ+\frac{7}{2}+(m+1)\overline{d}(P)-m\underline{d}(P)+(\frac{5}{2}-m)n, \tag{1.2}$$

(iii) $l=0$ and

$$(2mQ+6)\Theta(\infty,f)+4n\Theta(0,p(f))+2m\overline{d}(P)\delta(0,f)>2mQ+6+4m\overline{d}(P)-2m\underline{d}(P)+(4-m)n. \tag{1.3}$$

Then $P[f]=tp(f)$ for some t such that $t^m=1$.

2. Lemmas

In this section we state some lemmas which will be needed in the sequel.

Lemma 2.1. *[3] Let f be a nonconstant meromorphic function and $P[f]$ be a differential polynomial of f. Then*

$$m\left(r,\frac{P[f]}{f^{\overline{d}(P)}}\right)\leq\left(\overline{d}(P)-\underline{d}(P)\right)m\left(r,\frac{1}{f}\right)+S(r,f), \tag{2.1}$$

$$N\left(r,\frac{P[f]}{f^{\overline{d}(P)}}\right)\leq(\overline{d}(P)-\underline{d}(P))N\left(r,\frac{1}{f}\right)+Q\left[\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)\right]+S(r,f), \tag{2.2}$$

$$N\left(r,\frac{1}{P[f]}\right)\leq Q\overline{N}(r,f)+(\overline{d}(P)-\underline{d}(P))m\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^{\overline{d}(P)}}\right)+S(r,f). \tag{2.3}$$

Lemma 2.2. *[12] Let f and g be two nonconstant meromorphic functions. If f and g share $(1,0)$, then*

$$\overline{N}_L\left(r,\frac{1}{f-1}\right)\leq\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+S(r), \tag{2.4}$$

where $S(r)=\circ(T(r))$ as $r\to\infty$ with $T(r)=max\{T(r,f),T(r,g)\}$.

Lemma 2.3. *[1] Let f and g be two nonconstant meromorphic functions. If f and g share $(1,1)$, then*

$$\begin{aligned}2\overline{N}_L\left(r,\frac{1}{f-1}\right)+2\overline{N}_L\left(r,\frac{1}{g-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{f-1}\right)-\overline{N}_{f>2}\left(r,\frac{1}{g-1}\right)\\ \leq N\left(r,\frac{1}{g-1}\right)-\overline{N}\left(r,\frac{1}{g-1}\right)\end{aligned} \tag{2.5}$$

3. Proof of the Main Theorem 1.11

Proof. Let $p(z)=z^n+a_{n-1}z^{n-1}+...+a_1z$, where $a_1,a_2,...a_{n-1}$ are constants. Let $F_1=\frac{p(f)}{a}$ and $G_1=\frac{P[f]}{a}$. Set $F=(F_1)^m$, $G=(G_1)^m$. Then F and G share $(1,l)$ with the possible exception of the zeros and poles of $a(z)$. Also we have

$\overline{N}(r,F)=\overline{N}(r,f)+S(r,f)$ and $\overline{N}(r,G)=\overline{N}(r,f)+S(r,f)$

We define

$$\psi=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right) \tag{3.1}$$

Suppose that $\psi\not\equiv 0$. Then $m(r,\psi)=S(r,f)$.
By Second Fundamental Theorem of Nevanlinna, we have

$$\begin{aligned}T(r,F)+T(r,G)\ \leq\ & 2\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G}\right)\\ &+\overline{N}\left(r,\frac{1}{G-1}\right)-\overline{N_0}\left(r,\frac{1}{F'}\right)-\overline{N_0}\left(r,\frac{1}{G'}\right)+S(r,f)\end{aligned} \tag{3.2}$$

Case 1: $l\geq 1$. If z_0 is a common simple 1-point of F and G, then substituting their Taylor series at z_0 in $\psi(z)$, we see that z_0 is a *zero* of $\psi(z)$. Then we get

$$\begin{aligned}N_E^{1)}\left(r,\frac{1}{F-1}\right)\ &\leq\ N\left(r,\frac{1}{\psi}\right)+S(r,f)\\ &\leq\ T(r,\psi)+S(r,f)\\ &\leq\ N(r,\psi)+S(r,f)\\ &\leq\ \overline{N}(r,f)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)\\ &\quad+\overline{N}_L\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f)\end{aligned} \tag{3.3}$$

