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The Unifying Formula for all Tribonacci-type Octonions Sequences and Their Properties

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Abstract

Various families of octonion number sequences (such as Fibonacci octonion, Pell octonion and Jacobsthal octonion) have been established by a number of authors in many different ways. In addition, formulas and identities involving these number sequences have been presented. In this paper, we aim to establishing new classes of octonion numbers associated with the generalized Tribonacci numbers. In this sense, we introduce the Tribonacci and generalized Tribonacci octonions (such as Narayana octonion, Padovan octonion and third-order Jacobsthal octonion) and give some of their properties. We derive the relations between generalized Tribonacci numbers and Tribonacci octonions.

Keywords: *Generalized Tribonacci number, Tribonacci octonion, Generalized Tribonacci octonion, Octonion algebra.*
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1. Introduction

Recently, the topic of number sequences in real normed division algebras has attracted the attention of several researchers. It is worth noticing that there are exactly four real normed division algebras: real numbers ($\mathbb{R}$), complex numbers ($\mathbb{C}$), quaternions ($\mathbb{H}$) and octonions ($\mathbb{O}$). In [4] Baez gives a comprehensive discussion of these algebras.
The real quaternion algebra

$$\mathbb{H}=\{q=q_r+q_i\mathbf{i}+q_j\mathbf{j}+q_k\mathbf{k}:\ q_r,q_l\in\mathbb{R},\ l=i,j,k\}$$

is a 4-dimensional $\mathbb{R}$-vector space with basis $\{\mathbf{1}\simeq e_0,\mathbf{i}\simeq e_1,\mathbf{j}\simeq e_2,\mathbf{k}\simeq e_3\}$ satisfying multiplication rules $q_r\mathbf{1}=q_r$, $e_1e_2=-e_2e_1=e_3$, $e_2e_3=-e_3e_2=e_1$ and $e_3e_1=-e_1e_3=e_2$. Furthermore, the real octonion algebra denoted by $\mathbb{O}$ is an 8-dimensional real linear space with basis

$$\{e_0=\mathbf{1},e_1=\mathbf{i},e_2=\mathbf{j},e_3=\mathbf{k},e_4=\mathbf{e},e_5=\mathbf{ie},e_6=\mathbf{je},e_7=\mathbf{ke}\}, \tag{1.1}$$

where $e_0\cdot e_l=e_l$ $(l=1,...,7)$ and $q_re_0=q_r$ $(q_r\in\mathbb{R})$. The space $\mathbb{O}$ becomes an algebra via multiplication rules listed in the table 1, see [29].

Table 1: The multiplication table for the basis of $\mathbb{O}$.

$\times$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

A variety of new results on Fibonacci-like quaternion and octonion numbers can be found in several papers [7, 10, 11, 16, 17, 18, 19, 20, 21, 28]. The origin of the topic of number sequences in division algebra can be traced back to the works by Horadam in [18] and Iyer in [20]. In this sense, A. F. Horadam [18] defined the quaternions with the classic Fibonacci and Lucas number components as

$QF_n=F_n+F_{n+1}\mathbf{i}+F_{n+2}\mathbf{j}+F_{n+3}\mathbf{k}$

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and

$QL_n = L_n + L_{n+1}\mathbf{i} + L_{n+2}\mathbf{j} + L_{n+3}\mathbf{k},$

respectively, where F_n and L_n are the n-th classic Fibonacci and Lucas numbers, respectively, and the author studied the properties of these quaternions. Several interesting and useful extensions of many of the familiar quaternion numbers (such as the Fibonacci and Lucas quaternions [3, 16, 18], Pell quaternion [6, 10], Jacobsthal quaternions [28] and third order Jacobsthal quaternion [7]) have been considered by several authors. For example, in [8, 9] a new type of quaternion whose coefficients are generalized Tribonacci and fourth-order Jacobthal numbers are defined.

There has been an increasing interest on quaternions and octonions that play an important role in various areas such as computer sciences, physics, differential geometry, quantum physics, signal, color image processing and geostatics (for more, see [1, 5, 14, 15, 22, 23]).

In this paper, we define a family of the octonions, where the coefficients in the terms of the octonions are determined by the generalized Tribonacci numbers. These family of the octonions are called as the generalized Tribonacci octonions. Furthermore, we mention some of their properties, and apply them to the study of some identities and formulas of the generalized Tribonacci octonions.

Here, our approach for obtaining some fundamental properties and characteristics of generalized Tribonacci octonions is to apply the properties of the generalized numbers introduced by Shannon and Horadam [27], Yalavigi [30] and Pethe [26]. This approach was originally proposed by Horadam and Iyer in the articles [18, 20] for Fibonacci quaternions. The methods used by Horadam and Iyer in that papers have been recently applied to the other familiar octonion numbers by several authors [2, 6, 8, 11, 21, 24, 25].

This paper has three main sections. In Section 2, we provide the basic definitions of the octonions and the generalized Tribonacci numbers. Section 3 is devoted to introducing generalized Tribonacci octonions, and then obtain some fundamental properties and characteristics of these numbers.

