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On Uniqueness of Two Meromorphic Functions Sharing A Small Function

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Abstract

In this paper, we have investigated the uniqueness problems of entire and meromorphic functions concerning differential polynomials sharing a small function. Our results radically extended and improved the results of *Bhoosnurmath-Pujari* [6] and *Harina - Anand* [13] not only by sharing small function instead of fixed point but also reducing the lower bound of n. There are some miscalculation in the proof of a result of Harina-Anand [13]. We have corrected all of them in a more convenient way. At last some open questions have been posed for further study in this direction.

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1. Introduction, Definitions and Results

The Nevanlinna theory mainly describes the asymptotic distribution of solutions of the equation $f(z) = w$, as w varies. At the outset, we assume that readers are familiar with the basic Nevanlinna Theory [9]. First we explain the general sharing notion. Let f and g be two non-constant meromorphic functions in the complex plane $\mathbb{C}$. Two meromorphic functions f and g are said to share a value $w \in \mathbb{C} \cup \{\infty\}$ *IM* (ignoring multiplicities) if f and g have the same w-points counted with ignoring multiplicities. If multiplicities of w-points are counted, then f and g are said to share w *CM* (counting multiplicities).
When $w = \infty$ the zeros of $f - w$ means the poles of f.
It is well known that if two moromorphic functions f and g share four distinct values *CM*, then one is *Möbius Transformation* of the other.
In 1993, corresponding to one famous question of *Hayman* [10], *Yang-Hua* [16] showed that similar conclusions hold for certain types of differential polynomials when they share only one value.
Recently by using the same argument as in [16], *Fang-Hong* [7] the following result was obtained.

Theorem 1.1. *Let f and g be two transcendental entire functions, $n \geq 11$, an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f \equiv g$.*

The following example shows that in *Theorem A* one simply can not replace "entire" by "meromorphic " functions.

Example 1.2. *Let*

$$f(z) = \frac{(n+2)}{(n+1)} \frac{e^z + \ldots + e^{(n+1)z}}{1 + e^z + \ldots + e^{(n+1)z}}$$

and

$$f(z) = \frac{(n+2)}{(n+1)} \frac{1 + e^z + \ldots + e^{nz}}{1 + e^z + \ldots + e^{(n+1)z}}.$$

It is clear that $f(z) = e^z g(z)$. Also $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM but note that $f \not\equiv g$.

In 2004, *Lin-Yi* [11] extended Theorem A and obtained the following results.

Theorem 1.3. *[11] Let f and g be two transcendental entire functions, $n \geq 7$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share z CM, then $f \equiv g$.*

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Theorem 1.4. *[11] Let f and g be two transcendental meromorphic functions, $n \geq 12$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share z CM, then either $f \equiv g$ or*

$$g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}, \quad f = \frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})},$$

where h is a non-constant meromorphic function.

Theorem 1.5. *[11] Let f and g be two transcendental meromorphic functions, $n \geq 13$ an integer. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share z CM, then $f \equiv g$.*

To improve all the above mentioned results, natural questions arise as follows.

Question 1.6. *Keeping all other conditions intact, is it possible to reduce further the lower bounds of n in the above results ?*

Question 1.7. *Is it also possible to replace the transcendental meromorphic (entire) functions by a more general class of meromorphic (entire) functions in all the above mentioned results ?*

In 2013, *Bhoosnurmath-Pujari* [6], answered the above questions affirmatively and obtained the following results.

Theorem 1.8. *[6] Let f and g be two non-constant meromorphic functions, $n \geq 11$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share z CM, f and g share ∞ IM, then either $f \equiv g$ or*

$$g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}, \quad f = \frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})},$$

where h is a non-constant meromorphic function.

Theorem 1.9. *[6] Let f and g be two non-constant meromorphic functions, $n \geq 12$ an integer. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share z CM, f and g share ∞ IM, then $f \equiv g$.*

Theorem 1.10. *[6] Let f and g be two non-constant entire functions, $n \geq 7$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share z CM, then $f \equiv g$.*

In this direction, for the purpose of extending Theorem E and F, one may ask the following question.

Question 1.11. *Keeping all other conditions intact in Theorem E, F and G, is it possible to replace respectively $f^n(f-1)f'$ and $g^n(g-1)g'$ by $f^n(f-1)^m f'$ and $g^n(g-1)^m g'$?*

Next the following question is inevitable.

Question 1.12. *Is it possible to omit the second conclusions of Theorems C and E ?*

In 2016, *Waghmore-Anand* [13] answer the *Questions 1.11* and *1.12* affirmatively and obtained the following results.

Theorem 1.13. *[13] Let f and g be two non-constant meromorphic functions, $n \geq m+10$ be an integer. If $f^n(f-1)^m f'$ and $g^n(g-1)^m g'$ share z CM, f and g share ∞ IM, then $f \equiv g$.*

Theorem 1.14. *[13] Let f and g be any two non-constant entire functions, $n \geq m+6$ an integer. If $f^n(f-1)^m f'$ and $g^n(g-1)^m g'$ share z CM, then $f \equiv g$.*

Note 1.15. *We see that in the results of* Waghmore - Anand, *for $m = 2$,* Theorem H *reduces to* Theorem F *and for $m = 1$,* Theorem I *reduces to* Theorem G .

Remark 1.16. *We notice that in the proof of Theorem H and hence in the case of Theorem I also, we have found some miscalculation made by the authors* Waghmore-Anand *[13]. We mention below few of them.*

(i) In [13, page-947], just before Case 2, *the authors obtained that the coefficient of $T(r,g)$ is $(n-m-8)$, while actually it will be $(n+m-8)$.*

(ii) In [13, page-948], just before Case 3, *the authors finally obtained that "$h^{n+m}-1 = 0$, $h^{n+1}-1 = 0$, which imply $h = 1$". Note that this possible only when $\gcd(n+m, n+1) = 1$ but which is not true if one consider some suitable value of n and m. For example if we choose $n = 3$ and $m = 5$, we note that $\gcd(n+m, n+1) = \gcd(8,4) = 4 \neq 1$.*

(iii) We observe that in [13, equation (49), page-950], the coefficient of $T(r,g)$ is $\frac{m}{n+m-1}$ while actually it should be $\frac{m}{n+m+1}$.

In this paper, our aim is to correct all the mistakes made by *Waghmore-Anand* [13] and at the same time to get an improved and extended version results of all the above mentioned Theorems A - I.

To this end, throughout the paper, we will use the following transformations (see [5]). Let

$$\mathscr{P}(w) = w^{n+m} + \ldots + a_n w^n + \ldots + a_0 = a_{n+m}\prod_{i=1}^{s}(w - w_{p_i})^{p_i}$$

where $a_j (j = 0, 1, 2, \ldots, n+m-1)$ and $w_{p_i} (i = 1, 2, ..., s)$ are distinct finite complex numbers and $2 \leq s \leq n+m$ and $p_1, p_2, \ldots, p_s$, $s \geq 2$, n, m and k are all positive integers with $\sum_{i=1}^{s} p_i = n+m$. Also let $p > \max_{p \neq p_i, i=1,\ldots,r}\{p_i\}$, $r = s-1$, where s and r are two positive integers.