Now,

$$\begin{aligned}&\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\\ &=N_E^{1)}\left(r,\frac{1}{F-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right)\\ &+\overline{N}\left(r,\frac{1}{G-1}\right)+S(r,f)\\ &\leq\overline{N}(r,f)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+2\overline{N}_L\left(r,\frac{1}{F-1}\right)+2\overline{N}_L\left(r,\frac{1}{G-1}\right)\\ &+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f)\end{aligned} \tag{3.4}$$

Subcase 1.1: $l=1$.
We have,

$$\overline{N}_L\left(r,\frac{1}{F-1}\right)\leq\frac{1}{2}N\left(r,\frac{1}{F'}\mid F\neq 0\right)\leq\frac{1}{2}\overline{N}(r,F)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right) \tag{3.5}$$

where $N\left(r,\frac{1}{F'}\mid F\neq 0\right)$ denotes the zeros of F', which are not the zeros of F.

Now, from (2.5) and (3.5), we get

$$\begin{aligned}
&2\overline{N}_L\left(r,\frac{1}{F-1}\right)+2\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\\
&\le N\left(r,\frac{1}{G-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)+S(r,f)\\
&\le N\left(r,\frac{1}{G-1}\right)+\frac{1}{2}\overline{N}(r,f)+\frac{1}{2}\overline{N}(r,\frac{1}{F})+S(r,f) \qquad (3.6)
\end{aligned}$$

From (3.4) and (3.6), we have

$$\begin{aligned}
&\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\\
&\le\overline{N}(r,f)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\frac{1}{2}\overline{N}(r,f)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)\\
&+N\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f)\\
&\le\overline{N}(r,f)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\frac{1}{2}\overline{N}(r,f)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)\\
&+T(r,G)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f) \qquad (3.7)
\end{aligned}$$

From (2.3), (3.2) and (3.7), we get

$$\begin{aligned}
T(r,F)+T(r,G) \le\ & 2\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,f)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)\\
&+\frac{1}{2}\overline{N}(r,f)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+T(r,G)+S(r,f)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow T(r,F) \le\ & \frac{7}{2}\overline{N}(r,f)+\frac{5}{2}\overline{N}\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{G}\right)+S(r,f)\\
\le\ & \frac{7}{2}\overline{N}(r,f)+\frac{5}{2}\overline{N}\left(r,\frac{1}{p(f)}\right)+mN\left(r,\frac{1}{P[f]}\right)+S(r,f)\\
\le\ & \left(mQ+\frac{7}{2}\right)\overline{N}(r,f)+\frac{5}{2}\overline{N}\left(r,\frac{1}{p(f)}\right)+m\{\overline{d}(P)-\underline{d}(P)\}T(r,f)\\
&+\overline{d}(P)N\left(r,\frac{1}{f}\right)+S(r,f)\\
\le\ & \Big[\left(mQ+\frac{7}{2}\right)\{1-\Theta(\infty,f)\}+\frac{5n}{2}\{1-\Theta(0,p(f))\}\\
&+\overline{d}(P)\{1-\delta(0,f)\}\Big]T(r,f)+m\{\overline{d}(P)-\underline{d}(P)\}T(r,f)+S(r,f)
\end{aligned}$$

Now,

$$\begin{aligned}
mnT(r,f) =\ & T(r,F)+S(r,f)\\
\le\ & \Big[\left(mQ+\frac{7}{2}\right)\{1-\Theta(\infty,f)\}+\frac{5n}{2}\{1-\Theta(0,p(f))\}\\
&+\overline{d}(P)\{1-\delta(0,f)\}+m(\overline{d}(P)-\underline{d}(P))\Big]T(r,f)+S(r,f)
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow\Big[\left(mQ+\frac{7}{2}\right)\Theta(\infty,f)+\frac{5n}{2}\Theta(0,p(f))+\overline{d}(P)\delta(0,f)\\
&-mQ-\frac{7}{2}-\frac{5n}{2}+mn-(m+1)\overline{d}(P)+m\underline{d}(P)\Big]T(r,f)\le S(r,f)
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\left(mQ+\frac{7}{2}\right)\Theta(\infty,f)+\frac{5n}{2}\Theta(0,p(f))+\overline{d}(P)\delta(0,f)\\
&\le mQ+\frac{7}{2}+(m+1)\overline{d}(P)-m\underline{d}(P)+\left(\frac{5}{2}-m\right)n,
\end{aligned}$$

which contradict (1.2).