2. Preliminaries

We consider the generalized Tribonacci sequence $\{V_n(V_0,V_1,V_2;r,s,t)\}$, or briefly $\{V_n\}$, defined as

$$V_n = rV_{n-1} + sV_{n-2} + tV_{n-3},\ (n \geq 3), \tag{2.1}$$

where V_0, V_1, V_2 are arbitrary integers and r, s, t, are real numbers. This sequence has been studied by Shannon and Horadam [27], Yalavigi [30] and Pethe [26]. If we set $r = s = t = 1$ and $V_0 = 0$, $V_1 = V_2 = 1$, then $\{V_n\}$ is the well-known Tribonacci sequence which has been considered extensively (see, for example, [12]).

As the elements of this Tribonacci-type number sequence provide third order iterative relation, its characteristic equation is $x^3 - rx^2 - sx - t = 0$, whose roots are $\alpha = \alpha(r,s,t) = \frac{r}{3} + A_V + B_V$, $\omega_1 = \frac{r}{3} + \varepsilon A_V + \varepsilon^2 B_V$ and $\omega_2 = \frac{r}{3} + \varepsilon^2 A_V + \varepsilon B_V$, where

$$A_V = \sqrt[3]{\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}},\ B_V = \sqrt[3]{\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta}},$$

with $\Delta = \Delta(r,s,t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}$ and $\varepsilon = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$.

In this paper, $\Delta(r,s,t) > 0$, then the cubic equation $x^3 - rx^2 - sx - t = 0$ has one real and two nonreal solutions, the latter being conjugate complex. Thus, the Binet formula for the generalized Tribonacci numbers can be expressed as:

$$V_n = \frac{P\alpha^n}{(\alpha-\omega_1)(\alpha-\omega_2)} - \frac{Q\omega_1^n}{(\alpha-\omega_1)(\omega_1-\omega_2)} + \frac{R\omega_2^n}{(\alpha-\omega_2)(\omega_1-\omega_2)}, \tag{2.2}$$

where $P = V_2 - (\omega_1+\omega_2)V_1 + \omega_1\omega_2 V_0$, $Q = V_2 - (\alpha+\omega_2)V_1 + \alpha\omega_2 V_0$ and $R = V_2 - (\alpha+\omega_1)V_1 + \alpha\omega_1 V_0$.

In fact, the generalized Tribonacci sequence is the generalization of the well-known sequences like Tribonacci, Padovan, Narayana and third-order Jacobsthal. For example, $\{V_n(0,1,1;1,1,1)\}_{n\geq 0}$, $\{V_n(0,1,0;0,1,1)\}_{n\geq 0}$, are Tribonacci and Padovan sequences, respectively.

The Binet formula for the generalized Tribonacci sequence is expressed as follows:

Lemma 2.1. *The Binet formula for the generalized Tribonacci sequence $\{V_n\}_{n\geq 0}$ is:*

$$V_{n+1} = V_2 U_n + (sV_1 + tV_0)U_{n-1} + tV_1 U_{n-2},$$

and

$$U_n = \frac{\alpha^{n+1}}{(\alpha-\omega_1)(\alpha-\omega_2)} - \frac{\omega_1^{n+1}}{(\alpha-\omega_1)(\omega_1-\omega_2)} + \frac{\omega_2^{n+1}}{(\alpha-\omega_2)(\omega_1-\omega_2)},$$

where α, ω_1 and ω_2 are the roots of the cubic equation $x^3 - rx^2 - sx - t = 0$.

Proof. The validity of this formula can be confirmed using the recurrence relation. Furthermore, $\{U_n\}_{n\geq 0} = \{V_n(0,1,r;r,s,t)\}_{n\geq 0}$. □

In the following we will study the important properties of the octonions. We refer to [4] for a detailed analysis of the properties of the next octonions $p = \sum_{l=0}^{7} a_l e_l$ and $q = \sum_{l=0}^{7} b_l e_l$ where the coefficients $a_l, b_l \in \mathbb{R}$. We recall here only the following facts

- The sum and subtract of p and q is defined as

$$p \pm q = \sum_{l=0}^{7} (a_l \pm b_l) e_l, \tag{2.3}$$

  where $p \in \mathbb{O}$ can be written as $p = R_p + I_p$, and $R_p = a_0$ and $\sum_{l=1}^{7} a_l e_l$ are called the real and imaginary parts, respectively.

