A Study on Lorentzian $\alpha$–Sasakian Manifolds

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Abstract

The object of the present paper is to study the geometric properties of Concircular curvature tensor on Lorentzian $\alpha$–Sasakian manifold admitting a type of quarter-symmetric metric connection. In the last, we provide an example of 3-dimensional Lorentzian $\alpha$–Sasakian manifold endowed with the quarter-symmetric metric connection which is under consideration is an $\eta$–Einstein manifold with respect to the quarter-symmetric metric connection.

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1. Introduction

In 1975, Golab [5] defined and studied quarter-symmetric connection in differentiable manifolds. A linear connection $\bar{\nabla}$ on an $n$-dimensional Riemannian manifold $(M, g)$ is said to be a quarter-symmetric connection [5] if its torsion tensor $T$ defined by

$$T(X, Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y]$$

satisfies

$$T(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y,$$

where $\phi$ is a $(1, 1)$ tensor field, $\eta$ is a 1-form and $X, Y$ are vector fields on $\Gamma(TM)$. $\Gamma(TM)$ is the set of all differentiable vector fields on $M$. In particular, if $\phi X = X$, then the quarter-symmetric connection reduces to the semi-symmetric connection [4]. Thus the notion of the quarter-symmetric connection generalizes the notion of the semi-symmetric connection. If moreover, a quarter-symmetric connection $\bar{\nabla}$ satisfies the condition

$$\{\bar{\nabla}_X g\}(Y, Z) = 0,$$

for all $X, Y, Z$ on $\Gamma(TM)$, then $\bar{\nabla}$ is said to be a quarter-symmetric metric connection, otherwise it is said to be a quarter symmetric non-metric connection. Recently quarter-symmetric metric connection have been studied by several authors ([8], [9], [12]).

A differentiable manifold $M$ is said to be a Lorentzian manifold, if $M$ has a Lorentzian metric $g$, which is a symmetric non-degenerate $(0,2)$ tensor field of index 1. Since the Lorentzian metric $g$ is of index 1 therefore Lorentzian manifold $M$ has not only spacelike vector fields but also lightlike and timelike vector fields. On a Lorentzian manifold this difference with Riemannian case gives interesting results. In 1989, K. Matsumoto used a structure vector field $-\xi$ instead of $\xi$ in an almost para contact manifold and associated a Lorentzian metric with this resulting structure, called it as Lorentzian almost para contact manifold.

Yildiz et al. ([15]) studied Lorentzian $\alpha$–Sasakian manifolds in 2005 and obtained results for conformally flat and quasi-conformally flat Lorentzian $\alpha$–Sasakian manifolds. In 2009, Yildiz et al. ([16, 17]), further studied on three dimensional Lorentzian $\alpha$–Sasakian manifolds and a class of Lorentzian $\alpha$–Sasakian manifolds and obtained some important results. In 2013, U.C. De and K. De ([3]) studied on Lorentzian Trans-Sasakian manifolds, which is a generalization of Lorentzian $\alpha$–Sasakian manifolds. A concircular transformation ([7], [13]) on an $n$-dimensional Riemannian manifold $M$ is a transformation under which every geodesic circle of $M$ transforms into a geodesic circle. Every concircular transformation is always a conformal transformation [7]. Thus the concircular geometry, is a generalization of invariance geometry in the sense that the change of metric is more general than that induced by a circle.
An interesting invariant of a concircular transformation is the concircular curvature tensor $\mathcal{C}$. It is defined by ([13], [14])

$$\mathcal{C}(X, Y)Z = \mathring{R}(X, Y)Z - \frac{\mathring{r}}{2n(2n + 1)}[g(Y, Z)X - g(X, Z)Y],$$

(1.4)

for all vector fields $X, Y, Z \in \Gamma(TM)$, where $\mathring{R}$ and $\mathring{r}$ be the curvature tensor and scalar curvature with respect to the quarter-symmetric metric connection $\nabla$ respectively.

