Konuralp Journal of Mathematics, 7 (2) (2019) 268-273

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath
e-ISSN: 2147-625X

Hermite-Hadamard Type Inequalities for Generalized Fractional Integrals via Strongly Convex Functions

H. Fehmi Gidergelmez[1*], Abdullah Akkurt[2] and Hüseyin Yıldırım[3]

[1,2,3]*Department of Mathematics, Faculty of Science and Arts, University of Kahramanmaraş Sütçü İmam, 46100, Kahramanmaraş, Turkey*
[*]*Corresponding author E-mail: hfgidergelmez@hotmail.com*

Abstract

In this paper, the authors have obtained some new developments of Hermite-Hadamard type inequalities for generalized fractional integrals defined by Mubeen et. al. [6]. In the last part of the article, some results are given with the help of the definition of many fractional integral arising from the generalization.

Keywords: *Fractional integrals, Hermite-Hadamard inequality, Strongly convex functions.*
2010 Mathematics Subject Classification: *26D15, 26A51, 26A33, 26A42*

1. Introduction

Definition 1.1. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on the interval I if the following inequality*

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y) \tag{1.1}$$

holds for every $x,\ y \in I$ and $t \in [0,1]$.

In [2], Polyak defined strongly convexity as the following:

Definition 1.2. *A function $f : I \to \mathbb{R}$ is said to be strongly convex function in the classical sense with modulus $\eta \geq 0$, if*

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)-\eta t(1-t)(y-x)^2 \tag{1.2}$$

holds for every $x,\ y \in I$ and $t \in [0,1]$.

Theorem 1.3. *Let $f : I \to \mathbb{R}$ be a convex function, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.3}$$

Merentes and Nikodem [4] established a new refinement of Hermite–Hadamard's inequality via strongly convex functions. This result is as follows:

Theorem 1.4. *Let $f : I \to \mathbb{R}$ be a strongly convex function with modulus $\eta \geq 0$, then*

$$f\left(\frac{a+b}{2}\right)+\frac{\eta}{12}(b-a)^2 \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}-\frac{\eta}{6}(b-a)^2. \tag{1.4}$$

Integral inequalities have an important role in the theory of convex functions. Therefore Scientists have done many studies on integral inequalities. Studies involving integral inequalities are important in many areas of science, such as mathematics, physics, engineering, etc. In recent years, fractional integrals has attracted the attention of many scientists and many articles have been published in this field. For instance, see [2], [4], [6], [7]-[19]

Email addresses: **hfgidergelmez@hotmail.com (Hasan Fehmi Gidergelmez), abdullahmat@gmail.com (Abdullah Akkurt), hyildir@ksu.edu.tr (Hüseyin Yıldırım)**

Akkurt et al. [1] introduced the following integral operator which generalize many of the fractional integrals.

Definition 1.5. *([1]) Let $\omega:[a,b]\to R$ be an increasing and positive monotone function, having a continuous derivative $\omega'(x)$ on (a,b). The left and right sided fractional of f with respect to the function ω on $[a,b]$ of order $\alpha>0$ is defined by*

$$I_{a^+}^{\alpha}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_a^x\frac{\omega'(t)f(t)}{[\omega(x)-\omega(t)]^{1-\frac{\alpha}{k}}}dt,\quad x>a, \tag{1.5}$$

and

$$I_{b^-}^{\alpha}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_x^b\frac{\omega'(t)f(t)}{[\omega(t)-\omega(x)]^{1-\frac{\alpha}{k}}}dt,\quad b>x. \tag{1.6}$$

If by using the change of variable $s=\frac{t-a}{b-a}$ and $s=\frac{t-b}{a-b}$ respectively, then we have the following definition:

$$I_{a^+}^{\alpha}f(b)=\frac{b-a}{k\Gamma_k(\alpha)}\int_0^1\frac{\omega'((1-s)a+bs)}{[\omega(x)-\omega((1-s)a+bs)]^{1-\frac{\alpha}{k}}}f((1-s)a+bs)ds, \tag{1.7}$$

and

$$I_{b^-}^{\alpha}f(a)=\frac{b-a}{k\Gamma_k(\alpha)}\int_0^1\frac{\omega'(sa+(1-s)b)}{[\omega(sa+(1-s)b)-\omega(x)]^{1-\frac{\alpha}{k}}}f((sa+(1-s)b)ds. \tag{1.8}$$