- The conjugate of p is defined by

$$\overline{p}=R_p-I_p=a_0-\sum_{l=1}^{7}a_le_l \tag{2.4}$$

and this operation satisfies $\overline{\overline{p}}=p$, $\overline{p+q}=\overline{p}+\overline{q}$ and $\overline{p\cdot q}=\overline{q}\cdot\overline{p}$, for all $p,q\in\mathbb{O}$.
- The norm of an octonion, which agrees with the standard Euclidean norm on $\mathbb{R}^8$ is defined as

$$Nr^2(p)=\overline{p}\cdot p=p\cdot\overline{p}=\sum_{l=0}^{7}a_l^2,\ \left(Nr(p)=\sqrt{\sum_{l=0}^{7}a_l^2}\in\mathbb{R}_0^+\right). \tag{2.5}$$

- The inverse of $p\neq 0$ is given by $p^{-1}=\frac{\overline{p}}{Nr^2(p)}$. From the above two definitions it is deduced that

$$Nr^2(p\cdot q)=Nr^2(p)Nr^2(q) \text{ and } (p\cdot q)^{-1}=q^{-1}\cdot p^{-1}.$$

- $\mathbb{O}$ is non-commutative and non-associative but it is alternative, in other words

$$\begin{aligned}&p\cdot(p\cdot q)=p^2\cdot q,\\&(p\cdot q)\cdot q=p\cdot q^2,\\&(p\cdot q)\cdot p=p\cdot(q\cdot p)=p\cdot q\cdot p,\end{aligned}$$

where $\cdot$ denotes the product in the octonion algebra $\mathbb{O}$.

3. The Generalized Tribonacci Octonions

In this section, we define new kinds of sequences of octonion number called as generalized Tribonacci octonions. We study some properties of these octonions. We obtain various results for these classes of octonion numbers included recurrence relations, summation formulas, Binet's formulas and generating functions.
In [8], the author introduced the so-called generalized Tribonacci quaternions, which are a new class of quaternion sequences. They are defined by

$$Q_{v,n}=\sum_{l=0}^{3}V_{n+l}e_l=V_n+\sum_{l=1}^{3}V_{n+l}e_l,\ (V_n\mathbf{1}=V_n), \tag{3.1}$$

where V_n is the n-th generalized Tribonacci number, $e_1^2=e_2^2=e_3^2=-\mathbf{1}$ and $e_1e_2e_3=-\mathbf{1}$.
We now consider the usual generalized Tribonacci numbers, and similarly based on the definition (3.1) we give definition of a new kind of octonion numbers, which we called the generalized Tribonacci octonions. In this paper, we define the n-th generalized Tribonacci octonion number by the following recurrence relation

$$\begin{aligned}O_{v,n}&=V_n+\sum_{l=1}^{7}V_{n+l}e_l,\ n\geq 0\\&=V_n+V_{n+1}e_1+V_{n+2}e_2+V_{n+3}e_3\\&\quad+V_{n+4}e_4+V_{n+5}e_5+V_{n+6}e_6+V_{n+7}e_7,\end{aligned} \tag{3.2}$$

where V_n is the n-th generalized Tribonacci number. Here $\{e_l:\ l=1,...,7\}$ satisfies the multiplication rule given in the Table 1. Furthermore, the sequence $\{U_n\}$ is the special case of $\{V_n\}$ where $V_0=0$, $V_1=1$ and $V_2=r$. Then, we can write $O_{u,n}=U_n+\sum_{l=1}^{7}U_{n+l}e_l,\ (n\geq 0)$.
The equalities in (2.3) gives

$$O_{v,n}\pm O_{v,m}=\sum_{l=0}^{7}(V_{n+l}\pm V_{m+l})e_l\ (n,m\geq 0). \tag{3.3}$$

From (2.4), (2.5) and (3.2) an easy computation gives

$$\overline{O_{v,n}}=V_n-\sum_{l=1}^{7}V_{n+l}e_l, \text{ and } Nr(O_{v,n})=\sqrt{\sum_{l=0}^{7}V_{n+l}^2}\in\mathbb{R}_0^+.$$

By some elementary calculations we find the following recurrence relation for the generalized Tribonacci octonions from (3.2), (3.3) and (2.1):

$$\begin{aligned}rO_{v,n+1}+sO_{nv,}+tO_{v,n-1}&=\sum_{l=0}^{7}(rV_{n+l+1}+sV_{n+l}+tV_{n+l-1})e_l\\&=V_{n+2}+\sum_{l=1}^{7}V_{n+l+2}e_l\\&=O_{v,n+2}\ \ (n\geq 1).\end{aligned}$$

Now, we will state Binet's formula for the generalized Tribonacci octonions. Repeated use of (2.2) in (3.2) enables one to write for $\underline{\alpha} = \sum_{l=0}^{7} \alpha^l e_l$, $\underline{\omega_1} = \sum_{l=0}^{7} \omega_1^l e_l$ and $\underline{\omega_2} = \sum_{l=0}^{7} \omega_2^l e_l$:

$$\begin{aligned} O_{v,n} &= \sum_{l=0}^{7} V_{n+l} e_l \\ &= \sum_{l=0}^{7} \left(\frac{P\alpha^{n+l} e_l}{(\alpha-\omega_1)(\alpha-\omega_2)} - \frac{Q\omega_1^{n+l} e_l}{(\alpha-\omega_1)(\omega_1-\omega_2)} + \frac{R\omega_2^{n+l} e_l}{(\alpha-\omega_2)(\omega_1-\omega_2)} \right) \\ &= \frac{P\underline{\alpha}\alpha^n}{(\alpha-\omega_1)(\alpha-\omega_2)} - \frac{Q\underline{\omega_1}\omega_1^n}{(\alpha-\omega_1)(\omega_1-\omega_2)} + \frac{R\underline{\omega_2}\omega_2^n}{(\alpha-\omega_2)(\omega_1-\omega_2)}, \end{aligned} \tag{3.4}$$

where α, ω_1 and ω_2 are the roots of the cubic equation $x^3 - rx^2 - sx - t = 0$, and P, Q and R as before. The formula in (3.4) is called as Binet's formula for the generalized Tribonacci octonions.
In the following theorem we present the generating function for generalized Tribonacci octonions.