Let $\mathscr{Q}(w_*) = \prod_{i=1}^{s-1}(w_* + w_p - w_{p_i})^{p_i} = b_q w_*^q + b_{q-1} w_*^{q-1} + \ldots + b_0$, where $w_* = w - w_p$, $q = n+m-p$. So it is clear that $\mathscr{P}(w) = w_*^p \mathscr{Q}(w_*)$

In particular, if we choose $b_i = (-1)^i\, {}^qC_i$, for $i = 0, 1, \ldots, q$. Then we get, easily $\mathscr{P}_*(w) = w_*^p(w_* - 1)^q$.

Note that if $w_p = 0$ and $p = n$, then we get $w = w_*$ and $\mathscr{P}_*(w) = w^n(w-1)^m$.

Observing all the above mentioned results, we note that $h^n(h-1)h'$ or $h^n(h-1)^2h'$ $(h = f$ or $g)$ are a special form of $h^n(h-1)^m h'$, $m \geq 1$ be an integer.

Definition 1.17. *[3] A Meromorphic function $a \equiv a(z)(\not\equiv 0,\infty)$ is said to be a small function of f provided that $T(r,a) = S(r,f)$ i.e., $T(r,a) = O(T(r,f))$ as $r \to \infty$, outside of a possible exceptional set of finite linear measure.*

Studying two differential polynomials when sharing a small function (see [1, 2, 3, 5]) or some non-zero polynomial (see [4]) becomes an interesting part of modern value distribution theory. Since the extension of derivatives of a meromorphic functions is nothing but differential polynomials. So for the improvements and extensions of the above mentioned results further to a large extent, the following questions are inevitable.

Question 1.18. *Is it possible to replace $f^n(f-1)^m f'$ and $g^n(g-1)^m g'$ by a more general expressions of the form $\mathscr{P}_*(f)f'_* = f_*^p(f_*-1)^q f'_*$ and $\mathscr{P}_*(g)g'_* = g_*^p(g_*-1)^q g'_*$ respectively in all the above mentioned results ?*

If the answer of the *Question 1.18* is found to be affirmative, then one my ask the following questions.

Question 1.19. *Is it possible to reduce further the lower bounds of n in Theorems E, F, G and H ?*

Question 1.20. *Is it also possible to replace sharing z CM by sharing $\alpha(z)$ CM in Theorem G and H ?*

Answering all the above mentioned questions affirmatively is the main motivation of writing this paper.
Following two theorems are the main results of this paper.

Theorem 1.21. *Let f and g hence $f_* = f - w_p$ and $g_* = g - w_p$, $w_p \in \mathbb{C}$ be any two non-constant non- entire meromorphic functions, $n \geq q+9$, $q \in \mathbb{N}$, be an integer. If $\mathscr{P}_*(f)f'_* = f_*^p(f_*-1)^q f'_*$ and $\mathscr{P}_*(g)g'_* = g_*^p(g_*-1)^q g'_*$ share $\alpha \equiv \alpha(z)$ $(\not\equiv 0,\infty)$ CM, f_* and g_* share ∞ IM, then $f \equiv g$.*

Theorem 1.22. *Let f and g hence $f_* = f - w_p$ and $g_* = g - w_p$, $w_p \in \mathbb{C}$ be any two non-constant entire functions, $n \geq q+5$, $q \in \mathbb{N}$, be an integer. If $\mathscr{P}_*(f)f'_* = f_*^p(f_*-1)^q f'_*$ and $\mathscr{P}_*(g)g'_* = g_*^p(g_*-1)^q g'_*$ share $\alpha \equiv \alpha(z)$ $(\not\equiv 0,\infty)$ CM, then $f \equiv g$.*

2. Some lemmas

In this section we present some lemmas which will be needed in sequel.

Lemma 2.1. *[14] Let f_1, f_2 and f_3 be non constant meromorphic functions such that $f_1+f_2+f_3 = 1$. If f_1, f_2 and f_3 are linearly independent, then*

$$T(r,f_1) < \sum_{i=1}^{3} N_2\left(r,\frac{1}{f_i}\right) + \sum_{i=1}^{3} \overline{N}(r,f) + o(T(r)),$$

where $T(r) = \max\limits_{1\leq i\leq 3}\left\{T(r,f_i)\right\}$ and $r \notin E$.

Lemma 2.2. *[17] Let f_1 and f_2 be two non-constant meromorphic functions. If $c_1f_1 + c_2f_2 = c_3$, where c_i, $i = 1,2,3$ are non-zero constants, then*

$$T(r,f_1) \leq \overline{N}(r,f_1) + \overline{N}\left(r,\frac{1}{f_1}\right) + \overline{N}\left(r,\frac{1}{f_2}\right) + S(r,f_1).$$

Lemma 2.3. *[17] Let f be a non-constant meromorphic function and k be a non-negative integer, then*

$$N\left(r,\frac{1}{f^{(k)}}\right) \leq N(r,\frac{1}{f}) + k\overline{N}(r,f) + S(r,f).$$

Lemma 2.4. *[19] Suppose that f is a non-constant meromorphic function and $P(f) = a_nf^n + a_{n-1}f^{n-1} + \ldots + a_1f + a_0$, where $a_n(\not\equiv 0)$, $a_{n-1},\ldots,a_1,a_0$ are small meromorphic functions of $f(z)$. Then*

$$T(r,P(f)) = n\,T(r,f) + S(r,f).$$

Lemma 2.5. *[15] Let f_1, f_2 and f_3 be three meromorphic functions satisfying $\sum\limits_{i=1}^{3} f_i = 1$, then the functions $g_1 = -\dfrac{f_1}{f_2}$, $g_2 = \dfrac{1}{f_2}$ and $g_3 = -\dfrac{f_1}{f_2}$ are linearly independent when f_1, f_2 and f_3 are linearly independent.*

Lemma 2.6. *Let f and g and hence $f_* = f - w_p$ and $g_* = g - w_p$ be two non-constant meromorphic functions and $\alpha \equiv \alpha(z)$ $(\not\equiv 0,\infty)$ be a small function of f and g. If $\mathscr{P}_*(f)f'_* = f_*^p(f_*-1)^q f'_*$ and $\mathscr{P}_*(g)g'_* = g_*^p(g_*-1)^q g'_*$ share α CM and $p \geq 7$, then*

$$T(r,g_*) \leq \left(\frac{p+q+2}{p-6}\right)T(r,f_*) + S(r,g_*)$$

Proof. Applying *Second Fundamental Theorem* on $\mathscr{P}_*(g)g'_*$, we get

$$\begin{aligned} & T\left(r,\mathscr{P}_*(g)g'_*\right) \qquad (2.1)\\ \leq\ & \overline{N}(r,\mathscr{P}_*(g)g'_*) + \overline{N}\left(r,\frac{1}{\mathscr{P}_*(g)g'_*}\right) + \overline{N}\left(r,\frac{1}{\mathscr{P}_*(g)g'_*-\alpha}\right) + S(r,g_*)\\ \leq\ & \overline{N}\left(r,\frac{1}{g_*^p(g_*-1)^q g'_*}\right) + \overline{N}(r,g_*) + \overline{N}\left(r,\frac{1}{g_*^p(g_*-1)^q g'_*-\alpha}\right) + S(r,g_*) \end{aligned}$$

