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Mittag-Leffler-Hyers-Ulam Stability for Cauchy Fractional Differential Equation in the Unit Disk

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Abstract

In this paper, we prove the Mittag-Leffler-Hyers-Ulam stability of Cauchy fractional differential equations in the unit disk for the linear and non-linear cases.

Keywords: *Mittag-Leffler-Hyers-Ulam Stability, Fractional calculus, Cauchy fractional differential equation.*
2010 Mathematics Subject Classification: *Use about five key words or phrases in alphabetical order, Separated by Semicolon.*

1. Introduction

Fractional differential and integral equations can serve as excellent tools for description of mathematical modelling of systems. It also serves as an excellent tool for description of hereditary properties of various materials and processes. For more details on fractional calculus theory, one can see the monographs of Kilbas et al. [5], Miller and Ross [6]and Podlubny [7].
A classical problem in the theory of functional equations is that: Under what conditions there exists an additive mapping near an approximately additive mapping? (for more details see [10]).The first answer to the question of Ulam was given by Hyers [1] in 1941 in the case of Banach spaces: Let X_1, X_2 be two Banach spaces and $\varepsilon > 0$. Then for every mapping $f : X_1 \longrightarrow X_2$ satisfying $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in X_1$, there exists a unique additive mapping $g : X_1 \longrightarrow X_2$ with the property $\|f(x) - g(x)\| \leq \varepsilon, \quad \forall x \in X_1$.
This type of stability is called Hyers-Ulam stability. In 1978, Th. M. Rassias [9] provided a remarkable generalization of the Hyers-Ulam stability by considering variables on the right-hand side of the inequalities. Recently some authors ([2], [3], [4], [14], [15], [16], [17] and [19]) extended the Ulam stability problem from an integer-order differential equation to a fractional-order differential equation. For more results on Ulam type stability of fractional differential equations see [18, 11, 8, 13] and [12].
In this paper we present Mittag-Leffler-Hyers-Ulam stability and continue our study by imposing the Mittag-Leffler-Hyers-Ulam stability for Cauchy fractional differential equations in complex domain.

2. Conclusion

In this section, we introduce notations, definitions and preliminary facts which are used throughout this paper.

Definition 2.1. *The fractional derivative of order α is defined for a function $f(z)$ by*

$$D_z^{\alpha} f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_o^z \frac{f(\zeta)}{(z-\zeta)^{\alpha}} d\zeta, \qquad 0 \leq \alpha < 1,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 2.2. *The fractional integral of order α is defined for a function $f(z)$ by*

$$I_z^{\alpha} f(z) := \frac{1}{\Gamma(\alpha)} \int_o^z \frac{f(\zeta)}{(z-\zeta)^{1-\alpha}} d\zeta, \qquad \alpha > 0,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

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Remark 2.3. *From Definitions (2.1) and (2.2), we have*

$$D_z^{\alpha} z^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} z^{\mu-\alpha}, \mu > -1; 0 \leq \alpha < 1$$

and

$$I_z^{\alpha} z^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} z^{\mu+\alpha}, \mu > -1; \alpha > 0.$$

We need the following preliminaries in the squeal:
Let $U := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plan $\mathbb{C}$ and H denotes the space of all analytic functions on U. Also for $a \in \mathbb{C}$ and $m \in \mathbb{N}$, let $H[a,m]$ be the subspace of H consisting of functions of the form

$$f(z) = a + a_m z^m + a_{m+1} Z^{m+1} + \ldots, \quad z \in U.$$

Let A be the class of functions f, analytic in U and normalized by the conditions $f(0) = f'(0) - 1 = 0$. A function $f \in A$ is called univalent and denoted by class $\mathscr{S}$ if it is one to one in U.