For $k=1$, operator in (1.7) leads to fractional integral of f with respect to the function ω on $[0,1]$ of order $\alpha>0$. This relation is given by [6]

$$I_{a^+}^{\alpha}f(b)=\frac{b-a}{\Gamma(\alpha)}\int_0^1\frac{\omega'((1-s)a+bs)}{[\omega(x)-\omega((1-s)a+bs)]^{1-\alpha}}f((1-s)a+bs)ds$$

and

$$I_{b^-}^{\alpha}f(a)=\frac{b-a}{\Gamma(\alpha)}\int_0^1\frac{\omega'(sa+(1-s)b)}{[\omega(sa+(1-s)b)-\omega(x)]^{1-\alpha}}f((sa+(1-s)b)ds.$$

In this work, the authors obtained some new Hermite-Hadamard type inequalities with the help of strongly convex functions for generalized fractional integrals.

2. Main Results

Theorem 2.1. *Let $\omega:[a,b]\to R$ be an increasing and positive monotone function, having a continuous derivative $\omega'(x)$ on (a,b). Let $f:I\to\mathbb{R}$ be a strongly convex function with modulus $\eta\geq 0$, then*

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)+\frac{\eta\alpha(b-a)^3}{8[\omega(b)-\omega(a)]^{\alpha}}[G_1(\alpha,t)+G_2(\alpha,t)]\\
\leq\ &\frac{\Gamma(\alpha+1)}{4[\omega(b)-\omega(a)]^{\alpha}}[I_{a^+}^{\alpha}F(b)+I_{b^-}^{\alpha}F(a)]\\
\leq\ &\frac{f(a)+f(b)}{2}-\frac{\eta\alpha(b-a)^3}{2[\omega(b)-\omega(a)]^{\alpha}}[G_3(\alpha,t)+G_4(\alpha,t)],
\end{aligned}$$

where

$$\begin{aligned}
G_1(\alpha,t)&=\int_0^1\frac{\omega'((1-s)a+sb)}{[\omega(b)-\omega((1-s)a+bs)]^{1-\alpha}}(2s-1)^2ds\\
G_2(\alpha,t)&=\int_0^1\frac{\omega'(as+(1-s)b)}{[\omega(as+(1-s)b)-\omega(a)]^{1-\alpha}}(2s-1)^2ds\\
G_3(\alpha,t)&=\int_0^1\frac{\omega'((1-s)a+sb)}{[\omega(b)-\omega((1-s)a+bs)]^{1-\alpha}}s(1-s)ds\\
G_4(\alpha,t)&=\int_0^1\frac{\omega'(as+(1-s)b)}{[\omega(as+(1-s)b)-\omega((a)]^{1-\alpha}}s(1-s)ds.
\end{aligned}$$

Proof. Since f is a strongly convex function on $[a,b]$ for $t=\frac{1}{2}$

$$f\left(\frac{x+y}{2}\right)\leq\frac{f(x)+f(y)}{2}-\frac{\eta}{4}(y-x)^2 \tag{2.1}$$

obtained. For $s\in[0,1]$, $x=as+(1-s)b$ and $y=(1-s)a+sb$, then since f is convex, therefore

$$\begin{aligned}&f\left(\frac{as+(1-s)b+(1-s)a+sb}{2}\right)\\ \leq\ &\frac{f(as+(1-s)b)+f((1-s)a+sb)}{2}\\ &-\frac{\eta}{4}((1-s)a+sb-(as+(1-s)b))^2.\end{aligned} \tag{2.2}$$

From (2.2), we have

$$\begin{aligned}f\left(\frac{a+b}{2}\right)\ &\leq\ \frac{f(as+(1-s)b)+f((1-s)a+sb)}{2}\\ &-\frac{\eta}{4}(b-a)^2(2s-1)^2.\end{aligned}$$