Next by applying *First fundamental Theorem*,

$$\begin{aligned} & (p+q)T(r,g) \\ \leq\ & T(r,g_*^p(g_*-1)^q)+S(r,g_*) \\ \leq\ & T(r,g_*^p(g_*-1)^q g_*')+T\left(r,\frac{1}{g_*'}\right)+S(r,g_*). \end{aligned} \tag{2.2}$$

After combining (2.1) and (2.2), we get

$$\begin{aligned} & (p+q)T(r,g) \\ \leq\ & \overline{N}\left(r,\frac{1}{g_*}\right)+\overline{N}(r,0;g_*-1)+\overline{N}(r,g_*)+\overline{N}\left(r,\frac{1}{g_*'}\right)+\overline{N}\left(r,\frac{1}{f_*^p(f_*-1)^q f_*'-\alpha}\right) \\ & +S(r,g_*)+T(r,g_*'). \end{aligned} \tag{2.3}$$

Again since $S(r,g_*)=T(r,\alpha)=S(r,f_*)$, so we must have

$$\begin{aligned} & \overline{N}\left(r,\frac{1}{f_*^p(f_*-1)^q f_*'-\alpha}\right) \\ \leq\ & T\left(r,\alpha;f_*^p(f_*-1)^q f_*'\right)+O(1) \\ \leq\ & T(r,f_*^p)+T(r,(f_*-1)^q+T(r,f_*')+T(r,\alpha)+O(1) \\ \leq\ & p\,T(r,f_*)+q\,T(r,f_)+2T(r,f_*)+S(r,g_*) \\ =\ & (p+q+2)T(r,f_*)+S(r,g_*). \end{aligned} \tag{2.4}$$

By using (2.6) in (2.5), we get

$$\begin{aligned} & (p+q)T(r,g_*) \\ \leq\ & (q+6)T(r,g_*)+(p+q+2)T(r,f_*)+S(r,g_*). \end{aligned}$$

i.e.,

$$T(r,g)\leq\left(\frac{p+q+2}{p-6}\right)T(r,f)+S(r,g),$$

where $p\geq 7$. □

Lemma 2.7. *Let f and g and hence $f_*=f-w_p$ and $g_*=g-w_p$ be two non-constant entire functions and $\alpha\equiv\alpha(z)$ $(\not\equiv 0,\infty)$ be a small function of f and g. If $\mathscr{P}_*(f)f_*'=f_*^p(f_*-1)^q f_*'$ and $\mathscr{P}_*(g)g_*'=g_*^p(g_*-1)^q g_*'$ share α CM and $p\geq 5$, then*

$$T(r,g_*)\leq\left(\frac{p+q+2}{p-3}\right)T(r,f_*)+S(r,g_*)$$

Proof. Since f and g both are entire functions, so we must have $\overline{N}(r,f)=0=\overline{N}(r,g)$.
Proceeding exactly as in the line of the proof of Lemma 2.6, we can prove the lemma. □

Lemma 2.8. *Let $\Psi(z)=c^2(z^{p-q}-1)^2-4b(z^{p-2q}-1)(z^p-1)$, where $b,c\in\mathbb{C}-\{0\}$, $\frac{c^2}{4b}=\frac{p(p-2q)}{(p-q)^2}\neq 1$, , then $\Psi(z)$ has exactly one multiple zero of multiplicity 4 which is 1.*

Proof. We claim that $\Psi(1)=0$ with multiplicity 4 and all other zeros of $\Psi(w)$ are simple. Let $F(t)=\frac{1}{2}\Psi(e^t)e^{(q-p)t}$. Then

$$\begin{aligned} & F(t) \\ =\ & \frac{1}{2}\left\{4b(1-e^{pt})(1-e^{(p-2q)t})-c^2(1-e^{(p-q)t})\right\}e^{(q-p)t} \\ =\ & (4b-c^2)\cosh(q-p)t-4b\cosh qt+c^2. \end{aligned}$$

Next we see that for $t=0$, $F(t)=0$, $[F(t)]'=0$, $[F(t)]''=0$ since $\frac{c^2}{4b}=\frac{p(p-2q)}{(p-q)^2}$ and $[F(t)]'''=0$ but $[F(t)]^{(iv)}\neq 0$ where

$$[F(t)]'=(4b-c^2)(q-p)\sinh(q-p)t-4bq\sinh qt,$$

$$[F(t)]''=(4b-c^2)(q-p)^2\cosh(q-p)t-4bq^2\cosh qt,$$

$$[F(t)]'''=(4b-c^2)(q-p)^3\sinh(q-p)t-4bq^3\sinh qt$$

and

$$[F(t)]^{(iv)}=(4b-c^2)(q-p)^4\cosh(q-p)t-4bq^4\cosh qt.$$

Therefore it is clear that $F(0)=0$ with multiplicity 4 and hence $\Psi(1)=0$ with multiplicity 4.

Next we suppose that $\Psi(w)=0=\Psi'(w)$, for some $w\in\mathbb{C}$. Then $F(t)=0=F'(t)$ for every t satisfying $e^{qt}=w$. Now from $F(t)=0$ and $F'(t)=0$, we obtained respectively

$$(4b-c^2)\cosh(q-p)t-4b\cosh qt+c^2=0 \tag{2.5}$$

and

$$(4b-c^2)(q-p)\sinh(q-p)t-4qb\sinh qt=0. \tag{2.6}$$

Since $\cosh^2(q-p)t-\sinh^2(q-p)t=1$, so from (2.5) and (2.6), we get

$$\frac{(4b\cosh qt-c^2)^2}{(4b-c^2)^2}-\frac{16q^2b^2\sinh^2 qt}{(4b-c^2)^2(q-p)^2}=1.$$

i.e.,

$$(q-p)^2(4b\cosh^2 qt-c^2)^2-16q^2b^2(\cosh^2 qt-1)=(4b-c^2)^2(q-p)^2.$$

i.e.,

$$\left\{\cosh qt-1\right\}\left\{\cosh qt-\frac{a^2(q-p)^2}{2bp(p-2q)}+1\right\}=0. \tag{2.7}$$

Since $\dfrac{c^2}{4b}=\dfrac{p(p-2q)}{(p-q)^2}$, then $\dfrac{c^2(q-p)^2}{2bq(q-2p)}=2$, so we see that the equation (2.7) reduces to $\left\{\cosh qt-1\right\}^2=0$. i.e., we get $e^{qt}=1=w$. □