Using (1.4), we obtain

$$\mathring{\mathcal{C}}(X, Y, Z, W) = \mathring{\mathring{R}}(X, Y, Z, W) - \frac{\mathring{r}}{2n(2n + 1)}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)],$$

(1.5)

where $\mathring{\mathcal{C}}(X, Y, Z, W) = g(\mathring{\mathcal{C}}(X, Y)Z, W)$, $\mathring{\mathring{R}}(X, Y, Z, W) = g(\mathring{\mathring{R}}(X, Y)Z, W)$, where $X, Y, Z, W \in \Gamma(TM)$ and $\mathcal{C}$ is the concircular curvature tensor and $\mathring{r}$ is the scalar curvature with respect to the quarter-symmetric metric connection respectively. Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

In this paper, we study a type of quarter-symmetric metric connection on Lorentzian $\alpha$-Sasakian manifolds. The paper is organized as follows: After introduction section two gives some prerequisites of a Lorentzian $\alpha$-Sasakian manifold. In section three, we obtain a relation between the quarter-symmetric metric connection and Levi-civita connection. In section four, curvature tensor and Ricci tensor of Lorentzian $\alpha$–Sasakian manifold with respect to quarter-symmetric metric connection are given. Section five is devoted to the study of $\xi$-concircularly flat Lorentzian $\alpha$-Sasakian manifold with respect to the quarter-symmetric metric connection. Quasi-concircularly flat and $\phi$-concircularly flat Lorentzian $\alpha$-Sasakian manifolds with respect to the quarter-symmetric metric connection have been studied in section six and seven respectively. In the next section, we study a Lorentzian $\alpha$-Sasakian manifold satisfying $\mathcal{C} \cdot S = 0$ with respect to a quarter-symmetric metric connection. In the last, we construct an example of a 3-dimensional Lorentzian $\alpha$-Sasakian manifold endowed with the quarter-symmetric metric connection.

2. Preliminaries

An $(2n + 1)$-dimensional differentiable manifold $M$ is said to be a Lorentzian $\alpha$–Sasakian manifold, if it admits a structure $(\phi, \xi, \eta, g)$ consisting of a $(1, 1)$ tensor field $\phi$, vector field $\xi$, 1-form $\eta$ and a Lorentzian metric $g$ satisfying

$$\phi^2 X = X + \eta(X)\xi,$$

(2.1)

$$\phi \circ \xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = -1, \quad g(X, \xi) = \eta(X),$$

(2.2)

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

(2.3)

$$\{\nabla_X \phi\} Y = \alpha(g(X, Y)\xi + \eta(Y)X)$$

(2.4)

for any vector field $X, Y$ on $M$, where $\nabla$ denotes the covariant differentiation with respect to Lorentzian metric $g$.

Also a Lorentzian $\alpha$–Sasakian manifold satisfies [16]

$$\nabla_X \xi = \alpha \phi X,$$

(2.5)

$$\nabla_X \eta = \alpha g(X, \phi Y)$$

(2.6)

for $X, Y$ tangent to $M$.

Moreover, the curvature tensor $R$, the Ricci tensor $S$ and the Ricci operator $Q$ in a Lorentzian $\alpha$-Sasakian manifold $M$ with respect to the Levi-Civita connection $\nabla$, satisfies following relations [16]

$$R(\xi, X)Y = \alpha^2 \{g(X, Y)\xi - \eta(Y)X\},$$

(2.7)

$$R(X, Y)\xi = \alpha^2 \{\eta(Y)X - \eta(X)Y\},$$

(2.8)

$$R(\xi, X)\xi = -R(X, \xi)\xi = \alpha^2 \{X + \eta(X)\xi\},$$

(2.9)

$$S(X, \xi) = 2n\alpha^2 \eta(X),$$

(2.10)

$$S(\xi, \xi) = -2n\alpha^2, \quad Q\xi = 2n\alpha^2 \xi,$$

(2.11)

$$S(\phi X, \phi Y) = S(X, Y) - 2n\alpha^2 g(X, Y),$$

(2.12)

for all vector fields $X, Y \in \Gamma(TM)$. 


3. Relation Between the Quarter-Symmetric Metric Connection and Riemannian Connection

Let $\nabla$ be a Riemannian connection and $\bar{\nabla}$ be a linear connection on Lorentzian $\alpha$-Sasakian manifold $M$ such that

$$\bar{\nabla}_X Y = \nabla_X Y + H(X, Y),$$

where $H$ is a tensor of type $(1, 2)$. Now if $\bar{\nabla}$ be a quarter-symmetric connection on $M$, then we have [5]

$$H(X, Y) = \frac{1}{2} \left[ T(X, Y) + T(Y, X) + T(Y, X) \right],$$

where

$$g(T'(X, Y), Z) = g(T(Z, X), Y).$$

Using (1.2) in (3.3), we get

$$T'(X, Y) = \eta(X) \phi Y - g(\phi X, Y) \xi.$$

In view of (1.2) and (3.4), equation (3.2) gives

$$H(X, Y) = \eta(Y) \phi X - g(\phi X, Y) \xi.$$

Hence from (3.1), a quarter-symmetric connection $\bar{\nabla}$ on a Lorentzian $\alpha$-Sasakian manifold $M$ is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y) \phi X - g(\phi X, Y) \xi.$$