Since $F(x)=f(x)+f(a+b-x)$, then substituting $F((1-s)a+sb)=f((1-s)a+sb)+f(as+(1-s)b)$, we have

$$f\left(\frac{a+b}{2}\right)\leq\frac{F((1-s)a+sb)}{2}-\frac{\eta}{4}(b-a)^2(2s-1)^2. \tag{2.3}$$

Also, $F(as+(1-s)b)=f(as+(1-s)b)+f((1-s)a+sb)$, we have

$$f\left(\frac{a+b}{2}\right)\leq\frac{F(as+(1-s)b)}{2}-\frac{\eta}{4}(b-a)^2(2s-1)^2. \tag{2.4}$$

Multiplying both sides of inequality (2.3) with $\frac{b-a}{\Gamma(\alpha)}\frac{\omega'((1-s)a+sb)}{[\omega(b)-\omega((1-s)a+bs)]^{1-\alpha}}$ and integrating with respect to s from 0 to 1, we obtain

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\frac{b-a}{\Gamma(\alpha)}\int_0^1\frac{\omega'((1-s)a+sb)}{[\omega(b)-\omega((1-s)a+bs)]^{1-\alpha}}ds\\ \leq\ &\frac{b-a}{2\Gamma(\alpha)}\int_0^1\frac{\omega'((1-s)a+sb)}{[\omega(b)-\omega((1-s)a+bs)]^{1-\alpha}}F((1-s)a+sb)ds\\ &-\frac{\eta}{4}\frac{b-a}{\Gamma(\alpha)}(b-a)^2\int_0^1\frac{\omega'((1-s)a+sb)}{[\omega(b)-\omega((1-s)a+bs)]^{1-\alpha}}(2s-1)^2ds.\end{aligned} \tag{2.5}$$

Substituting $u=\omega(b)-\omega((1-s)a+bs)$, in the left hand side of (2.5) we get

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\frac{b-a}{\Gamma(\alpha)}\int_0^1\frac{\omega'((1-s)a+sb)}{[\omega(b)-\omega((1-s)a+bs)]^{1-\alpha}}ds\\ =\ &f\left(\frac{a+b}{2}\right)\frac{[\omega(b)-\omega(a)]^{\alpha}}{\Gamma(\alpha+1)}.\end{aligned}$$

Hence,

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\frac{[\omega(b)-\omega(a)]^{\alpha}}{\Gamma(\alpha+1)}\\ \leq\ &\frac{b-a}{2\Gamma(\alpha)}\int_0^1\frac{\omega'((1-s)a+sb)}{[\omega(b)-\omega((1-s)a+bs)]^{1-\alpha}}F((1-s)a+sb)ds\\ &-\frac{\eta}{4}\frac{(b-a)^3}{\Gamma(\alpha)}G_1(\alpha,t).\end{aligned}$$

Finally, we get

$$f\left(\frac{a+b}{2}\right)\frac{[\omega(b)-\omega(a)]^{\alpha}}{\Gamma(\alpha+1)}+\frac{\eta}{4}\frac{(b-a)^3}{\Gamma(\alpha)}G_1(\alpha,t)\leq\frac{1}{2}I_{a^+}^{\alpha}F(b). \tag{2.6}$$

Multiplying both sides of above inequality (2.4) with $\frac{b-a}{\Gamma(\alpha)}\frac{\omega'((as+(1-s)b)}{[\omega(as+(1-s)b)-\omega((a)]^{1-\alpha}}$ and integrating with respect to s from 0 to 1, we obtain

$$\begin{aligned}
&\frac{b-a}{\Gamma(\alpha)}f\left(\frac{a+b}{2}\right)\int_0^1\frac{\omega'((as+(1-s)b)}{[\omega(as+(1-s)b)-\omega((a)]^{1-\alpha}}ds\\
\leq\ &\frac{b-a}{2\Gamma(\alpha)}\int_0^1\frac{\omega'((as+(1-s)b)}{[\omega(as+(1-s)b)-\omega((a)]^{1-\alpha}}F(as+(1-s)b)ds \qquad (2.7)\\
&-\frac{\eta}{4}\frac{b-a}{\Gamma(\alpha)}(b-a)^2\int_0^1\frac{\omega'((as+(1-s)b)}{[\omega(as+(1-s)b)-\omega((a)]^{1-\alpha}}(2s-1)^2ds.
\end{aligned}$$