3. Proofs of the theorems

Proof of Theorem 1.21. Since $\mathscr{P}_*(f)f'_*$ and $\mathscr{P}_*(g)g'_*$ share $\alpha\equiv\alpha(z)$ CM, f and g share ∞ IM, so we suppose that

$$\mathscr{H}\equiv\frac{\mathscr{P}_*(f)f'_*-\alpha}{\mathscr{P}_*(g)g'_*-\alpha}\equiv\frac{f_*^p(f_*-1)^q f'_*-\alpha}{g_*^p(g_*-1)^q g'_*-\alpha}. \tag{3.1}$$

Then from (2.6) and (3.1), we get

$$\begin{aligned}
&T(r,\mathscr{H})\\
=\;& T\left(r,\frac{\mathscr{P}_*(f)f'_*-\alpha}{\mathscr{P}_*(g)g'_*-\alpha}\right)\\
\le\;& T(\mathscr{P}_*(f)f'_*-\alpha)+T(r,\mathscr{P}_*(g)g'_*-\alpha)+O(1)\\
\le\;& T(r,f_*^p(f_*-1)^q f'_*-\alpha)+T(r,g_*^p(g_*-1)^q g'_*-\alpha)+O(1)\\
\le\;& (p+q+2)(T(r,f_*)+T(r,g_*))+S(r,f_*)+S(r,g_*)\\
\le\;& 2(p+q+2)T_*(r)+S_*(r),
\end{aligned}$$

where $T_*(r)=\max\{T(r,f_*),T(r,g_*)\}$ and $S_*(r)=\max\{S(r,f_*),S(r,g_*)\}$.
i.e.,

$$T(r,\mathscr{H})=O(T_*(r)). \tag{3.2}$$

Again from (3.1), we see that the zeros and poles of $\mathscr{H}$ are multiple and hence

$$\overline{N}(r,\mathscr{H})\le\overline{N}_L(r,f),\quad \overline{N}\left(r,\frac{1}{\mathscr{H}}\right)\le\overline{N}_L(r,g). \tag{3.3}$$

Let $f_1=\dfrac{f_*^p(f_*-1)^q f'_*}{\alpha}$, $\quad f_2=\mathscr{H}$ $\quad$ and $\quad f_3=-\mathscr{H}\dfrac{g_*^p(g_*-1)^q g'_*}{\alpha}$.
Thus we get $f_1+f_2+f_3=1$. Next we denote $T(r)=\max\{T(r,f_1),T(r,f_2),T(r,f_3)\}$.
We have,

$$T(r,f_1)=O(T(r,f_*))$$

$$T(r,f_2)=O(T(r,f_*)+T(r,g_*))=T(r,f_3).$$

So we have $T(r,f_i)=O(T_*(r))$ for $i=1,2,3$ and hence $S(r,f_*)+S(r,g_*)=o(T_*(r))$.

Next we discuss the following cases.
Case 1. Suppose none of f_2 and f_3 is a constant. If f_1, f_2 and f_3 are linearly independent, then by Lemma 2.1 and 2.4, we have

$$\begin{aligned} & T(r,f_1) \\ \leq\ & \sum_{i=1}^{3} N_2\left(r,\frac{1}{f_i}\right)+\sum_{i=1}^{3}\overline{N}(r,f_i)+o(T(r)) \\ \leq\ & N_2\left(r,\frac{\alpha}{f_*^p(f_*-1)^q f_*'}\right)+N_2\left(r,\frac{1}{\mathscr{H}}\right)+N_2\left(r,\frac{\alpha}{\mathscr{H}g_*^p(g_*-1)^q g_*'}\right) \\ & +\overline{N}(r,f_*^p(f_*-1)^q f_*')+\overline{N}(r,\mathscr{H})+\overline{N}(r,\mathscr{H}g_*^p(g_*-1)^q g_*')+o(T(r)) \\ \leq\ & N_2\left(r,\frac{1}{f_*^p(f_*-1)^q f_*'}\right)+2N_2\left(r,\frac{1}{\mathscr{H}}\right)+N_2\left(r,\frac{1}{g_*^p(g_*-1)^q g_*'}\right)+\overline{N}(r,f_*) \\ & +2\overline{N}(r,\mathscr{H})+\overline{N}(r,g_*)+o(T(r)). \end{aligned} \tag{3.4}$$

We see that $N_2\left(r,\frac{1}{\mathscr{H}}\right)\leq 2\overline{N}\left(r,\frac{1}{\mathscr{H}}\right)\leq 2\overline{N}_L(r,g_*),\quad \overline{N}(r,\mathscr{H})\leq\overline{N}_L(r,f).$
Again since $\overline{N}_L(r,f_*)=0=\overline{N}_L(r,g_*)$ and note that $\overline{N}(r,f_*)=\overline{N}(r,g_*)$, so using all this facts, we get from (3.4) that

$$\begin{aligned} & T(r,f_1) \\ \leq\ & N_2\left(r,\frac{1}{f_*^p(f_*-1)^q f_*'}\right)+N_2\left(r,\frac{1}{g_*^p(g_*-1)^q g_*'}\right)+2\overline{N}(r,f_*)+o(T(r)) \\ \leq\ & N\left(r,\frac{1}{f_*^p(f_*-1)^q f_*'}\right)-\left[N_{(3}\left(r,\frac{1}{f_*^p(f_*-1)^q f_*'}\right)-2\overline{N}_{(3}\left(r,\frac{1}{f_*^p(f_*-1)^q f_*'}\right)\right] \\ +\ & N\left(r,\frac{1}{g_*^p(g_*-1)^q g_*'}\right)-\left[N_{(3}\left(r,\frac{1}{g_*^p(g_*-1)^q g_*'}\right)-2\overline{N}_{(3}\left(r,\frac{1}{g_*^p(g_*-1)^q g_*'}\right)\right] \\ & +2\overline{N}(r,f_*)+o(T(r)). \end{aligned} \tag{3.5}$$

Let z_0 be a zero of f_* of multiplicity r, then z_0 is a zero of $f_*^p(f_*-1)^q f_*'$ of multiplicity $pr+r-1\geq 3$. Thus we have

$$\begin{aligned} & N_{(3}\left(r,\frac{1}{f_*^p(f_*-1)^q f_*'}\right)-2\overline{N}_{(3}\left(r,\frac{1}{f_*^p(f_*-1)^q f_*'}\right) \\ \geq\ & (p-2)N\left(r,\frac{1}{f_*}\right). \end{aligned} \tag{3.6}$$

Similarly, we get

$$\begin{aligned} & N_{(3}\left(r,\frac{1}{g_*^p(g_*-1)^q g_*'}\right)-2\overline{N}_{(3}\left(r,\frac{1}{g_*^p(g_*-1)^q g_*'}\right) \\ \geq\ & (p-2)N\left(r,\frac{1}{g_*}\right). \end{aligned} \tag{3.7}$$

