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New Upper and Lower Bounds for the Trapezoid Inequality of Absolutely Continuous Functions and Applications

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Abstract

In this paper, new upper and lower bounds for the Trapezoid inequality of absolutely continuous functions are obtained. Applications to some special means are provided as well.

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1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following inequality, known as *Hermite–Hadamard inequality* for convex functions, holds:

$$f\left(\frac{a+b}{2}\right) \le \int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

On the other hand, a very related inequality to (1.1) was known in literature as the '*Trapezoid inequality*', which states that: if $f : [a,b] \to \mathbb{R}$ is twice differentiable such that $\|f''\|_\infty := \sup_{t\in(a,b)} |f''(t)| < \infty$, then

$$\left|\int_a^b f(x)\,dx - (b-a)\frac{f(a)+f(b)}{2}\right| \le \frac{(b-a)^3}{12}\|f''\|_\infty. \tag{1.2}$$

In 1992, Dragomir established a new approach to deal with (1.1). Namely, he considered two convex functions; the first one has a $\sup = \int_a^b f(x)\,dx$ and an $\inf = f\left(\frac{a+b}{2}\right)$. However, the second has a $\sup = \frac{f(a)+f(b)}{2}$ and an $\inf = \int_a^b f(x)\,dx$.
In 1998, Dragomir and Agarwal, proved an inequality for differentiable mapping whose derivative is convex, as follows:

Theorem 1.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{8}\left[|f'(a)| + |f'(b)|\right]. \tag{1.3}$$

In recent years many authors have established several inequalities connected to the Hermite-Hadamard's inequality. For recent results, refinements, counterparts, generalizations and new Hermite-Hadamard-type inequalities under various assumptions for the functions involved the reader may refer to [1] – [21] and the references therein.
In this paper, new upper and lower bounds for the Trapezoid inequality of absolutely continuous functions are established. Applications to some special means are provided as well.

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2. The result

Theorem 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}_+$ be an absolutely continuous mapping on I°, the interior of the interval I, where $a,b \in I$ with $a < b$. Then there exists $x \in (a,b)$ such that the double inequality*

$$\frac{(b-a)^2}{2M^2}\left[\frac{1}{b-a}\int_a^b f(s)\,ds - \frac{M^2}{(x-a)(b-x)}f(x)\right] \le \frac{f(b)+f(a)}{2} - \frac{1}{b-a}\int_a^b f(s)\,ds$$
$$\le \frac{(b-a)^2}{2m^2}\left[\frac{1}{b-a}\int_a^b f(s)\,ds - \frac{m^2}{(x-a)(b-x)}f(x)\right] \quad (2.1)$$

holds, where

$$M := \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right],$$

and

$$m := \left[\frac{b-a}{2} - \left|x - \frac{a+b}{2}\right|\right].$$

Proof. Consider the function $F : [a,b] \to (0,\infty)$ defined by

$$F(t) = \frac{1}{b-t}\int_t^b f(s)\,ds - \frac{1}{t-a}\int_a^t f(s)\,ds$$

for all $t \in [a,b]$. Since f is absolutely continuous on $[a,b]$, then it is easy to see that F is differentiable on (a,b). So that by the Pompiue's Value Theorem there exits $\eta \in (x_1,x_2) \subseteq (a,b)$, $x_1 < x_2$ such that

$$F(\eta) - \eta F'(\eta) = \frac{x_1 F(x_2) - x_2 F(x_1)}{x_1 - x_2}. \quad (2.2)$$

Simple calculations yield that

$$F(\eta) - \eta F'(\eta) = \frac{1}{b-\eta}\int_\eta^b f(s)\,ds - \frac{1}{\eta-a}\int_a^\eta f(s)\,ds + \frac{\eta(b-a)f(\eta)}{(\eta-a)(b-\eta)}$$
$$- \frac{\eta}{(b-\eta)^2}\int_\eta^b f(s)\,ds - \frac{\eta}{(\eta-a)^2}\int_a^\eta f(s)\,ds$$
$$= \frac{\eta(b-a)f(\eta)}{(\eta-a)(b-\eta)} + \frac{b-2\eta}{(b-\eta)^2}\int_\eta^b f(s)\,ds - \frac{2\eta+a}{(\eta-a)^2}\int_a^\eta f(s)\,ds$$

and

$$\frac{aF(b) - bF(a)}{a-b} = \frac{1}{a-b}\left\{af(b) + bf(a) - \frac{a}{b-a}\int_a^b f(s)\,ds - \frac{b}{b-a}\int_a^b f(s)\,ds\right\}$$

therefore, we have

$$\frac{b-2\eta}{(b-\eta)^2}\int_\eta^b f(s)\,ds - \frac{2\eta+a}{(\eta-a)^2}\int_a^\eta f(s)\,ds + \frac{af(b)+bf(a)}{b-a}$$
$$= \frac{a}{(b-a)^2}\int_a^b f(s)\,ds - \frac{b}{(b-a)^2}\int_a^b f(s)\,ds + \frac{\eta(b-a)f(\eta)}{(\eta-a)(b-\eta)}. \quad (2.3)$$