Also we have

$$\left( \bar{\nabla}_X g(Y, Z) \right) = Xg(Y, Z) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z).$$

With the help of (3.6), after simplification (3.7) gives

$$\left( \bar{\nabla}_X g(Y, Z) \right) = 0, \quad \forall Y, Z \in \Gamma(TM).$$

By virtue of (3.6) and (3.8), we conclude that $\bar{\nabla}$ is a quarter-symmetric metric connection. Therefore (3.6) is the relation between Riemannian connection and quarter-symmetric metric connection on a Lorentzian $\alpha$-Sasakian manifold.

4. Curvature Tensor and Ricci Tensor of Lorentzian $\alpha$–Sasakian Manifold with respect to the Quarter-Symmetric Metric Connection

Let $R(X, Y) Z$ and $\bar{R}(X, Y) Z$ be the curvature tensors of a Lorentzian $\alpha$-Sasakian manifold $M$ with respect to the Riemannian connection $\nabla$ and quarter-symmetric metric connection $\bar{\nabla}$ respectively, then relation between $R(X, Y) Z$ and $\bar{R}(X, Y) Z$ is given by

$$\bar{R}(X, Y) Z = R(X, Y) Z + \alpha \eta(Z) \left[ \eta(Y) X - \eta(X) Y \right]$$

$$+ (2\alpha - 1) g(\phi X, Z) \phi Y - g(\phi Y, Z) \phi X$$

$$- \alpha g(X, Z) \eta(Y) - g(Y, Z) \eta(X) \xi.$$

From (4.1), we have

$$\bar{R}(\xi, X) Y = (\alpha^2 - \alpha) \left[ g(\phi X, Y) \xi - \eta(Y) X \right].$$

$$\bar{R}(X, Y) \xi = \left( \alpha^2 - \alpha \right) \left[ \eta(Y) X - \eta(X) Y \right],$$

$$\bar{R}(\xi, Y) \xi = \left( \alpha^2 - \alpha \right) \left[ Y + \eta(Y) \xi \right].$$

Let $\{e_1, e_2, \ldots, e_{2n}, e_{2n+1} = \xi \}$ be a local orthonormal basis of vector fields in $M$. Since on a semi-Riemannian manifold, we have [10]

$$\sum_{i=1}^{2n+1} e_ig(\bar{R}(e_i, Y) Z, e_i) = S(Y, Z),$$

$$\sum_{i=1}^{2n+1} e_is(e_i, Y) g(e_i, Z) = S(Y, Z),$$
\[ \sum_{i=1}^{2n+1} \varepsilon_ig(e_i,Y)g(e_i,Z) = g(Y,Z), \]

and

\[ \sum_{i=1}^{2n+1} \varepsilon_ig(\phi e_i, e_i) = \text{trace}(\phi), \]

where \( \varepsilon_i = g(e_i, e_i), \ i = 1, 2, \ldots, 2n+1. \) Using above results on a Lorentzian \( \alpha \)-Sasakian manifold, it can be easily verify that

\[ \sum_{i=1}^{2n} \bar{g}(\bar{R}(e_i,Y)Z, e_i) = S(Y,Z) - \alpha^2 g(\phi Y, \phi Z), \]  \hspace{1cm} (4.5)

\[ \sum_{i=1}^{2n} S(e_i,Y)g(e_i,Z) = S(Y,Z) + 2n\alpha^2 \eta(Y)\eta(Z), \]  \hspace{1cm} (4.6)

\[ \sum_{i=1}^{2n} g(e_i, e_i) = 2n, \]  \hspace{1cm} (4.7)

\[ \sum_{i=1}^{2n} g(e_i, Y)g(e_i, Z) = g(\phi Y, \phi Z), \]  \hspace{1cm} (4.8)

\[ \sum_{i=1}^{2n} g(\phi e_i, e_i) = \text{trace}(\phi) \]  \hspace{1cm} (4.9)

and

\[ \sum_{i=1}^{2n} g(\bar{R}(e_i,Y)Z, e_i) = \bar{S}(Y,Z) - \left( \alpha^2 - \alpha \right) g(\phi Y, \phi Z). \]  \hspace{1cm} (4.10)