Substituting $u=\omega(as+(1-s)b)-\omega((a)$, in the left hand side of (2.7) we get

$$\begin{aligned}
&\frac{b-a}{\Gamma(\alpha)}f\left(\frac{a+b}{2}\right)\int_0^1\frac{\omega'((as+(1-s)b)}{[\omega(as+(1-s)b)-\omega((a)]^{1-\alpha}}ds\\
=\ &f\left(\frac{a+b}{2}\right)\frac{[\omega(b)-\omega(a)]^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)\frac{[\omega(b)-\omega(a)]^\alpha}{\Gamma(\alpha+1)}\\
\leq\ &\frac{b-a}{2\Gamma(\alpha)}\int_0^1\frac{\omega'((as+(1-s)b)}{[\omega(as+(1-s)b)-\omega((a)]^{1-\alpha}}F(as+(1-s)b)ds\\
&-\frac{\eta}{4}\frac{(b-a)^3}{\Gamma(\alpha)}G_2(\alpha,t).
\end{aligned}$$

Finally, we get

$$f\left(\frac{a+b}{2}\right)\frac{[\omega(b)-\omega(a)]^\alpha}{\Gamma(\alpha+1)}+\frac{\eta}{4}\frac{(b-a)^3}{\Gamma(\alpha)}G_1(\alpha,t)\leq\frac{1}{2}I^\alpha_{a^+}F(b),$$

and

$$f\left(\frac{a+b}{2}\right)\frac{[\omega(b)-\omega(a)]^\alpha}{\Gamma(\alpha+1)}+\frac{\eta}{4}\frac{(b-a)^3}{\Gamma(\alpha)}G_2(\alpha,t)\leq\frac{1}{2}I^\alpha_{b^-}F(a). \qquad (2.8)$$

Adding (2.6) and (2.8), we have

$$\begin{aligned}
&2f\left(\frac{a+b}{2}\right)\frac{[\omega(b)-\omega(a)]^\alpha}{\Gamma(\alpha+1)}+\frac{\eta}{4}\frac{(b-a)^3}{\Gamma(\alpha)}[G_1(\alpha,t)+G_2(\alpha,t)]\\
\leq\ &\frac{1}{2}\left[I^\alpha_{a^+}F(b)+I^\alpha_{b^-}F(a)\right]. \qquad (2.9)
\end{aligned}$$

Now, multiplying both sides of above inequality (2.9) with $\frac{\Gamma(\alpha+1)}{2[\omega(b)-\omega(a)]^\alpha}$, we have

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)+\frac{\eta\alpha(b-a)^3}{8[\omega(b)-\omega(a)]^\alpha}[G_1(\alpha,t)+G_2(\alpha,t)] \qquad (2.10)\\
\leq\ &\frac{\Gamma(\alpha+1)}{4[\omega(b)-\omega(a)]^\alpha}[I^\alpha_{a^+}F(b)+I^\alpha_{b^-}F(a)],
\end{aligned}$$

and the first inequality is proved. For the proof of the second inequality in (2.1) we first note that if f is a convex function on $[a,b]$, then we have

