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On Complex Pulsating Fibonacci Sequence

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Abstract

In this study, we defined complex pulsating Fibonacci sequence of the second kind which is a generalization of first kind. For this newly defined complex sequence we give some summation formulas. In addition, we defined multiplicative version of complex pulsating Fibonacci sequence of second kind.

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1. Introduction

Son of a merchant, Leonardo Fibonacci had known making calculations in Roman and Indo-Arabic numeration systems. He also studied al-Khowarizmi's *Hisâb al-jabr w'al-muqabâlah* in Algeria. In 1202, Fibonacci published his classical book named *Liber abaci*. In *Liber abaci*, he mentioned and demonstrated superiority of Indo-Arabic numeration system. Also, *Liber abaci* contains elementary problems. One of the problems is about rabbits which leads to sequence named after Fibonacci [7].

Fibonacci sequence is well known sequence and studied by many authors. In addition, there are some journals which are about properties of Fibonacci sequences.

In this paper we focus on pulsating Fibonacci sequence which is first defined by K. T. Atanassov in [8]. After his definition there are some papers which studies different aspects of pulsating Fibonacci sequence such as [10, 2].

Definition of pulsating Fibonacci sequence is as follows. Let a and b be two initial values. Using these values two sequences can be constructed

$$\alpha_0 = a, \beta_0 = b,$$
$$\alpha_{2k+1} = \beta_{2k+1} = \alpha_{2k} + \beta_{2k},$$
$$\alpha_{2k+2} = \alpha_{2k+1} + \beta_{2k},$$
$$\beta_{2k+2} = \beta_{2k+1} + \alpha_{2k},$$

for $k \in \mathbb{N}$. This pair of sequences called pulsating Fibonacci sequence [8]. Atanassov gave the formulas below to find desired element of the sequences α or β.

$$\alpha_{2k+1} = \alpha_{2k+1} = 3^k a + 3^k b,$$
$$\alpha_{4k} = \frac{3^{2k}+1}{2}a + \frac{3^{2k}-1}{2}b,$$
$$\beta_{4k} = \frac{3^{2k}-1}{2}a + \frac{3^{2k}+1}{2}b,$$
$$\alpha_{4k+2} = \frac{3^{2k+1}-1}{2}a + \frac{3^{2k+1}+1}{2}b$$

and

$$\beta_{4k+2} = \frac{3^{2k+1}+1}{2}a + \frac{3^{2k+1}-1}{2}b$$

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for $k \in \mathbb{N}$. In addition he gave two identities;

$$\alpha_{4k+1} = \beta_{4k+1} = \alpha_{4k} + \beta_{4k}$$

and

$$\alpha_{4k+1} = \beta_{4k+1} = \alpha_{4k} + \beta_{4k-1}.$$

However, second identity must be written mistakenly because explicit form of second identity indicate $\beta_{4k+2} = \alpha_{4k} + \beta_{4k+1}$.

After Atanassov's paper Bhatnagar and Sikhwal defined pulsating Fibonacci sequence of second kind [10]. Their definition is as follows,

$$\alpha_0 = a, \beta_0 = b,$$
$$\alpha_1 = \beta_1 = c,$$
$$\alpha_{2k+2} = \alpha_{2k+1} + \beta_{2k},$$
$$\beta_{2k+2} = \beta_{2k+1} + \alpha_{2k}$$

and

$$\alpha_{2k+1} = \beta_{2k+1} = \alpha_{2k} + \beta_{2k}.$$

For second kind if we take $c = a + b$ we will get original definition of pulsating Fibonacci sequence or pulsating Fibonacci sequence of first kind. Also they gave formulas to get needed element of sequences as follows

$$\alpha_{2k} = \frac{3^{k-1} + (-1)^k}{2} a + \frac{3^{k-1} + (-1)^{k-1}}{2} b + 3^{k-1} c,$$
$$\beta_{2k} = \frac{3^{k-1} + (-1)^{k-1}}{2} a + \frac{3^{k-1} + (-1)^k}{2} b + 3^{k-1} c$$

but third equation is written inaccurately in [10]. The true explicit formula is given below,

$$\alpha_{2k+1} = \beta_{2k+1} = 3^{k-1} a + 3^{k-1} b + 3^{k-1} 2c.$$

From inspiration of papers like [8, 10, 2], we will define complex pulsating Fibonacci sequence. Also we study multiplicative version of complex pulsating Fibonacci sequence. Furthermore we give some identities involving complex pulsating Fibonacci sequence.

2. Complex Pulsating Fibonacci Sequence

In this section we define complex pulsating Fibonacci sequence for second kind which generalizes first kind. Firstly, let us give some basic concepts. A complex sequence is a function from N into C such that f is a complex sequence iff $dom f = N$ and for every $x \in N$ holds $f(x)$ is an element of C [1]. Complex numbers and sequences are widely used in various branches of science and for that reason they are also very important. For example it is used in the coding and decoding, mobile robot localization, image store systems, neural networks, information theory, etc [4, 5, 3, 9, 6].