Then from (4.1), we obtain

\[ \bar{S}(Y,Z) = S(Y,Z) + \left\{(2n+1)\alpha - 1\right\} \eta(Y)\eta(Z) \]
\[ + (\alpha - 1)g(Y,Z) - (2\alpha - 1)\text{trace}(\phi)\Phi(Y,Z), \]  \hspace{1cm} (4.11)

\[ \bar{S}(Y,\xi) = 2n \left( \alpha^2 - \alpha \right) \eta(Y), \]  \hspace{1cm} (4.12)

\[ \bar{S}(\xi,\xi) = -2n \left( \alpha^2 - \alpha \right), \]  \hspace{1cm} (4.13)

\[ \bar{S}(\phi Y, \phi Z) = \bar{S}(Y,Z) - 2n\alpha^2 g(Y,Z) - 2n\alpha \eta(Y)\eta(Z). \]  \hspace{1cm} (4.14)

where \( \bar{S} \) and \( \bar{r} \) be the Ricci tensor and scalar curvature with respect to the quarter-symmetric metric connection \( \bar{\nabla} \) respectively.

5. \( \xi \)-Concircularly Flat Lorentzian \( \alpha \)-Sasakian Manifold with Respect to the Quarter-Symmetric Metric Connection

**Definition 5.1.** A Lorentzian \( \alpha \)-Sasakian manifold is said to be \( \xi \)-concircularly flat \([1]\) with respect to the quarter-symmetric metric connection if \( C(X, Y)\xi = 0, \) where \( X, Y \in \Gamma(TM) \).

**Theorem 5.2.** A Lorentzian \( \alpha \)-Sasakian manifold admitting a quarter-symmetric metric connection \( \bar{\nabla} \) is \( \xi \)-concircularly flat if and only if the scalar curvature \( \bar{r} \) with respect to the quarter-symmetric metric connection is equal to \( 2n(2n+1) \left( \alpha^2 - \alpha \right) \).
Proof. From (1.4), we have
\[ \tilde{\mathcal{C}}(X,Y)\xi = \tilde{\mathcal{R}}(X,Y)\xi - \frac{\tilde{r}}{2n(2n+1)}[\eta(Y)X - \eta(X)Y]. \] (5.1)
Using (4.3) in (5.1), we have
\[ \tilde{\mathcal{C}}(X,Y)\xi = \left(\alpha^2 - \alpha\right)[\eta(Y)X - \eta(X)Y] - \frac{\tilde{r}}{2n(2n+1)}[\eta(Y)X - \eta(X)Y]. \] (5.2)
From (5.2), we have
\[ \tilde{\mathcal{C}}(X,Y)\xi = \left(\alpha^2 - \alpha\right) \frac{\tilde{r}}{2n(2n+1)}[\eta(Y)X - \eta(X)Y]. \] (5.3)

Thus from (5.3), if \( \tilde{\mathcal{C}}(X,Y)\xi = 0 \), then \( \tilde{r} = 2n(2n+1) \left(\alpha^2 - \alpha\right) \) or \( \eta(Y)X - \eta(X)Y = 0 \), implies that \( \eta(X) = 0 \) which is not possible. Conversely, if \( \tilde{r} = 2n(2n+1) \left(\alpha^2 - \alpha\right) \), then from (5.3), it follows that \( \tilde{\mathcal{C}}(X,Y)\xi = 0 \). This completes the proof of the theorem.

6. Quasi-Concircularly Flat Lorentzian \( \alpha \)-Sasakian Manifold with Respect to the Quarter-Symmetric Metric Connection

Definition 6.1. A Lorentzian \( \alpha \)-Sasakian manifold is said to be quasi–concircularly flat with respect to the quarter-symmetric metric connection if
\[ \tilde{\mathcal{C}}(\phi X, Y, Z, \phi W) = 0 \] (6.1)
where \( X, Y, Z, W \in \Gamma(TM) \).

Definition 6.2. A Lorentzian \( \alpha \)-Sasakian manifold is said to be an \( \eta \)–Einstein manifold if its Ricci tensor \( S \) of the Levi-Civita connection is of the form
\[ S(X, Y) = a g(X, Y) + b \eta(X) \eta(Y), \] (6.2)
where \( a \) and \( b \) are smooth functions on the manifold.