Theorem 3.1. *The generating function for the generalized Tribonacci octonion $O_{v,n}$ is*

$$g(x) = \frac{O_{v,0} + (O_{v,1} - rO_{v,0})x + (O_{v,2} - rO_{v,1} - sO_{v,0})x^2}{1 - rx - sx^2 - tx^3}. \tag{3.5}$$

Proof. Assuming that the generating function of the generalized Tribonacci octonion sequence $\{O_{v,n}\}_{n\geq 0}$ has the form $g(x) = \sum_{n\geq 0} O_{v,n} x^n$, we obtain that

$$\begin{aligned} &(1 - rx - sx^2 - tx^3) \sum_{n\geq 0} O_{v,n} x^n \\ &\quad = O_{v,0} + O_{v,1}x + O_{v,2}x^2 + O_{v,3}x^3 + \cdots \\ &\quad\quad - rO_{v,0}x - rO_{v,1}x^2 - rO_{v,2}x^3 - rO_{v,3}x^4 - \cdots \\ &\quad\quad - sO_{v,0}x^2 - sO_{v,1}x^3 - sO_{v,2}x^4 - sO_{v,3}x^5 - \cdots \\ &\quad\quad - tO_{v,0}x^3 - tO_{v,1}x^4 - tO_{v,2}x^5 - tO_{v,3}x^6 - \cdots \\ &\quad = O_{v,0} + (O_{v,1} - rO_{v,0})x + (O_{v,2} - rO_{v,1} - sO_{v,0})x^2, \end{aligned}$$

since $O_{v,n} = rO_{v,n-1} + sO_{v,n-2} + tO_{v,n-3}$, $n \geq 3$ and the coefficients of x^n for $n \geq 3$ are equal with zero. Then, we get

$$g(x) = \frac{O_{v,0} + (O_{v,1} - rO_{v,0})x + (O_{v,2} - rO_{v,1} - sO_{v,0})x^2}{1 - rx - sx^2 - tx^3}.$$

The theorem is proved. □

In Table 2, we examine some special cases of generating functions given in Eq. (3.5).

Table 2: Generating functions according to initial values.

Narayana	$\dfrac{\left\{\begin{array}{c} x + e_1 + (1+x^2)e_2 + (1+x+x^2)e_3 \\ +(2+x+x^2)e_4 + (3+x+2x^2)e_5 \\ +(4+2x+3x^2)e_6 + (6+3x+4x^2)e_7 \end{array}\right\}}{1-x-x^3}$
Tribonacci	$\dfrac{\left\{\begin{array}{c} x + e_1 + (1+x+x^2)e_2 + (2+2x+x^2)e_3 \\ +(4+3x+2x^2)e_4 + (7+6x+4x^2)e_5 \\ +(13+11x+7x^2)e_6 + (24+20x+13x^2)e_7 \end{array}\right\}}{1-x-x^2-x^3}$
Padovan	$\dfrac{\left\{\begin{array}{c} x + e_1 + (x+x^2)e_2 + (1+x)e_3 \\ +(1+x+x^2)e_4 + (1+2x+x^2)e_5 \\ +(2+2x+x^2)e_6 + (2+3x+2x^2)e_7 \end{array}\right\}}{1-x^2-x^3}$
Third-Order Jacobsthal	$\dfrac{\left\{\begin{array}{c} x + e_1 + (1+x+x^2)e_2 + (2+3x+2x^2)e_3 \\ +(5+4x+4x^2)e_4 + (9+9x+10x^2)e_5 \\ +(18+19x+18x^2)e_6 + (37+36x+36x^2)e_7 \end{array}\right\}}{1-x-x^2-2x^3}$

Now, let us write the formula which gives the summation of the first n generalized Tribonacci numbers and then octonions.

Lemma 3.2 ([8])**.** *For every integer $n \geq 0$, we have:*

$$\sum_{l=0}^{n} V_l = \frac{1}{\delta_{r,s,t}} \left\{ \begin{array}{c} V_{n+2} + (1-r)V_{n+1} + tV_n \\ +(r-s-1)V_0 + (r-1)V_1 - V_2 \end{array} \right\}, \tag{3.6}$$

where $\delta = \delta_{r,s,t} = r + s + t - 1$ and V_n denote the n-th term of the generalized Tribonacci numbers.