Let

$$\mathscr{F}=\frac{f_*^{p+q+1}}{p+q+1}-\frac{{}^qC_1}{p+q}f_*^{p+q}+\frac{{}^qC_2}{p+q-1}f_*^{p+q-1}+\ldots+(-1)^q\frac{1}{p+q}f_*^{p+1}$$

and

$$\mathscr{G}=\frac{g_*^{p+q+1}}{p+q+1}-\frac{{}^qC_1}{p+q}g_*^{p+q}+\frac{{}^qC_2}{p+q-1}g_*^{p+q-1}+\ldots+(-1)^q\frac{1}{p+q}g_*^{p+1}.$$

By Lemma 2.4, we have

$$T(r,\mathscr{F})=(p+q+1)T(r,f_*)+S(r,f_*).$$

It is clear $\mathscr{F}'=\alpha f_1$. So we have

$$m\left(r,\frac{1}{\mathscr{F}}\right)\leq m\left(r,\frac{1}{\alpha f_1}\right)+m\left(r,\frac{\mathscr{F}'}{\mathscr{F}}\right)\leq m\left(r,\frac{1}{f_1}\right)+S(r,f_*). \tag{3.8}$$

By using *First fundamental Theorem* and (3.8), we obtained

$$\begin{aligned} & T(r,\mathscr{F}) \\ =\ & m\left(r,\frac{1}{\mathscr{F}}\right)+N\left(r,\frac{1}{\mathscr{F}}\right) \\ \leq\ & T(r,f_1)+N\left(r,\frac{1}{\mathscr{F}}\right)-N\left(r,\frac{1}{f_1}\right)+S(r,f_*) \\ \leq\ & T(r,f_1)+(p+1)N\left(r,\frac{1}{f_*}\right)+\sum_{i=1}^{q}N\left(r,\frac{1}{f_*-a_i}\right)-N\left(r,\frac{1}{f_1}\right)+S(r,f_*), \end{aligned} \tag{3.9}$$

where a_i $(i = 1,2,\ldots,q)$ are the roots of the algebraic equation

$$\frac{1}{p+q+1}z^q - \frac{{}^qC_1}{p+q}z^{q-1} + \frac{{}^qC_2}{p+q-1}z^{q-2} + \ldots + (-1)^q\frac{1}{p+1} = 0.$$

Using (3.5) - (3.8) in (3.9), we get

$$\begin{aligned} & T(r,\mathscr{F}) \\ \leq\; & N\left(r,\frac{1}{f_*^p(f_*-1)^q f_*'}\right) + (2-p)N\left(r,\frac{1}{f_*}\right) + N\left(r,\frac{1}{g_*^p(g_*-1)^q g_*'}\right) + (2-p)N\left(r,\frac{1}{g_*}\right) \\ & +2\overline{N}(r,f_*) + (p+1)N\left(r,\frac{1}{f_*}\right) + \sum_{i=1}^{q} N\left(r,\frac{1}{f_*-a_i}\right) - N\left(r,\frac{1}{f_*^p(f_*-1)^q f_*'}\right) + o(T(r)). \end{aligned}$$

i.e.,

$$\begin{aligned} & (p+q+1)T(r,f_*) \\ \leq\; & 3N\left(r,\frac{1}{f_*}\right) + 3N\left(r,\frac{1}{g_*}\right) + \overline{N}(r,g_*) + q\,N\left(r,\frac{1}{g_*-1}\right) + 2\overline{N}(r,f_*) \\ & +\sum_{i=1}^{q} N\left(r,\frac{1}{f_*-a_i}\right) + o(T(r)) \\ \leq\; & (q+5)T(r,f_*) + (q+4)T(r,g_*) + o(T(r)). \end{aligned}$$

i.e.,

$$(p-4)T(r,f_*) \leq (q+4)T(r,g_*) + o(T(r)). \tag{3.10}$$

Let $g_1 = -\dfrac{f_3}{f_2} = \dfrac{g_*^p(g_*-1)^q g_*'}{\alpha}$, $g_2 = \dfrac{1}{f_2} = \dfrac{1}{\mathscr{H}}$ and $g_3 = -\dfrac{f_1}{f_2} = -\dfrac{f_*^p(f_*-1)^q f'}{\alpha\mathscr{H}}$.
Then we get $g_1+g_2+g_3 = 1$. By Lemma 2.5, g_1, g_2 and g_3 are linearly independent since f_1, f_2 and f_3 are linearly independent. Proceeding exactly same way as done in above, we get

$$(p-4)T(r,g_*) \leq (q+4)T(r,g_*) + o(T(r)). \tag{3.11}$$

Let $T_*(r) = \max\{T(r,f_*),T(r,g_*)\}$. After combining (3.10) and (3.11), we get

$$(p-q-8)T_*(r) \leq o(T(r)),$$

which contradicts $p \geq q+9$.
Thus f_1, f_2 and f_3 must be linearly dependent. Therefore there exists three constants c_1, c_2 and c_3, at least one of them are non-zero such that

$$c_1f_1 + c_2f_2 + c_3f_3 = 0. \tag{3.12}$$

Subcase 1.1. If $c_1 = 0$, $c_2 \neq 0$ and $c_3 \neq 0$, then from (3.12) we get $f_3 = -\dfrac{c_2}{c_3}f_2$ which implies $g_*^p(g_*-1)^q g_*' = \dfrac{c_2}{c_3}\alpha$.
On integrating, we get

$$\frac{g_*^{p+q+1}}{p+q+1} - \frac{{}^qC_1\,g_*^{p+q}}{p+q} + \frac{{}^qC_2\,g_*^{p+q-1}}{p+q-1} \ldots + (-1)^q\frac{g_*^{p+1}}{p+1} = \frac{c_2}{c_3}\alpha + c, \tag{3.13}$$

where c is an arbitrary constant.
Thus we see that

$$T\left(r,\frac{g_*^{p+q+1}}{p+q+1} - \frac{{}^qC_1\,g_*^{p+q}}{p+q} + \frac{{}^qC_2\,g_*^{p+q-1}}{p+q-1} \ldots + (-1)^q\frac{g_*^{p+1}}{p+1}\right) \leq T(r,\alpha) + O(1).$$

i.e.,

$$(p+q+1)T(r,g_*) \leq S(r,g_*).$$

Since $p \geq q+9$, so we get a contradiction.
Subcase 1.2. Let $c_1 \neq 0$. Then from (3.12), we get

$$f_1 = \left(-\frac{c_2}{c_1}\right)f_2 + \left(-\frac{c_3}{c_1}\right)f_3.$$

After substituting this in the relation $f_1+f_2+f_3 = 1$, we get

$$\left(1-\frac{c_2}{c_1}\right)f_2 + \left(1-\frac{c_3}{c_1}\right)f_3 = 1,$$

where $(c_1-c_2)(c_1-c_3)\neq 0$. So we get

$$\left(1-\frac{c_3}{c_1}\right)\frac{g_*^p(g_*-1)^q f_*'}{\alpha}+\frac{1}{\mathscr{H}}=\left(1-\frac{c_2}{c_1}\right). \tag{3.14}$$