Lemma 2.4. *[20]. Let the function $f(z)$ be in the class $\mathscr{S}$. Then*

$$|D_z^{\alpha} f(z)| \leq r^{1-\alpha} \Gamma(1-\alpha) \int_0^1 \frac{1+rt}{(1-t)^{\alpha}(1-rt)^3} dt$$

$$(r = |z|, \quad z \in U, \quad 0 < \alpha < 1).$$

Lemma 2.5. *[20]. Let the function $f(z)$ be in the class $\mathscr{S}$. Then*

$$|D_z^{1+\alpha} f(z)| \leq r^{1-\alpha} \Gamma(1-\alpha) (rF(2,1;1-\alpha;r))'$$

$$(r = |z|, \quad z \in U \backslash \{0\}, \quad 0 < \alpha < 1).$$

Lemma 2.6. *[20]. Let the function $f(z)$ be in the class $\mathscr{S}$. Then*

$$|D_z^{\alpha} f(z)| \leq \frac{r^{1-\alpha}}{\Gamma(1-\alpha)} \int_0^1 \frac{1+rt}{(1-t)^{\alpha}(1-rt)^3} dt \qquad (r = |z|, z \in U, 0 < \alpha < 1).$$

2.1. Mittag-Lefller-Hyers-Ulam stability for fractional problems

In this section, we shall study the Mittag-Lefller-Ulam-Hyers stability for two different types of fractional problems involving the differential operator in definition 2.1. The first initial value problem is

$$D_z^{\alpha} u(z) = \rho(z) u(z) \qquad (1)$$

$$(u(0) = 0, z \in U, 0 < \alpha < 1)$$

where $u(z), \rho(z) \in H[U, \mathbb{C}]$ (the space of analytic function on the unit disk).
While the second problem is

$$D_z^{\alpha} u(z) = f(z, u(z)) \qquad (2)$$

$$(u(0) = 0, z \in U, 0 < \alpha < 1)$$

where $f : U \times \mathbb{C} \longrightarrow \mathbb{C}$ is analytic in $z \in U$.
And the third one id the form

$$D_z^{1+\alpha} u(z) = f(z, u(z)) \qquad (3)$$

$$(u(z_0) = c, z_0 \in U \backslash \{0\}, 0 < \alpha < 1)$$

where $f : U \times \mathbb{C} \longrightarrow \mathbb{C}$ is analytic in $z \in U$ and $u(z) \in H[U, \mathbb{C}]$.

Definition 2.7. *problem (1) has the Mittag-Leffler-Ulam-Hyers stability if there exists a positive constant K with the following property : for every $\varepsilon > 0, u \in H[U, \mathbb{C}]$ if*

$$|D_z^{\alpha} u(z) - \rho(z) u(z)| \leq \varepsilon E_q(z^q)$$

then there exists some $v \in H[U, \mathbb{C}]$ satisfying $D_z^{\alpha} v(z) = \rho(z) v(z)$ with $v(0) = 0$ such that $|u(z) - v(z)| < K \varepsilon E_q(z^q)$.

Definition 2.8. *problem (2) has the Mittag-Leffler-Ulam-Hyers stability if there exists a positive constant K with the following property : for every $\varepsilon > 0, u \in H[U, \mathbb{C}]$ if*

$$|D_z^{\alpha} u(z) - f(z, u(z))| \leq \varepsilon E_q(z^q)$$

then there exists some $v \in H[U, \mathbb{C}]$ satisfying $D_z^{\alpha} v(z) = f(z, v(z))$ with $v(0) = 0$ such that $|u(z) - v(z)| < K \varepsilon E_q(z^q)$.

Definition 2.9. *problem (3) has the Mittag-Leffler-Ulam-Hyers stability if there exists a positive constant K with the following property : for every $\varepsilon > 0, u \in H[U, \mathbb{C}]$ if*

$$|D_z^{1+\alpha} u(z) - f(z, u(z))| \leq \varepsilon E_q(z^q)$$

then there exists some $v \in H[U, \mathbb{C}]$ satisfying $D_z^{1+\alpha} v(z) = f(z, v(z))$ with $v(z_0) = c, \quad z_0 \in U \backslash \{0\}$ such that $|u(z) - v(z)| < K \varepsilon E_q(z^q)$.