When a, b and c are real values we can define a new sequence by using these values as follows.

$$P_0 = a + ci,$$
$$Q_0 = b + ci,$$
$$Re(P_{n+1}) = Im(P_n),$$
$$Re(Q_{n+1}) = Im(Q_n),$$
$$Im(P_{2n+2}) = Im(Q_{2n+2}) = Im(P_{2n+1} + Q_{2n+1}),$$
$$Im(P_{2n+1}) = Im(P_{2n}) + Re(Q_{2n})$$

and

$$Im(Q_{2n+1}) = Im(Q_{2n}) + Re(P_{2n})$$

for $n \in \mathbb{N}$. In the equations $Re()$ is used for real part of the complex number and $Im()$ is used for imaginary part of the complex number. This pair of complex sequences are called complex pulsating Fibonacci sequence.

In order to obtain the value of desired element of sequences we will give following theorem.

Theorem 2.1. *For every $n \in \mathbb{N}$,*

$$Im(P_{2n+1}) = \frac{3^n + (-1)^{n+1}}{2} a + \frac{3^n + (-1)^n}{2} b + 3^n c,$$
$$Im(Q_{2n+1}) = \frac{3^n + (-1)^n}{2} a + \frac{3^n + (-1)^{n+1}}{2} b + 3^n c$$

and

$$Im(P_{2n+2}) = Im(Q_{2n+2}) = 3^n a + 3^n b + 3^n 2c.$$

Proof. For $n=0$ the assertion is valid. Now, we assume that

$$Im(P_{2n+1}) = \frac{3^n+(-1)^{n+1}}{2}a + \frac{3^n+(-1)^n}{2}b + 3^n c,$$

$$Im(Q_{2n+1}) = \frac{3^n+(-1)^n}{2}a + \frac{3^n+(-1)^{n+1}}{2}b + 3^n c$$

and

$$Im(P_{2n+2}) = Im(Q_{2n+2}) = 3^n a + 3^n b + 3^n 2c.$$

Then, first let us prove the odd case of sequence P

$$\begin{aligned} Im(P_{2n+3}) &= Im(P_{2n+2}) + Re(Q_{2n+2}), \\ &= Im(P_{2n+1}) + 2Im(Q_{2n+1}) \end{aligned}$$

$$= \frac{3^n+(-1)^{n+1}}{2}a + \frac{3^n+(-1)^n}{2}b + 3^n c + 2\left[\frac{3^n+(-1)^n}{2}a + \frac{3^n+(-1)^{n+1}}{2}b + 3^n c\right]$$

$$= \frac{3^{n+1}+(-1)^n}{2}a + \frac{3^{n+1}+(-1)^{n+1}}{2}b + 3^{n+1}c.$$

Odd case of sequence Q is as follows,

$$\begin{aligned} Im(Q_{2n+3}) &= Im(Q_{2n+2}) + Re(P_{2n+2}), \\ &= Im(Q_{2n+1}) + 2Im(P_{2n+1}) \end{aligned}$$

$$= \frac{3^n+(-1)^n}{2}a + \frac{3^n+(-1)^{n+1}}{2}b + 3^n c + 2\left[\frac{3^n+(-1)^{n+1}}{2}a + \frac{3^n+(-1)^n}{2}b + 3^n c\right]$$

$$= \frac{3^{n+1}+(-1)^{n+1}}{2}a + \frac{3^{n+1}+(-1)^n}{2}b + 3^{n+1}c.$$

Finally, for even case of the theorem

$$Im(P_{2n+4}) = Im(Q_{2n+4}) = Im(P_{2n+3}) + Im(Q_{2n+3}),$$

$$= 3\left[\frac{3^n+(-1)^{n+1}}{2}a + \frac{3^n+(-1)^n}{2}b + \frac{3^n+(-1)^n}{2}a + \frac{3^n+(-1)^{n+1}}{2}b + 3^n 2c\right],$$

$$= 3^{n+1}a + 3^{n+1}b + 3^{n+1}2c.$$

With the equation above the proof is completed.

□

For the real parts of sequences, the values can be obtained from equations

$$Re(P_{n+1}) = Im(P_n)$$

and

$$Re(Q_{n+1}) = Im(Q_n).$$

Our definition of complex pulsating Fibonacci sequence is compatible with the definitions in [8, 10].

In the following corollaries we present summation formulas for imaginary parts of complex pulsating Fibonacci sequence without proof.