$$f(as+(1-s)b)\leq sf(a)+(1-s)f(b)-\eta s(1-s)(b-a)^2$$

and

$$f((1-s)a+sb)\leq(1-s)f(a)+sf(b)-\eta s(1-s)(b-a)^2$$

adding this inequalities we have

$$f(as+(1-s)b)+f((1-s)a+sb)\leq f(a)+f(b)-2\eta s(1-s)(b-a)^2.$$

Here, since $F(as+(1-s)b)=f(as+(1-s)b)+f((1-s)a+sb)$, then we have

$$F(as+(1-s)b)\leq f(a)+f(b)-2\eta s(1-s)(b-a)^{2}, \tag{2.11}$$

and, since $F((1-s)a+sb)=f((1-s)a+sb)+f(as+(1-s)b)$, then we have

$$F((1-s)a+sb)\leq f(a)+f(b)-2\eta s(1-s)(b-a)^{2}. \tag{2.12}$$

Multiplying both sides of the inequality (2.12) with $\frac{b-a}{\Gamma(\alpha)}\frac{\omega'((1-s)a+sb)}{[\omega(b)-\omega((1-s)a+bs)]^{1-\alpha}}$ and integrating with respect to s from 0 to 1, we obtain

$$\begin{aligned}
&\frac{b-a}{\Gamma(\alpha)}\int_{0}^{1}\frac{\omega'((1-s)a+sb)}{[\omega(b)-\omega((1-s)a+bs)]^{1-\alpha}}F((1-s)a+sb)ds\\
\leq\;&[f(a)+f(b)]\frac{b-a}{\Gamma(\alpha)}\int_{0}^{1}\frac{\omega'((1-s)a+sb)}{[\omega(b)-\omega((1-s)a+bs)]^{1-\alpha}}ds\\
&-2\eta\frac{(b-a)^{3}}{\Gamma(\alpha)}\int_{0}^{1}\frac{\omega'((1-s)a+sb)}{[\omega(b)-\omega((1-s)a+bs)]^{1-\alpha}}s(1-s)ds.
\end{aligned} \tag{2.13}$$

Substituting $u=\omega(b)-\omega((1-s)a+bs)$, the first integral in the right hand side of (2.13) we get

$$\begin{aligned}
&[f(a)+f(b)]\frac{b-a}{\Gamma(\alpha)}\int_{0}^{1}\frac{\omega'((1-s)a+sb)}{[\omega(b)-\omega((1-s)a+bs)]^{1-\alpha}}ds\\
=\;&[f(a)+f(b)]\frac{[\omega(b)-\omega(a)]^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{b-a}{\Gamma(\alpha)}\int_{0}^{1}\frac{\omega'((1-s)a+sb)}{[\omega(b)-\omega((1-s)a+bs)]^{1-\alpha}}F((1-s)a+sb)ds\\
\leq\;&[f(a)+f(b)]\frac{[\omega(b)-\omega(a)]^{\alpha}}{\Gamma(\alpha+1)}\\
&-2\eta\frac{(b-a)^{3}}{\Gamma(\alpha)}\int_{0}^{1}\frac{\omega'((1-s)a+sb)}{[\omega(b)-\omega((1-s)a+bs)]^{1-\alpha}}s(1-s)ds.
\end{aligned}$$

Finally, we get

$$I_{a^{+}}^{\alpha}F(b)\leq[f(a)+f(b)]\frac{[\omega(b)-\omega(a)]^{\alpha}}{\Gamma(\alpha+1)}-2\eta\frac{(b-a)^{3}}{\Gamma(\alpha)}G_{3}(\alpha,t). \tag{2.14}$$

Multiplying both sides of above inequality (2.11) with $\frac{b-a}{\Gamma(\alpha)}\frac{\omega'((as+(1-s)b)}{[\omega(as+(1-s)b)-\omega((a)]^{1-\alpha}}$ and integrating with respect to s from 0 to 1, we obtain

$$\begin{aligned}
&\frac{b-a}{\Gamma(\alpha)}\int_{0}^{1}\frac{\omega'(as+(1-s)b)}{[\omega(as+(1-s)b)-\omega((a))]^{1-\alpha}}F(as+(1-s)b)ds\\
\leq\;&[f(a)+f(b)]\frac{b-a}{\Gamma(\alpha)}\int_{0}^{1}\frac{\omega'(as+(1-s)b)}{[\omega(as+(1-s)b)-\omega((a)]^{1-\alpha}}ds\\
&-2\eta\frac{b-a}{\Gamma(\alpha)}(b-a)^{2}\int_{0}^{1}\frac{\omega'(as+(1-s)b)}{[\omega(as+(1-s)b)-\omega((a)]^{1-\alpha}}s(1-s)ds.
\end{aligned}$$