Theorem 6.3. If a Lorentzian \( \alpha \)-Sasakian manifold admitting a quarter-symmetric metric connection is quasi-concircularly flat, then the manifold with respect to the quarter-symmetric metric connection is an \( \eta \)–Einstein manifold.

Proof. From (1.4), we have
\[ \tilde{\mathcal{C}}(X, Y, Z, W) = \tilde{\mathcal{R}}(X, Y, Z, W) - \frac{\tilde{r}}{2n(2n+1)} g(Y, Z) g(X, W) \] (6.3)
where \( \tilde{\mathcal{C}}(X, Y, Z, W) = g(\tilde{\mathcal{C}}(X, Y)Z, W) \) and \( \tilde{\mathcal{R}}(X, Y, Z, W) = g(\tilde{\mathcal{R}}(X, Y)Z, W) \). Now putting \( X = \phi X \) and \( W = \phi W \) in (6.3), we get
\[ \tilde{\mathcal{C}}(\phi X, Y, Z, \phi W) = \tilde{\mathcal{R}}(\phi X, Y, Z, \phi W) - \frac{\tilde{r}}{2n(2n+1)} g(Y, Z) g(\phi X, \phi W) \] (6.4)
Using (6.1) in (6.4), we get
\[ \tilde{\mathcal{R}}(\phi X, Y, Z, \phi W) = \frac{\tilde{r}}{2n(2n+1)} [g(Y, Z) g(\phi X, \phi W) - g(\phi X, Z) g(Y, \phi W)]. \] (6.5)

Let \( \{e_1, e_2, \ldots, e_{2n}, \xi\} \) be a local orthonormal basis of vector fields in \( M \), then \( \{\phi e_1, \phi e_2, \ldots, \phi e_{2n}, \xi\} \) is also a local orthonormal basis. Putting \( X = W = e_i \) in (6.5) and summing over \( i = 1 \) to \( 2n \), we obtain
\[ \sum_{i=1}^{2n} \tilde{\mathcal{R}}(\phi e_i, Y, Z, \phi e_i) = \frac{\tilde{r}}{2n(2n+1)} \sum_{i=1}^{2n} [g(Y, Z) g(\phi e_i, \phi e_i) - g(\phi e_i, Z) g(Y, \phi e_i)]. \] (6.6)

So by virtue of (2.3), (4.7), (4.8) and (4.10), the equation (6.6) takes the form
\[ \tilde{S}(Y, Z) = \left[ \frac{\tilde{r}(2n-1)}{2n(2n+1)} + (\alpha^2 - \alpha) \right] g(Y, Z) - \frac{\tilde{r}}{2n(2n+1)} \left[ (\alpha^2 - \alpha) \right] \eta(Y) \eta(Z), \] (6.7)
or
\[ \tilde{S}(Y, Z) = a g(Y, Z) + b \eta(Y) \eta(Z), \]
where \( a = \frac{\tilde{r}(2n-1)}{2n(2n+1)} + (\alpha^2 - \alpha) \) and \( b = -\frac{\tilde{r}}{2n(2n+1)} - (\alpha^2 - \alpha) \).

From which it follows that the manifold is an \( \eta \)–Einstein manifold with respect to the quarter-symmetric metric connection. This completes the proof of the theorem.
7. \( \phi \)-Concircularly Flat Lorentzian \( \alpha \)-Sasakian Manifold with Respect to the Quarter-Symmetric Metric Connection

**Definition 7.1.** A Lorentzian \( \alpha \)-Sasakian manifold is said to be \( \phi \)-concircularly flat \([11]\) with respect to the quarter-symmetric metric connection if

\[
\tilde{C}(\phi X, \phi Y, \phi Z, \phi W) = 0,
\]

where \( \tilde{C} \) and \( \phi \) denote the concircular curvature tensor and the Levi-Civita connection, respectively. \((\tilde{C}, \phi)\) is the quarter-symmetric metric connection.

**Theorem 7.2.** If a Lorentzian \( \alpha \)-Sasakian manifold admitting a quarter-symmetric metric connection is \( \phi \)-concircularly flat, then the manifold is an Einstein manifold.