Corollary 2.2. *Summation formula for imaginary parts of odd elements of P*

$$\sum_{j=1}^{2n+1} P_{2j+1} = \frac{3^{n+1}-1}{4}(a+b+2c) \quad \textit{if } n \textit{ is odd}$$

and

$$\sum_{j=1}^{2n+1} P_{2j+1} = \frac{3^{n+1}-1}{4}(a+b+2c) + (b+c) \quad \textit{if } n \textit{ is even.}$$

Corollary 2.3. *Summation formula for imaginary parts of odd elements of Q*

$$\sum_{j=1}^{2n+1} Q_{2j+1} = \frac{3^{n+1}-1}{4}(a+b+2c) \quad \textit{if } n \textit{ is odd}$$

and

$$\sum_{j=1}^{2n+1} Q_{2j+1} = \frac{3^{n+1}-1}{4}(a+b+2c)+(a+c) \quad \textit{if } n \textit{ is even.}$$

Corollary 2.4. *Summation formula for imaginary parts of even elements of P (or Q, because imaginary parts of even elements are equal)*

$$\sum_{j=0}^{2n} P_{2j} = \frac{3^{n}-1}{2}(a+b+2c)+c.$$

Proof of Corollary 2.2, Corollary 2.3 and Corollary 2.4 can be made easily using the Theorem 2.1 which can be used to calculate any element of complex pulsating Fibonacci sequence.

3. Complex Multiplicative Pulsating Fibonacci Sequence

In this section, we will define the multiplicative version of complex pulsating Fibonacci sequence which is not studied earlier in literature. One of the earliest papers about multiplicative pulsating Fibonacci sequence belongs to A. Suvarnamani and S. Koyram [2]. Our complex multiplicative pulsating Fibonacci sequence definition is perfectly compatible with their work.

For real values a,b and c, a complex multiplicative pulsating Fibonacci sequences R and S are defined with following equations;

$$R_0 = a+ci,$$
$$S_0 = b+ci,$$
$$Re(R_{n+1}) = Im(R_n),$$
$$Re(S_{n+1}) = Im(S_n),$$
$$Im(R_{2n+2}) = Im(S_{2n+2}) = Im(R_{2n+1})Im(S_{2n+1}),$$
$$Im(R_{2n+1}) = Im(R_{2n})Re(S_{2n})$$

and

$$Im(S_{2n+1}) = Im(S_{2n})Re(R_{2n})$$

for $n \in \mathbb{N}$.

In the following theorem we give a formula which can be used to find desired element of this new sequence.

Theorem 3.1. *For every $n \in \mathbb{N}$,*

$$Im(R_{2n+1}) = a^{\frac{3^n+(-1)^{n+1}}{2}} b^{\frac{3^n+(-1)^n}{2}} c^{3^n},$$

$$Im(S_{2n+1}) = a^{\frac{3^n+(-1)^n}{2}} b^{\frac{3^n+(-1)^{n+1}}{2}} c^{3^n}$$

and

$$Im(R_{2n+2}) = Im(S_{2n+2}) = a^{3^n} b^{3^n} c^{2.3^n}.$$

Proof. The proof of the Theorem 3.1 can be seen analogously to the Theorem 2.1. With the help of this theorem, every element of complex multiplicative pulsating Fibonacci sequences can be calculated easily. □

For complex multiplicative pulsating Fibonacci sequences we can give multiplication formulas which are multiplicative counterpart of Corollaries $2.2, 2.3$ and 2.4.

Corollary 3.2. *Multiplication formula for imaginary parts of odd elements of R is as follows;*

$$\prod_{j=1}^{2n+1} R_{2j+1} = (abc^2)^{\frac{3^{n+1}-1}{4}} \quad \textit{if } n \textit{ is odd}$$

and

$$\prod_{j=1}^{2n+1} R_{2j+1} = (abc^2)^{\frac{3^{n+1}-1}{4}} bc \quad \textit{if } n \textit{ is even.}$$

Corollary 3.3. *Multiplication formula for imaginary parts of odd elements of S*

$$\prod_{j=1}^{2n+1} S_{2j+1} = (abc^2)^{\frac{3^{n+1}-1}{4}} \quad \text{if } n \text{ is odd}$$

and

$$\prod_{j=1}^{2n+1} S_{2j+1} = (abc^2)^{\frac{3^{n+1}-1}{4}} ac \quad \text{if } n \text{ is even.}$$

Corollary 3.4. *Multiplication formula for imaginary parts of even elements of R or S*

$$\prod_{j=0}^{2n} R_{2j} = (abc^2)^{\frac{3^n-1}{2}} c$$

4. Conclusion

In this study, we defined complex pulsating Fibonacci sequence of the second kind which is a generalization of first kind. The first kind equals to original definition of pulsating Fibonacci sequence which is made by Atanassov in [8]. For this newly defined complex sequence we give some summation formulas. In addition, we defined multiplicative complex pulsating Fibonacci sequence of second kind. For further studies, additional identities for complex pulsating Fibonnaci sequence can be calculated.

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