Similarly, we have

$$I_{b^{-}}^{\alpha}F(a)\leq[f(a)+f(b)]\frac{[\omega(b)-\omega(a)]^{\alpha}}{\Gamma(\alpha+1)}-2\eta\frac{(b-a)^{3}}{\Gamma(\alpha)}G_{4}(\alpha,t). \tag{2.15}$$

Adding (2.14) and (2.15), we have

$$\begin{aligned}
&[I_{a^{+}}^{\alpha}F(b)+I_{b^{-}}^{\alpha}F(a)]\\
\leq\;&2[f(a)+f(b)]\frac{[\omega(b)-\omega(a)]^{\alpha}}{\Gamma(\alpha+1)}-2\eta\frac{(b-a)^{3}}{\Gamma(\alpha)}[G_{3}(\alpha,t)+G_{4}(\alpha,t)].
\end{aligned} \tag{2.16}$$

Now, multiplying both sides of above inequality (2.16) with $\frac{\Gamma(\alpha+1)}{4[\omega(b)-\omega(a)]^{\alpha}}$, we have

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{4[\omega(b)-\omega(a)]^{\alpha}}[I_{a^+}^{\alpha}F(b)+I_{b^-}^{\alpha}F(a)] \\ \leq\ & \frac{[f(a)+f(b)]}{2}-\frac{\eta\alpha(b-a)^3}{2[\omega(b)-\omega(a)]^{\alpha}}[G_3(\alpha,t)+G_4(\alpha,t)], \end{aligned} \tag{2.17}$$

and the second inequality is proved. If combined (2.9) and (2.17)

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)+\frac{\eta\alpha(b-a)^3}{8[\omega(b)-\omega(a)]^{\alpha}}[G_1(\alpha,t)+G_2(\alpha,t)] \\ \leq\ & \frac{\Gamma(\alpha+1)}{4[\omega(b)-\omega(a)]^{\alpha}}[I_{a^+}^{\alpha}F(b)+I_{b^-}^{\alpha}F(a)] \\ \leq\ & \frac{[f(a)+f(b)]}{2}-\frac{\eta\alpha(b-a)^3}{2[\omega(b)-\omega(a)]^{\alpha}}[G_3(\alpha,t)+G_4(\alpha,t)]. \end{aligned}$$

The proof is completed. □

Corollary 2.2. *If we take $\eta=0$ in Theorem 2.1, we have*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha+1)}{4[\omega(b)-\omega(a)]^{\alpha}}[I_{a^+}^{\alpha}F(b)+I_{b^-}^{\alpha}F(a)]\leq\frac{f(a)+f(b)}{2}.$$

Remark 2.3. *If we choose $\eta=0$ and $\omega(x)=x$ in Theorem 2.1, then the general results of the article [7] are obtained.*

Corollary 2.4. *If we take $\omega(x)=x$ in Theorem 2.1, we have*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)+\frac{\eta\alpha(b-a)^{3-\alpha}}{8}[G_1(\alpha,t)+G_2(\alpha,t)] \\ \leq\ & \frac{\Gamma(\alpha+1)}{4\,(b-a)^{\alpha}}[I_{a^+}^{\alpha}F(b)+I_{b^-}^{\alpha}F(a)] \\ \leq\ & \frac{f(a)+f(b)}{2}-\frac{\eta\alpha(b-a)^{3-\alpha}}{2}[G_3(\alpha,t)+G_4(\alpha,t)]. \end{aligned}$$

Corollary 2.5. *If we take $\alpha=1$ and $\omega(x)=x$ in Theorem 2.1, then we have the Theorem 1.4,*

$$f\left(\frac{a+b}{2}\right)+\frac{\eta(b-a)^2}{12}\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2}-\frac{\eta(b-a)^2}{6}.$$