Theorem 3.3. *The summation formula for generalized Tribonacci octonions is as follows:*

$$\sum_{l=0}^{n} O_{v,l} = \frac{1}{\delta_{r,s,t}}(O_{v,n+2} + (1-r)O_{v,n+1} + tO_{v,n} + \omega_{r,s,t}), \tag{3.7}$$

where $\omega_{r,s,t} = \lambda_{r,s,t} + e_1(\lambda_{r,s,t} - \delta V_0) + \cdots + e_7(\lambda_{r,s,t} - \delta(V_0 + \cdots + V_6))$, $\lambda_{r,s,t} = (r+s-1)V_0 + (r-1)V_1 - V_2$ *and* $\delta = \delta_{r,s,t} = r+s+t-1$.

Proof. Using Eq. (3.2), we have

$$\begin{aligned}\sum_{l=0}^{n} O_{v,l} &= \sum_{l=0}^{n} V_l + e_1 \sum_{l=0}^{n} V_{l+1} + e_2 \sum_{l=0}^{n} V_{l+2} + \cdots + e_7 \sum_{l=0}^{n} V_{l+7} \\ &= (V_0 + \cdots + V_n) + e_1(V_1 + \cdots + V_{n+1}) \\ &\quad + e_2(V_2 + \cdots + V_{n+2}) + \cdots + e_7(V_7 + \cdots + V_{n+7}).\end{aligned}$$

Since from Eq. (3.6) and using the notation $\lambda_{r,s,t} = (r+s-1)V_0 + (r-1)V_1 - V_2$, we can write

$$\begin{aligned}\delta_{r,s,t} \sum_{l=0}^{n} O_{v,l} &= V_{n+2} + (1-r)V_{n+1} + tV_n + \lambda_{r,s,t} \\ &\quad + e_1(V_{n+3} + (1-r)V_{n+2} + tV_{n+1} + \lambda_{r,s,t} - \delta V_0) \\ &\quad \vdots \\ &\quad + e_7(V_{n+9} + (1-r)V_{n+8} + tV_{n+7} + \lambda_{r,s,t} - \delta(V_0 + \cdots + V_6)) \\ &= O_{v,n+2} + (1-r)O_{v,n+1} + tO_{v,n} + \omega_{r,s,t},\end{aligned}$$

where $\omega_{r,s,t} = \lambda_{r,s,t} + e_1(\lambda_{r,s,t} - \delta V_0) + \cdots + e_7(\lambda_{r,s,t} - \delta(V_0 + \cdots + V_6))$. Finally,

$$\sum_{l=0}^{n} O_{v,l} = \frac{1}{\delta_{r,s,t}}(O_{v,n+2} + (1-r)O_{v,n+1} + tO_{v,n} + \omega_{r,s,t}).$$

The theorem is proved. □

The summation formula in Eq. (3.7) gives the sum of the elements in the octonion sequences which have not been found in the studies conducted so far. This can be seen in Table 3.

Table 3: Summation formulas according to initial values.

Narayana	$O_{v,n+3} - \left\{ \begin{array}{c} 1 + e_1 + 2e_2 + 3e_3 \\ +4e_4 + 6e_5 + 9e_6 + 13e_7 \end{array} \right\}$
Tribonacci	$\frac{1}{2}\left(O_{v,n+2} + O_{v,n} - \left\{ \begin{array}{c} 1 + e_1 + 3e_2 \\ +5e_3 + 9e_4 + 17e_5 \\ +31e_6 + 57e_7 \end{array} \right\} \right)$
Padovan	$O_{v,n+5} - \left\{ \begin{array}{c} 1 + e_1 + 2e_2 + 2e_3 \\ +3e_4 + 4e_5 + 5e_6 + 7e_7 \end{array} \right\}$
Third-order Jacobsthal	$\frac{1}{3}\left(O_{v,n+2} + 2O_{v,n} - \left\{ \begin{array}{c} 1 + e_1 + 4e_2 \\ +7e_3 + 13e_4 + 28e_5 \\ +55e_6 + 109e_7 \end{array} \right\} \right)$

Now, we present the formula which gives the norms for generalized Tribonacci octonions.

Theorem 3.4. *The norm value for generalized Tribonacci octonions is given with the following formula:*

$$Nr^2(O_{v,n}) = \frac{1}{\phi^2}\left\{ \begin{array}{c} (\omega_1 - \omega_2)^2 P^2 \overline{\alpha} \alpha^{2n} + (\alpha - \omega_2)^2 Q^2 \overline{\omega_1} \omega_1^{2n} \\ +(\alpha - \omega_1)^2 R^2 \overline{\omega_2} \omega_2^{2n} - 2K \end{array} \right\}, \tag{3.8}$$

where $K = (\omega_1 - \omega_2)(\alpha - \omega_2)PQ\underline{\alpha\omega_1}(\alpha\omega_1)^n + (\omega_1 - \omega_2)(\alpha - \omega_1)PR\underline{\alpha\omega_2}(\alpha\omega_2)^n + (\alpha - \omega_1)(\alpha - \omega_2)QR\underline{\omega_1\omega_2}(\omega_1\omega_2)^n$.