Subcase 1.2: $l \geq 2$.
In this case, we have,

$$2\overline{N}_L\left(r,\frac{1}{F-1}\right)+2\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\leq N\left(r,\frac{1}{G-1}\right)+S(r,f)$$

Therefore from (3.4), we obtain

$$\begin{aligned}&\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\\&\leq\overline{N}(r,f)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+N\left(r,\frac{1}{G-1}\right)\\&+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f)\\&\leq\overline{N}(r,f)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+T(r,G)\\&+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f)\end{aligned}\tag{3.8}$$

From (2.3), (3.2) and (3.8), we have

$$\begin{aligned}T(r,F)&\leq 3\overline{N}(r,f)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,f)\\&\leq 3\overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{p(f)}\right)+mN\left(r,\frac{1}{P[f]}\right)+S(r,f)\\&\leq (mQ+3)\overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{p(f)}\right)+m\{\overline{d}(P)-\underline{d}(P)\}T(r,f)+\\&\quad m\overline{d}(P)N\left(r,\frac{1}{f}\right)+S(r,f)\\&\leq [(mQ+3)\{1-\Theta(\infty,f)\}+2n\{1-\Theta(0,p(f))\}\\&\quad +m\overline{d}(P)\{1-\delta(0,f)\}]T(r,f)+m(\overline{d}(P)-\underline{d}(P))T(r,f)+S(r,f)\end{aligned}$$

Now

$$\begin{aligned}mnT(r,f)&=T(r,F)+S(r,f)\\&\leq [(mQ+3)\{1-\Theta(\infty,f)\}+2n\{1-\Theta(0,p(f))\}\\&\quad +m\overline{d}(p)\{1-\delta(0,f)\}+m(\overline{d}(p)-\underline{d}(p))]T(r,f)+S(r,f)\\&\Rightarrow\big[\{(mQ+3)\Theta(\infty,f)+2n\Theta(0,p(f))+m\overline{d}(P)\delta(0,f)\}\\&\quad -\{(mQ+3)+2n-mn+2m\overline{d}(P)-m\underline{d}(P)\}\big]T(r,f)\leq S(r,f)\end{aligned}$$

i.e.,

$$(mQ+3)\Theta(\infty,f)+2n\Theta(0,p(f))+m\overline{d}(P)\delta(0,f)\leq (mQ+3)+2m\overline{d}(P)-m\underline{d}(P)-(m-2)n,$$

which contradict (1.1).
Case 2: $l=0$.
In this case, we have

$$\begin{aligned}&\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\\&\leq N_E^{1)}\left(r,\frac{1}{F-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)\\&+\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)+S(r,f)\\&\leq\overline{N}(r,f)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+2\overline{N}_L\left(r,\frac{1}{F-1}\right)\\&+\overline{N}_L\left(r,\frac{1}{G-1}\right)+N\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f)\end{aligned}\tag{3.9}$$

From (2.3), (2.4), (3.2) and (3.9), we obtain

$$\begin{aligned} T(r,F) &\leq 3\overline{N}(r,f)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right) \\ &\quad +2\overline{N}_{L}\left(r,\frac{1}{F-1}\right)+\overline{N}_{L}\left(r,\frac{1}{G-1}\right)+S(r,f) \\ &\leq 6\overline{N}(r,f)+4\overline{N}\left(r,\frac{1}{F}\right)+2N\left(r,\frac{1}{G}\right)+S(r,f) \\ &\leq 6\overline{N}(r,f)+4\overline{N}\left(r,\frac{1}{p(f)}\right)+2mN\left(r,\frac{1}{P[f]}\right)+S(r,f) \\ &\leq \big[(2mQ+6)\{1-\Theta(\infty,f)\}+4n\{1-\Theta(0,p(f))\}+2m\{\overline{d}(P)-\underline{d}(P)\} \\ &\quad +2m\overline{d}(P)\{1-\delta(0,f)\}\big]T(r,f)+S(r,f) \end{aligned}$$

Now,

$$\begin{aligned} mnT(r,F) &= T(r,F)+S(r,f) \\ &\leq \big[\{(2mQ+6)\Theta(\infty,f)+4n\Theta(0,p(f))+2m\overline{d}(P)\delta(0,f)\} \\ &\quad +\{2mQ+6+4n+2m(\overline{d}(P)-\underline{d}(P))+2m\overline{d}(P)\}\big]T(r,f)+S(r,f) \\ &\Rightarrow \big[\{(2mQ+6)\Theta(\infty,f)+4n\Theta(0,p(f))+2m\overline{d}(P)\delta(0,f)\} \\ &\quad -\{2mQ+6+4n-mn+4m\overline{d}(P)-2m\underline{d}(P)\}\big]T(r,f)\leq S(r,f) \end{aligned}$$

i.e.,

$$(2mQ+6)\Theta(\infty,f)+4n\Theta(0,p(f))+2m\overline{d}(P)\delta(0,f)\leq 2mQ+6+4m\overline{d}(P)-2m\underline{d}(P)+(4-m)n$$

which contradict (1.3).