**Proof.** From (1.4), we have

\[
\tilde{C}(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) - \frac{f}{2n(2n+1)} [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)].
\]

Now putting \( X = \phi X, Y = \phi Y, Z = \phi Z, W = \phi W \) in (7.2), we get

\[
\tilde{C}(\phi X, \phi Y, \phi Z, \phi W) = \tilde{R}(\phi X, \phi Y, \phi Z, \phi W) - \frac{f}{2n(2n+1)} [g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)].
\]

Using (7.1) in (7.3), we get

\[
\tilde{R}(\phi X, \phi Y, \phi Z, \phi W) = \frac{f}{2n(2n+1)} [g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)].
\]

Let \( \{e_1, e_2, \ldots, e_{2n}, \xi\} \) be a local orthonormal basis of vector fields in \( M \), then \( \{\phi e_1, \phi e_2, \ldots, \phi e_{2n}, \xi\} \) is also a local orthonormal basis. Putting \( X = W = e_i \) in (7.4) and summing over \( i = 1 \) to \( 2n \), we obtain

\[
\sum_{i=1}^{2n} \tilde{R}(\phi e_i, \phi Y, \phi Z, \phi e_i) = \frac{f}{2n(2n+1)} \sum_{i=1}^{2n} [g(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)g(\phi Y, \phi e_i)].
\]

So by virtue of (4.7), (4.8) and (4.10), the equation (7.5) takes the form

\[
\delta(\phi Y, \phi Z) = \frac{f(2n-1)}{2n(2n+1)} + [\alpha^2 - \alpha]g(\phi Y, \phi Z).
\]

By making use of (2.3) and (4.14) in equation (7.6), we obtain

\[
\delta(X, Z) = \frac{f(2n-1)}{2n(2n+1)} + [\alpha^2 - \alpha] + 2n\alpha^2 \]

or

\[
\delta(X, Z) = a g(X, Z) + b \eta(Y) \eta(Z),
\]

where \( a = \frac{f(2n-1)}{2n(2n+1)} + [\alpha^2 - \alpha] + 2n\alpha^2 \) and \( b = \frac{f(2n-1)}{2n(2n+1)} + [\alpha^2 - \alpha] + 2n\alpha \).

From which it follows that the manifold is an \( \eta \)-Einstein manifold with respect to the quarter-symmetric metric connection. This completes the proof of the theorem.

8. Lorentzian \( \alpha \)-Sasakian Manifold Satisfying \( \tilde{C} \cdot \tilde{S} = 0 \) with Respect to the Quarter-Symmetric Metric Connection

**Definition 8.1.** A Lorentzian \( \alpha \)-Sasakian manifold is said to be an Einstein manifold if its Ricci tensor \( S \) of the Levi-Civita connection is of the form

\[
S(X, Y) = ag(X, Y),
\]

where \( a \) is a constant on the manifold.

**Theorem 8.2.** If Lorentzian \( \alpha \)-Sasakian manifold satisfying \( \tilde{C} \cdot \tilde{S} = 0 \) with respect to a quarter-symmetric metric connection, then the manifold is an Einstein manifold with respect to the quarter-symmetric metric connection.
Proof. We consider Lorentzian $\alpha$-Sasakian manifolds with respect to a quarter-symmetric metric connection $\tilde{V}$ satisfying the curvature condition $\tilde{C} \cdot \tilde{S} = 0$. Then

$$(\tilde{C}(X,Y) \cdot \tilde{S})(Z,W) = 0.$$  \hfill (8.2)

So,

$$\tilde{S} \tilde{C}(X,Y) Z, W) + \tilde{S}(Z, \tilde{C}(X,Y) W) = 0.$$  \hfill (8.3)

Putting $X = \xi$ in (8.3), we get

$$\tilde{S} \tilde{C}(\xi,Y) Z, W) + \tilde{S}(Z, \tilde{C}(\xi,Y) W) = 0.$$  \hfill (8.4)

From equation (1.4), we have

$$\tilde{C}(\xi,Y) Z = \tilde{R}(\xi,Y) Z - \frac{\tilde{r}}{2n(2n+1)} [g(Y,Z) \xi - \eta(Z) Y].$$  \hfill (8.5)

Using (4.2) in the equation (8.5), we obtain

$$\tilde{C}(\xi,Y) Z = \{\alpha^2 - \alpha - \frac{\tilde{r}}{2n(2n+1)}\} [g(Y,Z) \xi - \eta(Z) Y].$$  \hfill (8.6)

Using (8.6) and putting $Z = \xi$ in (8.4) and using the equations (2.2), (4.12), we obtain

$$\{\alpha^2 - \alpha - \frac{\tilde{r}}{2n(2n+1)}\} [\tilde{S}(Y,W) - 2n(\alpha^2 - \alpha) g(Y,W)] = 0.$$  \hfill (8.7)

Therefore,

$$\tilde{S}(Y,W) = 2n(\alpha^2 - \alpha) g(Y,W)$$

provided $\tilde{r} \neq 2n(2n+1)(\alpha^2 - \alpha)$.