Again we see that

$$\begin{aligned}&T(r,g_*^p(g_*-1)^q g_*')\\ \leq\ & T\left(r,\frac{g_*^p(g_*-1)^q g_*'}{\alpha}\right)+T(r,\alpha)\\ \leq\ & T\left(r,\frac{g_*^p(g_*-1)^q g_*'}{\alpha}\right)+S(r,g_*).\end{aligned}$$

Next applying Lemma 2.2 to the equation (3.14), we get

$$\begin{aligned}&T\left(r,\frac{g_*^p(g_*-1)^q g_*'}{\alpha}\right)\\ \leq\ & \overline{N}\left(r,\frac{g_*^p(g_*-1)^q g_*'}{\alpha}\right)+\overline{N}\left(r,\frac{\alpha}{g_*^p(g_*-1)^q g_*'}\right)+\overline{N}(r,\mathscr{H})+S(r,g).\end{aligned}$$

So combining the above two we get,

$$T(r,g_*^p(g_*-1)^q g_*')\leq \overline{N}\left(r,\frac{1}{g_*^p(g_*-1)^q g_*'}\right)+2\overline{N}(r,g_*)+S(r,g_*). \tag{3.15}$$

By applying Lemmas 2.3, 2.4 and (3.15), we have

$$\begin{aligned}&(p+q)T(r,g_*)\\ \leq\ & T(r,g_*^p(g_*-1)^q)+S(r,g_*)\\ \leq\ & T(r,g_*^p(g_*-1)^q g_*')+T\left(r,\frac{1}{g_*'}\right)+S(r,g_*)\\ \leq\ & \overline{N}\left(r,\frac{1}{g_*^p(g_*-1)^q g_*'}\right)+2\overline{N}(r,g_*)+T\left(r,\frac{1}{g_*'}\right)+S(r,g_*)\\ \leq\ & 8T(r,g_*)+S(r,g_*),\end{aligned}$$

which contradicts $p\geq q+9$.

Subcase 2. If $f_2=k$, where k is a constant.

Subcase 2.1 If $k\neq 1$, then from the relation $f_1+f_2+f_3=1$, we get

$$\frac{f_*^p(f_*-1)^q f_*'}{\alpha}-k\frac{g_*^p(g_*-1)^q g_*'}{\alpha}=1-k. \tag{3.16}$$

Next we apply Lemma 2.2 to the equation (3.16), we get

$$\begin{aligned}&T\left(r,\frac{f_*^p(f_*-1)^q f_*'}{\alpha}\right)\\ \leq\ & \overline{N}(r,g_*)+\overline{N}\left(r,\frac{1}{f_*^p(f_*-1)^q f_*'}\right)++\overline{N}\left(r,\frac{1}{g_*^p(g_*-1)^q g_*'}\right)+S(r,f_*).\end{aligned} \tag{3.17}$$

By applying Lemma 2.3, 2.4 and using equation (3.17), we get

$$\begin{aligned}&(p+q)T(r,f_*)\\ =\ & T(r,f_*^p(f_*-1)^q)+S(r,f_*)\\ \leq\ & T(r,f_*^p(f_*-1)^q f_*')+T\left(r,\frac{1}{f_*'}\right)+S(r,f_*)\\ \leq\ & T\left(r,\frac{f_*^p(f_*-1)^q f_*'}{\alpha}\right)+T\left(r,\frac{1}{f_*'}\right)+S(r,f_*).\end{aligned}$$

i.e.,

$$(p+q-7)T(r,f_*)\leq 4T(r,g_*)+S(r,g_*).$$

Using Lemma 2.6, we get

$$(p+q-4)T(r,f_*)\leq 4\left(\frac{p+q+2}{p-6}\right)T(r,f_*)+S(r,g_*),$$

which contradicts $p\geq q+9$.

Subcase 2.2 Let $k=1$ i.e., $\mathscr{H}=1$ i.e.,

$$f_*^p(f_*-1)^q f_*'\equiv g_*^p(g_*-1)^q g_*'.$$

On integrating, we get

$$\frac{f_*^{p+q+1}}{p+q+1}-\frac{{}^qC_1 f_*^{p+q}}{p+q}+\ldots+(-1)^q\frac{f_*^{p+1}}{p+1}\equiv\frac{g_*^{p+q+1}}{p+q+1}-\frac{{}^qC_1 g_*^{p+q}}{p+q}+\ldots+(-1)^q\frac{g_*^{p+1}}{p+1}+c,$$

where c is an arbitrary constant. i.e.,

$$\mathscr{F}\equiv\mathscr{G}+c. \tag{3.18}$$

Subcase 2.2.1 Let if possible $c\neq 0$. Next we get

$$\Theta(0,\mathscr{F})+\Theta(c,\mathscr{F})+\Theta(\infty,\mathscr{F})=\Theta(0,\mathscr{F})+\Theta(0,\mathscr{G})+\Theta(\infty,\mathscr{F}).$$

We have,

$$\overline{N}\left(r,\frac{1}{\mathscr{F}}\right)=\overline{N}\left(r,\frac{1}{f_*}\right)+\overline{N}\left(r,\frac{1}{f_*-a_1}\right)+\ldots+\overline{N}\left(r,\frac{1}{f_*-a_q}\right)\leq(q+1)\,T(r,f_*).$$

Similarly, we get $N\left(r,\frac{1}{\mathscr{G}}\right)\leq(q+1)\,T(r,g_*)$.

Again note that $\overline{N}(r,\mathscr{F})=\overline{N}(r,f_*)\leq T(r,f_*)$. Again

$$T(r,\mathscr{F})=(p+q+1)\,T(r,f_*)+S(r,f_*).$$

$$T(r,\mathscr{G})=(p+q+1)\,T(r,g_*)+S(r,g_*).$$

Thus

$$\Theta(0,\mathscr{F})=1-\limsup_{r\to\infty}\frac{\overline{N}\left(r,\frac{1}{\mathscr{F}}\right)}{T(r,\mathscr{F})}\geq 1-\frac{(q+1)T(r,f_*)}{(p+q+1)T(r,f_*)}=\frac{p}{p+q+1}.$$

Similarly

$$\Theta(0,\mathscr{H})\geq\frac{p}{p+q+1}\quad\text{and}\quad\Theta(\infty,\mathscr{F})\geq\frac{p+q}{p+q+1}.$$

Therefore

$$\Theta(0,\mathscr{F})+\Theta(c,\mathscr{F})+\Theta(\infty;\mathscr{F})\geq\frac{3p+q}{p+q+1}>2,$$

since $p\geq q+9$, which is a contradiction.
Subcase 2.2.2 Thus we get $c=0$. Thus we get

$$\mathscr{F}\equiv\mathscr{G}. \tag{3.19}$$

Let $h=\frac{f_*}{g_*}$. Then substituting in (3.19), we get

$$\begin{aligned}&(p+q)(p+q-1)\ldots(p+1)g_*^q(h^{p+q-1}-1)\\&-{}^qC_1(p+q+1)(p+q-1)\ldots(p+1)g_*^{q-1}(h^{p+q}-1)\\&+\ldots+(-1)^q(p+q+1)(p+q)\ldots p(h^{p+1}-1)=0.\end{aligned} \tag{3.20}$$