Now, for $x \in (a,b)$, we set

$$M := \max\{x-a, b-x\} = \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right],$$

and

$$m := \min\{x-a, b-x\} = \left[\frac{b-a}{2} - \left|x - \frac{a+b}{2}\right|\right],$$

so that, since f is positive we have

$$\frac{1}{M^2}\int_a^x f(s)\,ds \le \frac{1}{(x-a)^2}\int_a^x f(s)\,ds \le \frac{1}{m^2}\int_a^x f(s)\,ds$$

and

$$\frac{1}{M^2}\int_x^b f(s)\,ds \le \frac{1}{(b-x)^2}\int_x^b f(s)\,ds \le \frac{1}{m^2}\int_x^b f(s)\,ds$$

By adding the above two inequalities we get

$$\frac{1}{M^2}\int_a^b f(s)\,ds \le \frac{1}{(b-x)^2}\int_x^b f(s)\,ds + \frac{1}{(x-a)^2}\int_a^x f(s)\,ds \le \frac{1}{m^2}\int_a^b f(s)\,ds \tag{2.4}$$

and by (2.3) and (2.4) we get

$$\begin{aligned}\frac{1}{M^2}\int_a^b f(s)\,ds - \frac{(b-a)f(x)}{(x-a)(b-x)} \le \frac{2}{b-a}\left[\frac{f(b)+f(a)}{2} - \frac{1}{b-a}\int_a^b f(s)\,ds\right] \\ = \frac{1}{(b-x)^2}\int_c^b f(s)\,ds + \frac{1}{(x-a)^2}\int_a^x f(s)\,ds - \frac{(b-a)f(x)}{(x-a)(b-x)} \\ \le \frac{1}{m^2}\int_a^b f(s)\,ds - \frac{(b-a)f(x)}{(x-a)(b-x)}.\end{aligned}$$

Hence, by multiplying the above inequality by the quantity $\frac{b-a}{2}$ we get

$$\frac{b-a}{2}\left[\frac{1}{M^2}\int_a^b f(s)\,ds - \frac{(b-a)f(x)}{(x-a)(b-x)}\right] \le \frac{f(b)+f(a)}{2} - \frac{1}{b-a}\int_a^b f(s)\,ds \le \frac{b-a}{2}\left[\frac{1}{m^2}\int_a^b f(s)\,ds - \frac{(b-a)f(x)}{(x-a)(b-x)}\right].$$

Rearranging the terms we may write,

$$\begin{aligned}\frac{(b-a)^2}{2M^2}\left[\frac{1}{b-a}\int_a^b f(s)\,ds - \frac{M^2}{(x-a)(b-x)}f(x)\right] \le \frac{f(b)+f(a)}{2} - \frac{1}{b-a}\int_a^b f(s)\,ds \\ \le \frac{(b-a)^2}{2m^2}\left[\frac{1}{b-a}\int_a^b f(s)\,ds - \frac{m^2}{(x-a)(b-x)}f(x)\right],\end{aligned}$$

for some $x\in(a,b)$; which proves the inequality (2.1). □

Here, it is convenient to note that

$$\begin{aligned}0 \le \Delta &:= \text{The right-hand side of (2.1)} - \text{The left-hand side of (2.1)} \\ &= \frac{(b-a)^2}{2m^2}\left[\frac{1}{b-a}\int_a^b f(s)\,ds - \frac{m^2}{(x-a)(b-x)}f(x)\right]\frac{(b-a)^2}{2M^2}\left[\frac{1}{b-a}\int_a^b f(s)\,ds - \frac{M^2}{(x-a)(b-x)}f(x)\right] \\ &= \left(\frac{M^2-m^2}{2m^2M^2}\right)(b-a)\int_a^b f(s)\,ds,\end{aligned} \tag{2.5}$$