Therefore,

$$\psi\equiv 0, i.e., \frac{F''}{F'}-\frac{2F'}{F-1}=\frac{G''}{G'}-\frac{2G'}{G-1}$$

Integrating, we get

$$\frac{1}{F-1}=\frac{C}{G-1}+D, \tag{3.10}$$

where $C\neq 0$ and D are constant.
We consider the following three cases.
Case I: $D\neq 0,-1$
Rewriting (3.10) as

$$\frac{G-1}{C}=\frac{F-1}{D+1-DF}$$

we have

$$\overline{N}(r,G)=\overline{N}\left(r,\frac{1}{F-(D+1)/D}\right)$$

By Second Fundamental Theorem of Nevanlinna, we have

$$\begin{aligned} mnT(r,f) &= T(r,F)+S(r,f) \\ &\leq \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-(D+1)/D}\right)+S(r,f) \\ &\leq \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{p(f)}\right)+\overline{N}(r,G)+S(r,f) \\ &\leq 2\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{p(f)}\right)+S(r,f) \\ &\leq \big[2\{1-\Theta(\infty,f)\}+n\{1-\Theta(0,p(f))\}\big]T(r,f)+S(r,f) \end{aligned}$$

$$i.e., \big[2\Theta(\infty,f)+n\Theta(0,p(f))+(m-1)n-2\big]T(r,f)\leq S(r,f)$$

which contradicts (1.1), (1.2), and (1.3).
Case II: $D=0$
From (3.10), we obtain

$$G=CF-(C-1) \tag{3.11}$$

Therefore if $C \neq 1$, we have

$$\overline{N}\left(r, \frac{1}{G}\right) = \overline{N}\left(r, \frac{1}{F - (C-1)/C}\right)$$

By the Second Fundamental Theorem of Nevanlinna, we have

$$\begin{aligned}
mnT(r,f) &= T(r,F) + S(r,f) \\
&\leq \overline{N}(r,F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F-(C-1)/C}\right) + S(r,f) \\
&\leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) + S(r,f) \\
&\leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{p(f)}\right) + \overline{N}\left(r, \frac{1}{P[f]}\right) + S(r,f) \\
&\leq (Q+1)\overline{N}(r,f) + \overline{N}\left(r, \frac{1}{p(f)}\right) + \{\overline{d}(P) - \underline{d}(P)\}T(r,f) \\
&\quad + \overline{d}(P)N\left(r, \frac{1}{f}\right) + S(r,f) \\
&\leq \big[(Q+1)\{1-\Theta(\infty,f)\} + n\{1-\Theta(0,p(f))\} + \overline{d}(P)\{1-\delta(0,f)\}\big]T(r,f) \\
&\quad + \{\overline{d}(P) - \underline{d}(P)\}T(r,f) + S(r,f)
\end{aligned}$$

$$\begin{aligned}
&i.e., \\
&\big[(Q+1)\Theta(\infty,f) + n\Theta(0,p(f)) + \overline{d}(P)\delta(0,f) \\
&\quad -\{Q+1-n+mn+2\overline{d}(P) - \underline{d}(P)\}\big]T(r,f) \leq S(r,f)
\end{aligned}$$

which contradicts (1.1), (1.2), and (1.3).

Therefore $C = 1$ and from (3.11), we have

$F \equiv G, i.e., \ P[f] = tp(f),$

for some t such that $t^m = 1$.

Case III: $D = -1$
From (3.10), we obtain

$$\frac{1}{F-1} = \frac{C}{G-1} - 1 \tag{3.12}$$

Therefore if $C \neq -1$, then

$$\overline{N}\left(r, \frac{1}{G}\right) = \overline{N}\left(r, \frac{1}{F - (C+1)/C}\right)$$

and proceeding as in **Case II**, we arrived at a contradiction.

Therefore for $C = -1$ and from (3.12), we obtain

$$FG = 1, i.e., \left(\frac{p(f)}{a}.\frac{P[f]}{a}\right)^m = 1$$

So

$$\frac{p(f)P[f]}{a^2} = t, i.e., \ p(f)P[f] = ta^2$$

for some t such that $t^m = 1$.