Subcase 2.2.2.1. If h is a non-constant, then using Lemma 2.8 and proceeding exactly same way as done in [12, p-1272], we arrive at a contradiction.
Subcase 2.2.2.2. Let h is constant, then from (3.20), we get $h^{p+q+1}-1=0$, $h^{p+q}-1=0$, ..., $h^{p+1}-1=0$. i.e., $h^d-1=0$, where $d=gcd(p+q+1,p+q,\ldots,p+1)=1$. i.e., $h=1$.
Hence $f_*\equiv g_*$. i.e., $f\equiv g$.
Subcase 3. Suppose $f_3=c$, where c is a constant.
Subcase 3.1. If $c\neq 1$, then from the relation $f_1+f_2+f_3=1$, we get

$$\frac{f_*^p(f_*-1)^q f_*'}{\alpha}-\frac{c\alpha}{g_*^p(g_*-1)^q g_*'}=1-c. \tag{3.21}$$

Applying Lemma 2.2 to the above equation, we get

$$\begin{aligned}&T(r,f_*^p(f_*-1)^q f_*')\\\leq\;&T\left(r,\frac{f_*^p(f_*-1)^q f_*'}{\alpha}\right)+S(r,f_*)\\\leq\;&\overline{N}\left(r,\frac{f_*^p(f_*-1)^q f_*'}{\alpha}\right)+\overline{N}\left(r,\frac{\alpha}{f_*^p(f_*-1)^q f_*'}\right)+\overline{N}\left(r,\frac{g_*^p(g_*-1)^q g_*'}{\alpha}\right)\\&+S(r,f_*)\\\leq\;&\overline{N}(r,f_*)+\overline{N}\left(r,\frac{1}{f_*^p(f_*-1)^q f_*'}\right)+\overline{N}(r,g_*)+S(r,f_*).\end{aligned} \tag{3.22}$$

Using Lemma 2.3, 2.4 and (3.22), we have

$$\begin{aligned} & (p+q)T(r,f_*) \\ \leq\ & T(r,f_*^p(f_*-1)^q)+S(r,f_*) \\ \leq\ & T\left(r,\frac{1}{f_*^p(f_*-1)^q f_*'}\right)+T\left(r,\frac{1}{f_*'}\right)+S(r,f_*) \\ \leq\ & 7\,T(r,f_*)+T(r,g_*)+S(r,f_*). \end{aligned}$$

Next by applying Lemma 2.6, we get

$$\begin{aligned} & (p+q-7)\,T(r,f_*) \\ \leq\ & T(r,g_*)+S(r,f_*) \\ \leq\ & \left(\frac{p+q+2}{p-6}\right)T(r,f_*)+S(r,f_*), \end{aligned}$$

which contradicts $p\geq q+9$.
Subcase 3.2. Let $c=1$. Then from (3.21), we get

$$f_*^p(f_*-1)^q f_*' g_*^p(g_*-1)^q g_*'=\alpha^2. \tag{3.23}$$

Let z_0 be a zero of f_* of order r_0. Then from (3.23), we see that z_0 is a pole of g_* of order s_0 (say). Then from (3.23), we get $pr_0+r_0-1=ps_0+qs_0+s_0+1$. i.e., $(p+1)(r_0-s_0)=qs_0+2\geq p+1$. i.e.,

$$r_0\geq\frac{p+q+1}{q}.$$

Again let z_1 be a zero of f_*-1 of order r_1. Then from (3.23), we see that z_1 will be a pole of g_* of order s_1 (say). So we have $r_1+r_1-1=ps_1+qs_1+s_1+1$. i.e.,

$$r_1\geq\frac{p+q+3}{2}.$$

Let z_2 be a zero of f_*' of order r_2 which are not the zero of $f_*(f_*-1)$, so from (3.23) we see that z_2 will be a pole of g_* of order s_2 (say). Then from (3.23), we get $r_2=ps_2+qs_2+s_2+1$. i.e.,

$$r_2\geq p+q+2.$$

The similar explanations hold for the zeros of $g_*^p(g_*-1)^q g_*'$ also. Next we see from (3.23), we have

$$\overline{N}\left(r,f_*^p(f_*-1)^q f_*'\right)=\overline{N}\left(r,\frac{\alpha^2}{g_*^p(g_*-1)^q g_*'}\right).$$

i.e.,

$$\begin{aligned} & \overline{N}(r,f_*) \\ \leq\ & \overline{N}\left(r,\frac{1}{g_*}\right)+\overline{N}\left(r,\frac{1}{g_*-1}\right)+\overline{N}\left(r,\frac{1}{g_*'}\right) \\ \leq\ & \left(\frac{q}{p+q+1}\right)N\left(r,\frac{1}{g_*}\right)+\left(\frac{2}{p+q+3}\right)N\left(r,\frac{1}{g_*-1}\right)+\left(\frac{1}{p+q+2}\right)N\left(r,\frac{1}{g_*'}\right) \\ \leq\ & \left(\frac{q}{p+q+1}+\frac{2}{p+q+3}+\frac{2}{p+q+2}\right)T(r,g_*)+S(r,g_*). \end{aligned}$$

By applying Second Fundamental Theorem, we get

$$\begin{aligned} & T(r,f_*) \\ \leq\ & \overline{N}\left(r,\frac{1}{f_*}\right)+\overline{N}\left(r,\frac{1}{f_*-1}\right)+\overline{N}(r,f_*)+S(r,f_*) \\ \leq\ & \left(\frac{q}{p+q+1}+\frac{2}{p+q+3}\right)T(r,f_*)+\left(\frac{q}{p+q+1}+\frac{2}{p+q+3}+\frac{2}{p+q+2}\right) \\ \times\ & T(r,g_*)+S(r,f_*)+S(r,g_*). \end{aligned} \tag{3.24}$$

Similarly, we get

$$\begin{aligned} & T(r,g_*) \\ \leq\ & \left(\frac{q}{p+q+1}+\frac{2}{p+q+3}\right)T(r,g_*)+\left(\frac{q}{p+q+1}+\frac{2}{p+q+3}+\frac{2}{p+q+2}\right) \\ \times\ & T(r,f_*)+S(r,f_*)+S(r,g_*). \end{aligned} \tag{3.25}$$