thus, it is clear that $\left(\frac{M^2-m^2}{2m^2M^2}\right)\ge 0$ and so that the difference $\Delta\ge 0$ iff $f(t)\ge 0\ \forall t\in[a,b]$.
Finally, we note that another interesting form of the inequality (2.1) may be deduced by rewriting the terms of (2.1), to get:

$$\begin{aligned}\frac{2M^2(b-a)}{\left(2M^2+(b-a)^2\right)}\left[\frac{(b-a)^2}{2(x-a)(b-x)}f(x) + \frac{f(a)+f(b)}{2}\right] \ge \int_a^b f(s)\,ds \\ \ge \frac{2m^2(b-a)}{\left(2m^2+(b-a)^2\right)}\left[\frac{(b-a)^2}{2(x-a)(b-x)}f(x) + \frac{f(a)+f(b)}{2}\right],\end{aligned} \tag{2.6}$$

and so that, we have

$$0 \le \int_a^b f(s)\,ds - \frac{2m^2(b-a)}{\left(2m^2+(b-a)^2\right)}\Psi_f(a,b;x) \le \left[\frac{2M^2(b-a)}{\left(2M^2+(b-a)^2\right)} - \frac{2m^2(b-a)}{\left(2m^2+(b-a)^2\right)}\right]\Psi_f(a,b;x)$$

where,

$$\Psi_f(a,b;x) := \frac{(b-a)^2}{2(x-a)(b-x)}f(x) + \frac{f(a)+f(b)}{2}$$

for some $x\in(a,b)$.
In an interesting particular case, let $\mathscr{F}$ be the set of all functions $f: I\subset\mathbb{R}\to\mathbb{R}_+$ that satisfy the assumptions of Theorem 2.1 such that the required $x\in(a,b)$ is $x=\frac{a+b}{2}$, (in this case we have $M=m=\frac{b-a}{2}$) thus from (2.7) every such f satisfies that

$$\int_a^b f(s)\,ds = \frac{1}{3}(b-a)\left[2f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right], \tag{2.7}$$

where,

$$\Psi_f\left(a,b;\tfrac{a+b}{2}\right) = 2f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}.$$

On the other hand, it is well-known that the error term in Simpson's quadrature rule (2.8) involves a fourth derivatives, however using the above observation and for every $f\in\mathscr{F}$; $\int_a^b f(s)\,ds$ can be evaluated 'exactly' using the Simpson formula (2.7) with no errors, and without going through its higher derivatives which may not exists or hard to find; as in the classical result.

$$\int\limits_a^b f(x)\,dx = \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] + \frac{(b-a)^5}{90}\left\|f^{(4)}\right\|_\infty. \tag{2.8}$$

3. Applications to means

A function $M:\mathbb{R}_+^2\to\mathbb{R}_+$, is called a Mean function if it has the following properties:

1. Homogeneity: $M(ax,ay)=aM(x,y)$, for all $a>0$,
2. Symmetry : $M(x,y)=M(y,x)$,
3. Reflexivity : $M(x,x)=x$,
4. Monotonicity: If $x\le x'$ and $y\le y'$, then $M(x,y)\le M(x',y')$,
5. Internality: $\min\{x,y\}\le M(x,y)\le\max\{x,y\}$.

We shall consider the means for arbitrary positive real numbers α,β $(\alpha\neq\beta)$, see [8]–[9]. We take

1. The arithmetic mean :
$$A:=A(\alpha,\beta)=\frac{\alpha+\beta}{2},\quad \alpha,\beta\in\mathbb{R}_+.$$
2. The geometric mean :
$$G:=G(\alpha,\beta)=\sqrt{\alpha\beta},\quad \alpha,\beta\in\mathbb{R}_+$$
3. The harmonic mean :
$$H:=H(\alpha,\beta)=\frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}},\quad \alpha,\beta\in\mathbb{R}_+-\{0\}.$$
4. The power mean :
$$M_r(\alpha,\beta)=\left(\frac{\alpha^r+\beta^r}{2}\right)^{\frac{1}{r}},\quad r\ge1,\ \alpha,\beta\in\mathbb{R}_+$$
5. The identric mean:
$$I(\alpha,\beta)=\begin{cases}\frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right)^{\frac{1}{\beta-\alpha}}, & \alpha\neq\beta\\ \alpha, & \alpha=\beta\end{cases},\quad \alpha,\beta>0$$
6. The logarithmic mean :
$$L:=L(\alpha,\beta)=\frac{\alpha-\beta}{\ln|\alpha|-\ln|\beta|},\ |\alpha|\neq|\beta|,\ \alpha,\beta\neq0,\ \alpha,\beta\in\mathbb{R}_+.$$
7. The generalized log-mean:
$$L_p:=L_p(\alpha,\beta)=\left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}},\ p\in\mathbb{R}\backslash\{-1,0\},\ \alpha,\beta>0.$$

It is well known that L_p is monotonic nondecreasing over $p\in\mathbb{R}$, with $L_{-1}:=L$ and $L_0:=I$. In particular, we have the following inequality $H\le G\le L\le I\le A$.