Theorem 2.10. *Let $u \in A$ such that for all $z \in U$ we have* $\max|u(z)| \le \frac{h_\alpha}{2}$, *where*

$$h_\alpha = \frac{r^{1-\alpha}}{\Gamma(1-\alpha)} \int_0^1 \frac{1+rt}{(1-t)^\alpha (1-rt)^3} dt.$$

If $\max|\rho(z)| < 1$, *then problem (1) has the Mittag-Leffler-Ulam-Hyers stability.*

Proof. For every $\varepsilon > 0, u \in A$, we let

$$|D_z^\alpha u(z) - \rho(z)u(z)| \le \varepsilon E_q(z^q)$$

with $u(0) = 0$. In view of Lemma 2.1 we obtain $\max|D_z^\alpha u(z)| = h_\alpha$. consequently, we have:

$$\max|u(z)| \le \max|D_z^\alpha u(z) - \rho(z)u(z)| + \max|\rho(z)|\max|u(z)| \le \varepsilon E_q(z^q) + \max|\rho(z)|\max|u(z)|.$$

Hence we have impose that

$$\max|u(z)| \le \frac{\varepsilon E_q(z^q)}{1-\max|\rho(z)|} = K\varepsilon E_q(z^q).$$

Obviously $v(z) = 0$ is a solution of the problem (1) yields $|u(z)| \le K\varepsilon E_q(z^q)$. Hence (1) has the Mittag-Leffler-Ulam-Hyers stability. □

Theorem 2.11. *Let $u \in A$ such that for all $z \in U$ we have* $\max|u(z)| \le \frac{h_\alpha}{2}$, *where*

$$h_\alpha = \frac{r^{1-\alpha}}{\Gamma(1-\alpha)} \int_0^1 \frac{1+rt}{(1-t)^\alpha (1-rt)^3} dt.$$

If

$$\max|f(z,u(z))| \le M \max|u(z)|, \quad M \in (0,1),$$

then problem (2) has the Mittag-Leffler-Ulam-Hyers stability.

Proof. For every $\varepsilon > 0$ and $u \in A$, we let

$$|D_z^\alpha u(z) - f(z,u(z))| \le \varepsilon E_q(z^q)$$

with $u(0) = 0$. In view of Lemma 2.2 we obtain $\max|D_z^\alpha u(z)| = h_\alpha$. Therefore, we have:

$$\max|u(z)| \le \max|D_z^\alpha u(z) - f(z,u(z)| + \max|f(z,u(z))|$$

$$\le \varepsilon E_q(z^q) + \max|\, f(z,u(z))| \le \varepsilon E_q(z^q) + M \max|\, u(z)|$$

that is

$$\max|u(z)| \le \frac{\varepsilon E_q(z^q)}{1-M} = K\varepsilon E_q(z^q).$$

Obviously $v(z) = I_z^\alpha f(z,v(z))|_{z=0} = 0$ yields $|v(z)| \le K\varepsilon E_q(z^q)$. Hence problem (2) has the Mittag-Leffler-Ulam-Hyers stability. □

Theorem 2.12. *Let $u \in A$ such that for all $z \in U$ we have* $\max|u(z)| \le \dfrac{g_\alpha}{2}$, *where*

$$g_\alpha = \frac{r^{-\alpha}}{\Gamma(1-\alpha)} (rF(2,1;1-\alpha;r))'$$

and

$$|f(z,u(z)) - f(z,v(z))| \le L|u(z) - v(z)|.$$

If $L \in (0,1)$, *then problem (3) has the Mittag-Leffler-Ulam-Hyers stability.*

Proof. Since f is a contraction mapping, then the Banach fixed point theorem implies that problem (3) has a unique solution. For every $\varepsilon > 0$ and $u \in A$, we let

$$|D_z^{1+\alpha} u(z) - f(z,u(z))| < \varepsilon E_q(z^\alpha)$$

with $u(z_0) = c$ and $z_0 \in U \backslash \{0\}$. In view of Lemma 2.3, we impose

$$\max|D_z^{1+\alpha} u(z)| = g_\alpha$$

and consequently we have

$$\max|u(z) - v(z)| \le \max|D_z^\alpha (u(z) - v(z))|$$

$$\le |D_z^\alpha u(z) - D_z^\alpha v(z) - f(z,u(z)) + f(z,v(z))| + \max|f(z,u(z)) - f(z,v(z))| \le \varepsilon E_q(z^q) + L\max|u(z) - v(z)|.$$

Hence we receive

$$\max|u(z) - v(z)| \le \frac{\varepsilon E_q(z^q)}{1-L} = K\varepsilon E_q(z^q).$$

It is clear that $v(z_0) = c$ for some $z_0 \in U \backslash \{0\}$ yields

$$|u(z) - v(z)| \le K\varepsilon E_q(z^q).$$

Thus problem (3) has the Mittag-Leffler-Hyers-Ulam stability. □

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