This means that the manifold is an Einstein manifold with respect to the quarter-symmetric metric connection.

This completes the proof. \qed

9. Example

In this section we construct an example on Lorentzian $\alpha$-Sasakian manifold endowed with the quarter-symmetric metric connection. We consider the 3-dimensional manifold $M^3 = \{(x,y,z) : x,y,z \in \mathbb{R}\}$, where $(x,y,z)$ are the standard coordinates in $\mathbb{R}^3$. We choose the vector fields

$$e_1 = \frac{\partial}{\partial x}, \ e_2 = \frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \ e_3 = \alpha \frac{\partial}{\partial z} = \xi,$$

which are linearly independent at each point of $M^3$.

Let $g$ be a Lorentzian metric defined by

$g(e_1,e_1) = 1, \ g(e_2,e_2) = 1, \ g(e_3,e_3) = -1,$

and $g(e_i,e_j) = 0$ if $i \neq j$.

Let $\phi$ be the $(1,1)$-tensor field defined by

$$\phi e_1 = -e_1, \ \phi e_2 = -e_2, \ \phi e_3 = 0.$$

and $\eta$ be a 1-form defined by $\eta(X) = g(X,e_3)$ for any $X \in \Gamma(TM^3)$

Now using the linearity of $\phi$ and $g$, we obtain

$$\phi^3 X = X + \eta(X) \xi,$$

$$\eta(\xi) = -1,$$

and

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X) \eta(Y),$$

for any vector fields $X, Y \in \Gamma(TM^3)$. Thus for $e_3 = \xi$, the structure $(\phi, \xi, \eta, g)$ defines a Lorentzian para-contact metric structure on $M^3$.

Now, we have

$$[e_1,e_2] = 0, \ [e_2,e_3] = -\alpha e_2, \ [e_1,e_3] = -\alpha e_1.$$

Let $\nabla$ be the Levi-Civita connection of the Lorentzian metric $g$ which is given by Koszul’s formula defined by

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X, [Y,Z]) - g(Y, [X,Z]) + g(Z, [X,Y]).$$
Using Koszul’s formula, we obtain the following:

\[ \nabla_{\xi_1} e_1 = -\alpha e_3, \quad \nabla_{\xi_2} e_2 = 0, \quad \nabla_{\xi_3} e_3 = -\alpha e_1, \quad (9.1) \]
\[ \nabla_{\xi_1} e_1 = 0, \quad \nabla_{\xi_2} e_2 = -\alpha e_3, \quad \nabla_{\xi_2} e_3 = \alpha e_2, \]
\[ \nabla_{\xi_3} e_1 = 0, \quad \nabla_{\xi_1} e_2 = 0, \quad \nabla_{\xi_3} e_3 = 0. \]

In view of the above results, we see that

\[ (\nabla_X \eta) Y = \alpha g(\phi X, Y) \xi, \]
\[ \nabla_X \xi = \alpha \phi X, \]

for all \( X, Y \in \Gamma(TM^3) \) and \( \xi = e_3 \). Therefore the manifold is a Lorentzian \( \alpha \)-Sasakian manifold with the structure \((\phi, \xi, \eta, g)\).

It is known that

\[ R(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z. \]

Now using (9.1), we can easily obtain the non-zero components of the curvature tensor \( R \) as follows:

\[ R(e_1, e_2) e_1 = -\alpha^2 e_2, \quad R(e_1, e_2) e_2 = \alpha^2 e_1, \quad (9.2) \]
\[ R(e_1, e_1) e_1 = -\alpha^2 e_3, \quad R(e_1, e_3) e_3 = -\alpha^2 e_1 \]
\[ R(e_2, e_3) e_2 = -\alpha^2 e_3, \quad R(e_2, e_3) e_3 = -\alpha^2 e_2. \]