Proof. If we use the definition norm, then we obtain $Nr^2(O_{v,n}) = \sum_{l=0}^{7} V_{n+l}^2$. Moreover, by the Binet formula (3.4) we have

$$\phi V_n = (\omega_1 - \omega_2)P\alpha^n - (\alpha - \omega_2)Q\omega_1^n + (\alpha - \omega_1)R\omega_2^n,$$

where $\phi = \phi(\alpha, \omega_1, \omega_2) = (\alpha - \omega_1)(\alpha - \omega_2)(\omega_1 - \omega_2)$. Then,

$$\begin{aligned}\phi^2 V_n^2 &= (\omega_1 - \omega_2)^2 P^2 \alpha^{2n} + (\alpha - \omega_2)^2 Q^2 \omega_1^{2n} + (\alpha - \omega_1)^2 R^2 \omega_2^{2n} \\ &\quad - 2(\omega_1 - \omega_2)(\alpha - \omega_2)PQ(\alpha\omega_1)^n + 2(\omega_1 - \omega_2)(\alpha - \omega_1)PR(\alpha\omega_2)^n \\ &\quad - 2(\alpha - \omega_1)(\alpha - \omega_2)QR(\omega_1\omega_2)^n\end{aligned}$$

and

$$\begin{aligned}\phi^2 Nr^2(O_{v,n}) &= \phi^2(V_n^2+V_{n+1}^2+\cdots+V_{n+7}^2)\\ &= (\omega_1-\omega_2)^2P^2\overline{\alpha}\alpha^{2n}+(\alpha-\omega_2)^2Q^2\overline{\omega_1}\omega_1^{2n}\\ &\quad+(\alpha-\omega_1)^2R^2\overline{\omega_2}\omega_2^{2n}-2(\omega_1-\omega_2)(\alpha-\omega_2)PQ\underline{\alpha\omega_1}(\alpha\omega_1)^n\\ &\quad+2(\omega_1-\omega_2)(\alpha-\omega_1)PR\underline{\alpha\omega_2}(\alpha\omega_2)^n\\ &\quad-2(\alpha-\omega_1)(\alpha-\omega_2)QR\underline{\omega_1\omega_2}(\omega_1\omega_2)^n,\end{aligned}$$

where $\overline{\alpha}=1+\alpha^2+\alpha^4+\cdots+\alpha^{14}$, $\overline{\omega_{1,2}}=1+\omega_{1,2}^2+\omega_{1,2}^4+\cdots+\omega_{1,2}^{14}$, $\underline{\alpha\omega_{1,2}}=1+\alpha\omega_{1,2}+(\alpha\omega_{1,2})^2+\cdots+(\alpha\omega_{1,2})^7$ and $\underline{\omega_1\omega_2}=1+\omega_1\omega_2+(\omega_1\omega_2)^2+\cdots+(\omega_1\omega_2)^7$. □

Theorem 3.5. *For $n\geq 0$, $m\geq 3$ we have*

$$O_{v,n+m}=U_{m-1}O_{v,n+2}+(sU_{m-2}+tU_{m-3})O_{v,n+1}+tU_{m-2}O_{v,n}, \tag{3.9}$$

where $U_n=V_n(0,1,r;r,s,t)$.

Proof. For $m=3$, we have

$$\begin{aligned}O_{v,n+3}&=rO_{v,n+2}+sO_{v,n+1}+tO_{v,n}\\&=U_2O_{v,n+2}+(sU_1+tU_0)O_{v,n+1}+tU_1O_{v,n}.\end{aligned}$$

Suppose the equality holds for $m\leq l$. For $m=l+1$, we have

$$\begin{aligned}O_{v,n+l+1}&=rO_{v,n+l}+sO_{v,n+l-1}+tO_{v,n+l-2}\\&=r\left(U_{l-1}O_{v,n+2}+(sU_{l-2}+tU_{l-3})O_{v,n+1}+tU_{l-2}O_{v,n}\right)\\&\quad+s\left(U_{l-2}O_{v,n+2}+(sU_{l-3}+tU_{l-4})O_{v,n+1}+tU_{l-3}O_{v,n}\right)\\&\quad+t\left(U_{l-3}O_{v,n+2}+(sU_{l-4}+tU_{l-5})O_{v,n+1}+tU_{l-4}O_{v,n}\right)\\&=(rU_{l-1}+sU_{l-2}+tU_{l-3})O_{v,n+2}\\&\quad+\left(s(rU_{l-2}+sU_{l-3}+tU_{l-4})+t(rU_{l-3}+sU_{l-4}+tU_{l-5})\right)O_{v,n+1}\\&\quad+t(rU_{l-2}+sU_{l-3}+tU_{l-4})O_{v,n}\\&=U_lO_{v,n+2}+(sU_{l-1}+tU_{l-2})O_{v,n+1}+tU_{l-1}O_{v,n}.\end{aligned}$$

By induction on m, we get the result. □

Table 4: Convolution formulas $O_{v,n+m}$ according to initial values.