From (3.24) and (3.25), we get

$$T_*(r) \leq \left(\frac{2q}{p+q+1}+\frac{4}{p+q+3}+\frac{2}{p+q+2}\right)T_*(r)+S_*(r).$$

i.e.,

$$\left(1-\frac{2q}{p+q+1}-\frac{4}{p+q+3}-\frac{2}{p+q+2}\right)T_*(r)\leq S_*(r),$$

which contradicts $p\geq q+9$. □

Proof of Theorem 1.22. Since f_* and g_* both are non-constant entire functions, then we may consider the followings two cases.
Case 1. Let f_* and g_* are two transcendental entire functions. Then it is clear that $\overline{N}(r,f_*)=S(r,f_*)$ and $\overline{N}(r,g_*)=S(r,g_*)$. With this the rest of the proof can be carried out in the line of the proof of Theorem 1.21.
Case 2. Let f_* and g_* both are polynomials. Since $f_*^p(f_*-1)^qf_*'$ and $g_*^p(g_*-1)^qg_*'$ share α *CM*, then we must have

$$(f_*^p(f_*-1)^qf_*'-\alpha)=\kappa\,(g_*^p(g_*-1)^qg_*'-\alpha), \tag{3.26}$$

where κ is a non-zero constant.
Subcase 2.1. Suppose $\kappa\neq 1$, then from (3.26), we get

$$\frac{f_*^p(f_*-1)^qf_*'}{\alpha}-\kappa\frac{g_*^p(g_*-1)^qg_*'}{\alpha}=1-\kappa. \tag{3.27}$$

Applying Lemma 2.2, we get

$$\begin{aligned} & T(r,f_*^p(f_*-1)^qf_*') \qquad (3.28)\\ \leq\ & T\left(r,\frac{f_*^p(f_*-1)^qf_*'}{\alpha}\right)+S(r,f_*)\\ \leq\ & \overline{N}\left(r,\frac{f_*^p(f_*-1)^qf_*'}{\alpha}\right)+\overline{N}\left(r,\frac{\alpha}{f_*^p(f_*-1)^qf_*'}\right)+\overline{N}\left(r,\frac{\alpha}{g_*^p(g_*-1)^qg_*'}\right)\\ & +S(r,f_*)\\ \leq\ & \overline{N}(r,f_*)+\overline{N}\left(r,\frac{\alpha}{f_*^p(f_*-1)^qf_*'}\right)+\overline{N}\left(r,\frac{\alpha}{g_*^p(g_*-1)^qg_*'}\right)+S(r,f_*). \end{aligned}$$

Using Lemmas 2.3, 2.4 and (3.27), we get

$$\begin{aligned} & (p+q)T(r,f_*)\\ \leq\ & T(r,f_*^p(f_*-1)^q)+S(r,f_*)\\ \leq\ & T(r,f_*^p(f_*-1)^qf_*')+T\left(r,\frac{1}{f_*}\right)+S(r,f_*')\\ \leq\ & 4\,T(r,f_*)+3\,T(r,g_*)+S(r,f_*). \end{aligned}$$

i.e.,

$$(p+q-4)T(r,f_*)\leq 3\,T(r,g_*)+S(r,g_*).$$

Using Lemma 2.7, we get

$$(p+q-4)T(r,f_*)\leq 3\left(\frac{p+q+1}{p-3}\right)T(r,f_*)+S(r,f_*),$$

which contradicts $p\geq q+5$.
Subcase 2.2. Let $\kappa=1$. So from (3.27), we get

$$f_*^p(f_*-1)^qf_*'\equiv g_*^p(g_*-1)^qg_*'.$$

Next proceeding exactly same way as done in *Subcase 1.3.2* in the proof of *Theorem 1.21*, we get $f\equiv g$. □

4. Concluding remarks and some open questions

If we replace the condition "$f_*^p(f_*-1)^qf_*'$ and $g_*^p(g_*-1)^qg_*'$ share $\alpha(z)$ *CM*" by the condition "$f_*^p(f_*-1)^qf_*'$ and $g_*^p(g_*-1)^qg_*'$ share z *CM* ", then the conclusions of Theorems 1.21 and 1.22 still hold.
Thus we get the following results

Theorem 4.1. *Let f and g hence $f_*=f-w_p$ and $g_*=g-w_p$, $w_p\in\mathbb{C}$ be any two non-constant non- entire meromorphic functions, $n\geq q+9$, $q\in\mathbb{N}$, be an integer. If $\mathscr{P}_*(f)f_*'=f_*^p(f_*-1)^qf_*'$ and $\mathscr{P}_*(g)g_*'=g_*^p(g_*-1)^qf_*'$ share z CM, then $f\equiv g$.*

Theorem 4.2. *Let f and g hence $f_*=f-w_p$ and $g_*=g-w_p$, $w_p\in\mathbb{C}$ be any two non-constant entire functions, $n\geq q+5$, $q\in\mathbb{N}$, be an integer. If $\mathscr{P}_*(f)f_*'=f_*^p(f_*-1)^qf_*'$ and $\mathscr{P}_*(g)g_*'=g_*^p(g_*-1)^qf_*'$ share z CM, then $f\equiv g$.*

Note 4.3. *If we choose $q = m$, $w_p = 0$, then since $p = n+m-q$ and $f_* = f - w_p$, so we get $p = n$ and $f_* = f$, respectively. With this we see that $n \geq m+9$ in Theorem 4.1 and $n \geq m+5$ in Theorem 4.2.*

So from the above note, we observe that Theorem 4.1 and Theorem 4.2 are the direct improvement as well as extension of Theorem H and I respectively.

Remark 4.4. *We see from* Note 4.3 *that for $m = 1$ and $m = 2$, we get $n \geq 10$ and $n \geq 11$ respectively in* Theorem 4.1 *which is a direct improvement of* Theorem E *and* F.

Remark 4.5. *For $m = 1$, we see from* Note 4.3 *that $n \geq 6$ in* Theorem 4.2 *which is a direct improvement of* Theorem G.

Next for further research in this direction, one my glance over the following remarks.

Remark 4.6. *What worth noticing fact is that in [13, equation (39)], there is no term which is absent in the expression. So, for the case of h is constant, [13, equation (40)] implies $h^d - 1 = 0$, where $d = gcd(n+m+1, n+m, \ldots, n+1) = 1$. i.e., $h = 1$ and hence $f \equiv g$. But if we replace $(f-1)^m$ in the expression $f^n(f-1)^m f'$ by a more general expression $f^n P_m(f) f'$, where $P_m(f) = a_m f^m + a_{m-1} f^{m-1} + \ldots + a_1 f + a_0$, $a_i \in \mathbb{C}$, for $i = 0, 1, \ldots, m$. It is not always possible to handle the case of h is constant. If somehow one can do that, then from the case of h is constant, $h^d - 1 = 0$, where $d = gcd(n+m+1, n+m, \ldots, n+1) \neq 1$ in general. So we can't obtained $f \equiv g$ in general.*

Based on the above observations, we next pose the following open questions.

Question 4.7. *Is it possible to reduce further the lower bounds of p in* Theorem 1.21 *and* Theorem 1.22 *?*

Question 4.8. *To get the uniqueness between f and g is it possible to replace $f_*^p(f_*-1)^q f_*'$ and $g_*^p(g_*-1)^q g_*'$ respectively by $f_*^p P_m(f_*) f_*'$ and $g_*^p P_m(g_*) g_*'$, where $P_m(f_*) = P_m(f) = a_m f_*^m + a_{m-1} f_*^{m-1} + \ldots + a_1 f_* + a_0$ in* Theorem 1.21 *and* Theorem 1.22 *?*

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