Let \( X, Y \) and \( Z \) be any three vector fields given by

\[ X = X^1 e_1 + X^2 e_2 + X^3 e_3, \]
\[ Y = Y^1 e_1 + Y^2 e_2 + Y^3 e_3, \]
\[ Z = Z^1 e_1 + Z^2 e_2 + Z^3 e_3 \]

where \( X^i, Y^i \) and \( Z^i \), for all \( i = 1, 2, 3 \) are all non-zero real numbers. Then

\[ R(X, Y) Z = R(X^1 e_1 + X^2 e_2 + X^3 e_3, Y^1 e_1 + Y^2 e_2 + Y^3 e_3)(Z^1 e_1 + Z^2 e_2 + Z^3 e_3). \quad (9.4) \]

Using equation (9.2) in (9.4), we get

\[ R(X, Y) Z = \alpha^2 g(Y, Z) X - g(X, Z) Y. \quad (9.5) \]

Hence, the 3-dimensional Lorentzian \( \alpha \)-Sasakian manifold is of constant curvature \( \alpha^2 \). Also from (9.5), we obtain

\[ S(Y, Z) = 2\alpha^2 g(Y, Z) \]

which gives \( S(e_1, e_1) = S(e_2, e_2) = 2\alpha^2 \), \( S(e_3, e_3) = -2\alpha^2 \) and therefore the scalar curvature \( r = 6\alpha^2 \).

Now using (9.1) in (3.7), we obtain the components of quarter–symmetric metric connection \( \bar{\nabla} \) as follows:

\[ \bar{\nabla}_{e_1} e_1 = -(\alpha - 1)e_3, \quad \bar{\nabla}_{e_1} e_2 = 0, \quad \bar{\nabla}_{e_1} e_3 = -(\alpha - 1)e_1, \quad (9.7) \]
\[ \bar{\nabla}_{e_2} e_1 = 0, \quad \bar{\nabla}_{e_2} e_2 = -(\alpha - 1)e_3, \quad \bar{\nabla}_{e_2} e_3 = -(\alpha - 1)e_2, \]
\[ \bar{\nabla}_{e_3} e_1 = 0, \quad \bar{\nabla}_{e_3} e_2 = 0, \quad \bar{\nabla}_{e_3} e_3 = 0. \]

Using above results, we can easily obtain the components of curvature tensor \( \bar{R} \) with respect to quarter–symmetric metric connection \( \bar{\nabla} \) as follows:

\[ \bar{R}(e_1, e_2) e_1 = -(\alpha - 1)^2 e_2, \quad \bar{R}(e_1, e_2) e_2 = (\alpha - 1)^2 e_1, \quad \bar{R}(e_1, e_2) e_3 = 0, \quad (9.8) \]
\[ \bar{R}(e_1, e_1) e_1 = -\alpha(\alpha - 1)e_3, \quad \bar{R}(e_1, e_3) e_3 = -\alpha(\alpha - 1)e_1 \]
\[ \bar{R}(e_2, e_3) e_1 = 0, \quad \bar{R}(e_2, e_3) e_2 = -\alpha(\alpha - 1)e_3, \quad \bar{R}(e_2, e_3) e_3 = -\alpha(\alpha - 1)e_2, \]

With the help of (9.8), we find the Ricci tensors \( \bar{S} \) with respect to the quarter-symmetric metric connection as:

\[ \bar{S}(e_1, e_1) = \bar{S}(e_2, e_2) = (2\alpha - 1)(\alpha - 1), \quad \bar{S}(e_3, e_3) = -2\alpha(\alpha - 1). \]

From above results, it follows that the scalar curvature tensor with respect to the quarter-symmetric metric connection is \( \bar{r} = 2(3\alpha - 1)(\alpha - 1) \).

Using (4.11) and (9.6) in 3-dimensional Lorentzian \( \alpha \)-Sasakian manifold \( M^3 \), we have

\[ \bar{S}(Y, Z) = (2\alpha - 1)(\alpha - 1)g(Y, Z) - (\alpha - 1)\eta(Y)\eta(Z). \]

Thus the three dimensional Lorentzian \( \alpha \)-Sasakian manifold \( M^3 \) is an \( \eta \)-Einstein manifold with respect to the quarter-symmetric metric connection \( \bar{\nabla} \).

If we take \( \alpha = 1 \) in this example, then 3-dimensional Lorentzian \( \alpha \)-Sasakian manifold \( M^3 \) becomes flat with respect to the quarter-symmetric metric connection \( \bar{\nabla} \).
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References