Narayana	$\left\{\begin{array}{c}N_{m-1}O_{N,n+2}+N_{m-3}O_{N,n+1}\\+N_{m-2}O_{N,n}\end{array}\right\}$
Tribonacci	$\left\{\begin{array}{c}T_{m-1}O_{T,n+2}+(T_{m-2}+T_{m-3})O_{T,n+1}\\+T_{m-2}O_{T,n}\end{array}\right\}$
Padovan	$\left\{\begin{array}{c}P_{m-1}O_{P,n+2}+P_mO_{P,n+1}\\+P_{m-2}O_{v,n},\end{array}\right\}$
Third-order Jacobsthal	$\left\{\begin{array}{c}J_{m-1}O_{J,n+2}+(J_{m-2}+2J_{m-3})O_{J,n+1}\\+2J_{m-2}O_{J,n},\end{array}\right\}$

Now, we give the quadratic approximation of $\{O_{v,n}\}$.

Theorem 3.6. *Let $\{O_{v,n}\}_{n\geq 0}$, α, ω_1 and ω_2 be as above. Then, we have for all integer $n\geq 0$*

$$\begin{array}{c}\textit{Quadratic}\\ \textit{app. of } \{O_{v,n}\}\end{array}:\left\{\begin{array}{l}P\underline{\alpha}\alpha^{n+2}=\alpha^2O_{v,n+2}+\alpha(sO_{v,n+1}+tO_{v,n})+tO_{v,n+1},\\ Q\underline{\omega_1}\omega_1^{n+2}=\omega_1^2O_{v,n+2}+\omega_1(sO_{v,n+1}+tO_{v,n})+tO_{v,n+1},\\ R\underline{\omega_2}\omega_2^{n+2}=\omega_2^2O_{v,n+2}+\omega_2(sO_{v,n+1}+tO_{v,n})+tO_{v,n+1},\end{array}\right. \tag{3.10}$$

where $P=V_2-(\omega_1+\omega_2)V_1+\omega_1\omega_2V_0$, $Q=V_2-(\alpha+\omega_2)V_1+\alpha\omega_2V_0$ and $R=V_2-(\alpha+\omega_1)V_1+\alpha\omega_1V_0$. Furthermore, $\underline{\alpha}=\sum_{l=0}^{7}\alpha^le_l$ and $\underline{\omega_{1,2}}=\sum_{l=0}^{7}\omega_{1,2}^le_l$.

Proof. Using the Binet's formula Eq. (3.4), we have

$$\begin{aligned}&\alpha O_{v,n+2}+(s+\omega_1\omega_2)O_{v,n+1}+tO_{v,n}\\&\quad=\frac{P\underline{\alpha}\alpha^n(\alpha^3+(s+\omega_1\omega_2)\alpha+t)}{(\alpha-\omega_1)(\alpha-\omega_2)}-\frac{Q\underline{\omega_1}\omega_1^n(\alpha\omega_1^2+(s+\omega_1\omega_2)\omega_1+t)}{(\alpha-\omega_1)(\omega_1-\omega_2)}\\&\quad\quad+\frac{R\underline{\omega_2}\omega_2^n(\alpha\omega_2^2+(s+\omega_1\omega_2)\omega_2+t)}{(\alpha-\omega_2)(\omega_1-\omega_2)}\\&\quad=(V_2-(\omega_1+\omega_2)V_1+\omega_1\omega_2V_0)\underline{\alpha}\alpha^{n+1},\end{aligned}$$

the latter given that $\alpha\omega_1^2+(s+\omega_1\omega_2)\omega_1+t=0$ and $\alpha\omega_2^2+(s+\omega_1\omega_2)\omega_2+t=0$. Then, we get

$$\alpha O_{v,n+2}+(s+\omega_1\omega_2)O_{v,n+1}+tO_{v,n}=(V_2-(\omega_1+\omega_2)V_1+\omega_1\omega_2V_0)\underline{\alpha}\alpha^{n+1}. \tag{3.11}$$

Multiplying Eq. (3.11) by α and using $\alpha\omega_1\omega_2=t$, we have

$$\begin{aligned}P\underline{\alpha}\alpha^{n+2}&=\alpha^2O_{v,n+2}+\alpha(s+\omega_1\omega_2)O_{v,n+1}+\alpha tO_{v,n}\\&=\alpha^2O_{v,n+2}+\alpha(sO_{v,n+1}+tO_{v,n})+tO_{v,n+1},\end{aligned}$$

where $P=V_2-(\omega_1+\omega_2)V_1+\omega_1\omega_2V_0$. If we change α, ω_1 and ω_2 role above process, we obtain the desired result Eq. (3.10). □

The next theorem gives an alternative proof of the Binet's formula for the generalized Tribonacci octonions (see Eq. (3.4)).