References

[1] A. Akkurt, M. E. Yıldırım and H. Yıldırım, On some Integral Inequalities for $(k,h)-$Riemann-Liouville fractional integral, New Trends in Mathematical Sciences (NTMSCI), 4(1), (2016), 138-146.
[2] B.T. Polyak, Existence theorems and convergence of minimizing sequences in extremum problems with restrictions. Sov. Math. Dokl. 7, 72–75 (1966).
[3] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and Applications of Fractional Diferential Equations, Elsevier B.V., Amsterdam, Netherlands, 2006.
[4] N. Merentes and K. Nikodem, Remarks on strongly convex functions. Aequ. Math. 80, 193–199 (2010).
[5] S.G. Samko, A.A. Kilbas and O.I. Marichev, Fractional Integrals and Derivatives-Theory and Applications, Gordon and Breach, Linghorne, 1993.
[6] S. Mubeen, S. Iqbal and M. Tomar, Hermite-Hadamard Type Inequalities via Fractional Integrals of a Function With Respect to Another Function and $k-$parameter, J. Inequal. Math. Appl. 1 (2017), 1-9.
[7] M. Zeki Sarikaya, E. Set, H. Yaldiz and N. Basak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, Mathematical and Computer Modelling, doi:10.1016/j.mcm.2011.12.048, 57 (2013) 2403–2407.
[8] F. Ertuğral, M. Zeki Sarıkaya and H. Budak, On Refinements of Hermite-Hadamard-Fejer Type Inequalities for Fractional Integral Operators, Applications and Applied Mathematics,Vol. 13, Issue 1 (June 2018), pp. 426-442.
[9] M. Zeki Sarikaya and H. Yildirim, "On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals", Miskolc Mathematical Notes, 17(2), 1049-1059, 2016.
[10] H. Budak and M. Zeki Sarikaya, Hermite-Hadamard type inequalities for s-convex mappings via fractional integrals of a function with respect to another function, Fasciculi Mathematici, No.27, pp.25-36, 2016. DOI:10.1515/fascmath-2016-0014.
[11] M. Zeki Sarikaya and H. Budak, "Generalized Hermite-Hadamard type integral inequalities for fractional integrals", FILOMAT, 30:5 (2016), 1315–1326.
[12] J. Vanterler da C. Sousa, E. Capelas de Oliveira, The Minkowski's inequality by means of a generalized fractional integral. AIMS Mathematics, 2018, 3(1): 131-147. doi: 10.3934/Math.2018.1.131
[13] J. Sousa, D. S. Oliveira and E. C. de Oliveira, Grüss-type inequality by mean of a fractional integral, arXiv:1705.00965, (2017).
[14] E. Set, Z. Dahmani and İ. Mumcu, New extensions of Chebyshev type inequalities using generalized Katugampola integrals via Polya-Szegö inequality. An International Journal of Optimization and Control: Theories & Applications (IJOCTA), 8(2), 137-144. doi:http://dx.doi.org/10.11121/ijocta.01.2018.00541.
[15] S.K. Ntouyas, P. Agarwal and J. Tariboon, On Pólya-Szegö and Chebyshev type inequalities involving the Riemann-Liouville fractional integral op-erators.J. Math. Inequal,10(2), 491-504.
[16] P. Agarwal, M. Jleli and M. Tomar, Certain Hermite-Hadamard type inequalities via generalized $k-$fractional integrals. J. Inequal. Appl. 2017(1) (2017).
[17] P. Agarwal J. Tariboon, and S.K. Ntouyas, Some generalized Riemann-Liouville $k-$fractional integral inequalities. Journal of Inequalities and Applications, 2016, 122. https://doi.org/10.1186/s13660-016-1067-3.
[18] M. Tomar, S. Mubeen and J. Choi, Certain inequalities associated with Hadamard k-fractional integral operators, Journal of Inequalities and Applications20162016:234, https://doi.org/10.1186/s13660-016-1178-x.
[19] J. Vanterler da C. Sousa, E. Capelas de Oliveira., A Gronwall inequality and the Cauchy-type problem by means of $\psi-$Hilfer operator, arXiv:1709.03634, 2017.