Then

$$\overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) = S(r,f)$$

By using (2.1) and (2.2), we have

$$\begin{aligned}(n+\overline{d}(P))T(r,f) &\leq T\left(r,\frac{ta^2}{f^{n+\overline{d}(p)}}\right)+S(r,f)\\ &\leq T\left(r,\frac{p(f)}{f^n}.\frac{P[f]}{f^{\overline{d}(p)}}\right)+S(r,f)\\ &\leq (n-1)T(r,f)+T\left(r,\frac{P[f]}{f^{\overline{d}(p)}}\right)+S(r,f)\\ &\leq (n-1)T(r,f)++\{\overline{d}(P)-\underline{d}(P)\}T(r,f)+S(r,f)\\ i.e.,(1+\underline{d}(P))T(r,f) &\leq S(r,f),\end{aligned}$$

which contradict (1.1), (1.2) and (1.3). □

Remark 3.1. *For $m=1$ in Theorem 1.11, we get Theorem 1.8.*

Corollary 3.2. *Let f be a nonconstant meromorphic function and $p(z)$ be a polynomial in z of degree n (≥ 1) with $p(0)=0$. Let $a(z)$ $(\not\equiv 0,\infty)$ be an element of $S(f)$. Let $P[f]$ be a nonconstant homogeneous differential polynomial in f of degree d as defined in Definition 1.4. Suppose that $p(f)$ and $P[f]$ share (S_m,l) with one of the following conditions:*

(i) $l\geq 2$ and

$(Q+3)\Theta(\infty,f)+2n\Theta(0,p(f))+d\delta(0,f)>Q+d+n+3,$

(ii) $l=1$ and

$(Q+\frac{7}{2})\Theta(\infty,f)+\frac{5n}{2}\Theta(0,p(f))+d\delta(0,f)>Q+d+\frac{3n}{2}+\frac{7}{2},$

(iii) $l=0$ and

$(2Q+6)\Theta(\infty,f)+4n\Theta(0,p(f))+2d\delta(0,f)>2Q+2d+3n+6.$

Then $P[f]=tp(f)$ for some t such that $t^m=1$.

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References

[1] T. C. Alzahary and H.-X. Yi, Weighted value sharing and a question of I. Lahiri, Complex Variables. Vol:49, No.15 (2004), 1063-1078.
[2] A. Banerjee and S. Majumder, Some uniqueness results related to meromorphic function that share a small function with its derivative, Math Reports. Vol:66 (2014), 95-111.
[3] S. Bhoosnurmath and S.R. Kabbur, On entire and meromorphic functions that share one small function with their differential polynomial, Hindawi Publishing Corporation. Intl J Analysis 2013, Article ID 926340.
[4] K. S. Charak and B. Lal, Uniqueness of $p(f)$ and $P[f]$, Turk J Math. Vol:40 (2016), 569-581.
[5] A. A. Goldberg, I. V. Ostrovskii, Value Distribution of Meromorhic Functions, Translated from the 1970 Russian original by Mikhail Ostrovskii. With an appendix by Alexandre Eremenko and James K. Langley. Translations of Mathematical Monographs, 236. American Mathematical Society, Providence, RI, USA, 2008.
[6] G. G. Gundersen, Meromorphic functions that share finite values with their derivative, J. Math. Anal. Appl. Vol:75 (1980), 441-446.
[7] Hayman, W. K., Meromorphic Functions, Clarendon Press, Oxford, UK, 1964.
[8] I. Lahiri, Weighted sharing of two sets, Kyungpook Math. J. Vol:46 (2006), 79-87.
[9] E. Mues and N. Steinmetz, Meromorphe funktionen die unit ihrer ableitung werte teilen, Manuscripta Math. Vol:29 (1979), 195-206.
[10] Yang, L., Value Distribution Theory, Springer-Verlag, Berlin, 1993.
[11] Yang, C. C. and Yi, H. X., Uniqueness Theory of Meromorphic Functions, Kluwer Academic Publishers, Dordrecht, the Netherlands, 2003.
[12] H. X. Yi, Meromorphic function that share one or two value II, Kodai Math. J. Vol:22 (1999), 264-272.
[13] T. Zhang and W. Lü, Notes on a meromorphic function sharing one small function with its derivative, Complex Variables and Elliptic Equations. Vol:53 (2008), 857-867.