As a direct example on the inequality (2.1), consider $f:[a,b]\subset(0,\infty)\to(0,\infty)$ given by $f(s)=\frac{1}{s^2}$, , $s\in[a,b]$, it is easy to see that $F(t)=\frac{b-a}{ab}\cdot\frac{1}{t^2}$, and so that the required $x\in(a,b)$ is $x=\sqrt{ab}:=G(a,b)$. Applying (2.1), we get

$$\frac{(b-a)^2}{2M^2}\left[1-\frac{M^2}{(G(a,b)-a)(b-G(a,b))}\right]\le\frac{G^2(a,b)}{H(a^2,b^2)}-1\le\frac{(b-a)^2}{2m^2}\left[1-\frac{m^2}{(G(a,b)-a)(b-G(a,b))}\right],\tag{3.1}$$

where, $M:=\left[\frac{b-a}{2}+|G(a,b)-A(a,b)|\right]$, and $m:=\left[\frac{b-a}{2}-|G(a,b)-A(a,b)|\right]$.

In general, the reader may check that it is not easy to find the value of x that satisfies the inequality (2.1). For example, we consider $f:[a,b]\subset(0,\infty)\to(0,\infty)$ given by
(1-) $f(s)=\frac{1}{s}$, $s\in[a,b]\subset(0,\infty)$, so that $F(t)=\ln\left(\frac{(b/t)^{1/(b-t)}}{(t/a)^{1/(t-a)}}\right)$. Applying (2.1), we get

$$\frac{(b-a)^2}{2M^2}\left[\frac{1}{L(a,b)}-\frac{M^2}{(x-a)(b-x)}\frac{1}{x}\right]\le\frac{1}{H(a,b)}-\frac{1}{L(a,b)}\le\frac{(b-a)^2}{2m^2}\left[\frac{1}{L(a,b)}-\frac{m^2}{(x-a)(b-x)}\frac{1}{x}\right]\tag{3.2}$$

where $x\in(a,b)$ satisfying the equation (2.2), and m,M are defined above.

(2-) $f(s)=\ln(s)$, $s\in[a,b]\subset(0,\infty)$, so that

$$F(t)=\ln\left(\frac{(b^b/t^t)^{1/(b-t)}}{(t^t/a^a)^{1/(t-a)}}\right)=\ln\left(\frac{I(t,b)}{I(a,t)}\right),\quad t\in[a,b]$$

where $I(\cdot,\cdot)$ is the identric mean. Now, applying (2.1), we get

$$\frac{(b-a)^2}{2M^2}\left[\ln I(a,b)-\frac{M^2\ln x}{(x-a)(b-x)}\right]\le\ln G(a,b)-\ln I(a,b)\le\frac{(b-a)^2}{2m^2}\left[\ln I(a,b)-\frac{m^2\ln x}{(x-a)(b-x)}\right]\tag{3.3}$$

where $x \in (a,b)$ satisfying the equation (2.2), and m,M are defined above.

(3-) $f(s) = s^p$, $s \in [a,b] \subset (0,\infty)$ and $p \in \mathbb{R} \setminus \{-1,0\}$, so that

$$F(t) = \frac{b^{p+1} - t^{p+1}}{(b-t)(p+1)} - \frac{t^{p+1} - a^{p+1}}{(t-a)(p+1)} = L_p^p(t,b) - L_p^p(a,t), \ \ t \in [a,b]$$

where $L_p^p(\cdot,\cdot)$ is the generalized logarithmic mean. Applying (2.1), we get

$$\frac{(b-a)^2}{2M^2}\left[L_p^p(a,b) - \frac{M^2 x^p}{(x-a)(b-x)}\right] \le M_p^p(a,b) - L_p^p(a,b) \le \frac{(b-a)^2}{2m^2}\left[L_p^p(a,b) - \frac{m^2 x^p}{(x-a)(b-x)}\right] \tag{3.4}$$

where $x \in (a,b)$ satisfying the equation (2.2), and m,M are defined above.

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