Theorem 3.7. *For any integer $n\geq 0$, the n-th generalized Tribonacci octonion is*

$$O_{v,n}=\frac{P\underline{\alpha}\alpha^n}{(\alpha-\omega_1)(\alpha-\omega_2)}-\frac{Q\underline{\omega_1}\omega_1^n}{(\alpha-\omega_1)(\omega_1-\omega_2)}+\frac{R\underline{\omega_2}\omega_2^n}{(\alpha-\omega_2)(\omega_1-\omega_2)},$$

where P, Q and R as in Eq. (2.2), $\underline{\alpha}=\sum_{l=0}^{7}\alpha^l e_l$ and $\underline{\omega_{1,2}}=\sum_{l=0}^{7}\omega_{1,2}^l e_l$. If $V_0=V_1=0$, $V_2=1$ and $r=s=t=1$, we get the classic Tribonacci octonion.

Proof. For the Eq. (3.10), we have

$$\begin{aligned}&\alpha^2Q_{V,n+2}+\alpha\left(sQ_{V,n+1}+tQ_{V,n}\right)+tQ_{V,n+1}\\&=\alpha^2\left(V_{n+2}+V_{n+3}e_1+\cdots+V_{n+5}e_7\right)\\&\quad+\alpha\left(sV_{n+1}+tV_n+(sV_{n+2}+tV_{n+1})e_1+\cdots+(sV_{n+4}+tV_{n+3})e_7\right)\\&\quad+t\left(V_{n+1}+V_{n+2}e_1+\cdots+V_{n+4}e_7\right)\\&=\alpha^2V_{n+2}+\alpha\left(sV_{n+1}+tV_n\right)+tV_{n+1}+\left(\alpha^2V_{n+3}+\alpha\left(sV_{n+2}+tV_{n+1}\right)+tV_{n+2}\right)e_1\\&\quad+\left(\alpha^2V_{n+4}+\alpha\left(sV_{n+3}+tV_{n+2}\right)+tV_{n+3}\right)e_2\\&\quad\vdots\\&\quad+\left(\alpha^2V_{n+5}+\alpha\left(sV_{n+4}+tV_{n+3}\right)+tV_{n+4}\right)e_7.\end{aligned}$$

From the identity $P\alpha^{n+2}=\alpha^2V_{n+2}+\alpha(sV_{n+1}+tV_n)+tV_{n+1}$ for n-th generalized Tribonacci number V_n, we obtain

$$\alpha^2O_{v,n+2}+\alpha\left(sO_{v,n+1}+tO_{v,n}\right)+tO_{v,n+1}=P\underline{\alpha}\alpha^{n+2}. \tag{3.12}$$

Similarly, we have

$$\omega_1^2O_{v,n+2}+\omega_1\left(sO_{v,n+1}+tO_{v,n}\right)+tO_{v,n+1}=Q\underline{\omega_1}\omega_1^{n+2}, \tag{3.13}$$

$$\omega_2^2O_{v,n+2}+\omega_2\left(sO_{v,n+1}+tO_{v,n}\right)+tO_{v,n+1}=R\underline{\omega_2}\omega_2^{n+2}. \tag{3.14}$$

Subtracting Eq. (3.13) from Eq. (3.12) gives

$$(\alpha+\omega_1)O_{v,n+2}+\left(sO_{v,n+1}+tO_{v,n}\right)=\frac{P\underline{\alpha}\alpha^{n+2}-Q\underline{\omega_1}\omega_1^{n+2}}{\alpha-\omega_1}. \tag{3.15}$$

Similarly, subtracting Eq. (3.14) from Eq. (3.12) gives

$$(\alpha+\omega_2)O_{v,n+2}+\left(sO_{v,n+1}+tO_{v,n}\right)=\frac{P\underline{\alpha}\alpha^{n+2}-R\underline{\omega_2}\omega_2^{n+2}}{\alpha-\omega_2}. \tag{3.16}$$

Finally, subtracting Eq. (3.16) from Eq. (3.15), we obtain

$$\begin{aligned}O_{v,n+2}&=\frac{1}{\omega_1-\omega_2}\left(\frac{P\underline{\alpha}\alpha^{n+2}-Q\underline{\omega_1}\omega_1^{n+2}}{\alpha-\omega_1}-\frac{P\underline{\alpha}\alpha^{n+2}-R\underline{\omega_2}\omega_2^{n+2}}{\alpha-\omega_2}\right)\\&=\frac{P\underline{\alpha}\alpha^{n+2}}{(\alpha-\omega_1)(\alpha-\omega_2)}-\frac{Q\underline{\omega_1}\omega_1^{n+2}}{(\alpha-\omega_1)(\omega_1-\omega_2)}+\frac{R\underline{\omega_2}\omega_2^{n+2}}{(\alpha-\omega_2)(\omega_1-\omega_2)}.\end{aligned}$$

So, the theorem is proved. □

4. Conclusions

Octonions have great importance as they are used in quantum physics, applied mathematics, graph theory. In this work, we introduce the generalized Tribonacci octonion numbers and formulate the Binet-style formula, the generating function and some identities of the generalized Tribonacci octonion sequence. Thus, in our future studies we plan to examine different quaternion and octonion polynomials and their key features.

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