Konuralp Journal of Mathematics, 7 (2) (2019) 337-343

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath
e-ISSN: 2147-625X

A Weighted Companion of Ostrowski-Midpoint Inequality for Mappings of Bounded Variation

Mohammad W. Alomari[1*]

[1]*Department of Mathematics, Faculty of Science and Information Technology, Jadara University, P.O. Box 733, Irbid, P.C. 21110, Jordan.*
[*]*Corresponding author E-mail: mwomath@gmail.com*

Abstract

A weighted companion of Ostrowski–Midpoint type inequality is established. Application to a composite quadrature rule is provided.

Keywords: *Ostrowski's inequality, bounded variation, Midpoint inequality*
2010 Mathematics Subject Classification: *26D10, 26A15, 65D30, 65D32*

1. Introduction

In 1938, A. Ostrowski [6], proved the following inequality for differentiable mappings with bounded derivatives, as follows:

Theorem 1.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'(x)| \le M$, then the following inequality,*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \le M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right] \tag{1.1}$$

holds for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

In [7], Dragomir proved the following Ostrowski's inequality for mappings of bounded variation:

Theorem 1.2. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then we have the inequalities:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{2} + \left|\frac{x - \frac{a+b}{2}}{b-a}\right|\right] \cdot \bigvee_a^b (f), \tag{1.2}$$

for any $x \in [a,b]$, where $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$. The constant $\frac{1}{2}$ is best possible.

Motivated by [9], S.S. Dragomir in [8] has proved the following companion of the Ostrowski inequality for mappings of bounded variation:

Theorem 1.3. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then we have the inequalities:*

$$\left| \frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \left|\frac{x - \frac{3a+b}{4}}{b-a}\right|\right] \cdot \bigvee_a^b (f), \tag{1.3}$$

for any $x \in [a, \frac{a+b}{2}]$, where $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$. The constant $1/4$ is best possible.

In [10], Tseng et al. have proved the following weighted Ostrowski inequality for mappings of bounded variation:

***Email addresses*: mwomath@gmail.com (M. W. Alomari)**

Theorem 1.4. *Let $0 \leq \alpha \leq 1$, $g : [a,b] \to [0,\infty)$ continuous and positive on (a,b) and let $h : [a,b] \to \mathbb{R}$ be differentiable such that $h'(t) = g(t)$ on $[a,b]$. Let $c = h^{-1}\left(\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right)$ and $d = h^{-1}\left(\frac{\alpha}{2}h(a) + \left(1-\frac{\alpha}{2}\right)h(b)\right)$. Suppose that f is of bounded variation on $[a,b]$, then for all $x \in [c,d]$, we have*

$$\left|\int_a^b f(t)g(t)\,dt - \left[(1-\alpha)f(x) + \alpha\frac{f(a)+f(b)}{2}\right]\int_a^b g(t)\,dt\right| \leq M \cdot \bigvee_a^b(f) \tag{1.4}$$

where,

$$M = \begin{cases} \frac{1-\alpha}{2}\int_a^b g(t)\,dt + \left|h(x) + \frac{h(a)+h(b)}{2}\right|, & 0 \leq \alpha \leq \frac{1}{2} \\ \max\left\{\frac{1-\alpha}{2}\int_a^b g(t)\,dt + \left|h(x) + \frac{h(a)+h(b)}{2}\right|, \frac{\alpha}{2}\int_a^b g(t)\,dt\right\}, & \frac{1}{2} < \alpha < \frac{2}{3} \\ \frac{\alpha}{2}\int_a^b g(t)\,dt, & \frac{2}{3} \leq \alpha \leq 1 \end{cases}$$

and $\bigvee_a^b(f)$ is the total variation of f over $[a,b]$. The constant $\frac{1-\alpha}{2}$ for $0 \leq \alpha \leq \frac{1}{2}$ and the constant $\frac{\alpha}{2}$ for $\frac{2}{3} \leq \alpha \leq 1$ are the best possible.

for recent results concerning Ostrowski inequality for mappings of bounded variation see [4], [5] and [11]–[14].
In [1], Alomari and Dragomir have proved several inequalities for various Newton–Cotes formulae, among others they obtained the following inequality:

Theorem 1.5. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then for all $\lambda \in [0,1]$ and $a \leq x \leq \frac{a+b}{2}$, we have the inequality*

$$\left|(b-a)\left[\lambda\frac{f(x)+f(a+b-x)}{2} + (1-\lambda)f\left(\frac{a+b}{2}\right)\right] - \int_a^b f(t)\,dt\right|$$
$$\leq \frac{1}{2}\max\left\{2(x-a), \lambda(b-a) - 2(x-a), (1-\lambda)(b-a)\right\} \cdot \bigvee_a^b(f). \tag{1.5}$$

For instance, if we choose $x = a$, then we get

$$\left|(b-a)\left[\lambda\frac{f(a)+f(b)}{2} + (1-\lambda)f\left(\frac{a+b}{2}\right)\right] - \int_a^b f(t)\,dt\right| \leq \frac{(b-a)}{2}\left[\frac{1}{2} + \left|\frac{1}{2} - \lambda\right|\right] \cdot \bigvee_a^b(f). \tag{1.6}$$

Remark 1.6. *We note that the inequality (1.5) may be written as follows:*

$$\left|(b-a)\left[\lambda\frac{f(x)+f(a+b-x)}{2} + (1-\lambda)f\left(\frac{a+b}{2}\right)\right] - \int_a^b f(t)\,dt\right|$$
$$\leq \frac{1}{2}\max\left\{2(x-a), \lambda(b-a) - 2(x-a), (1-\lambda)(b-a)\right\} \cdot \bigvee_a^b(f)$$
$$\leq \max\left\{\frac{\lambda}{4}(b-a) + \left|(x-a) - \frac{\lambda(b-a)}{4}\right|, \frac{(1-\lambda)(b-a)}{2}\right\} \cdot \bigvee_a^b(f). \tag{1.7}$$

In the recent work [2], Alomari has proved the following generalization of companion of Ostrowski's inequality for mappings of bounded variation (see also [3]):

Theorem 1.7. *Under the assumptions of Theorem 1.4, we have*

$$\left|\left[\alpha\frac{f(a)+f(b)}{2} + (1-\alpha)\frac{f(x)+f(a+b-x)}{2}\right]\int_a^b g(t)\,dt - \int_a^b f(t)g(t)\,dt\right| \leq M' \cdot \bigvee_a^b(f) \tag{1.8}$$

where,

$$M' = \begin{cases} \max\left\{\frac{1-\alpha}{4}\int_a^b g(t)\,dt + \left|h(x) - \left[\frac{3-\alpha}{4}h(a) + \frac{\alpha+1}{4}h(b)\right]\right|,\right. \\ \left.\frac{1-\alpha}{4}\int_a^b g(t)\,dt + \left|h(a+b-x) - \left[\frac{1+\alpha}{4}h(a) + \frac{3-\alpha}{4}h(b)\right]\right|\right\}; & 0 \leq \alpha \leq \frac{1}{2} \\ \max\left\{\frac{1-\alpha}{4}\int_a^b g(t)\,dt + \left|h(x) - \left[\frac{3-\alpha}{4}h(a) + \frac{\alpha+1}{4}h(b)\right]\right|, \frac{1-\alpha}{4}\int_a^b g(t)\,dt\right. \\ \left.+\left|h(a+b-x) - \left[\frac{1+\alpha}{4}h(a) + \frac{3-\alpha}{4}h(b)\right]\right|, \frac{\alpha}{2}\int_a^b g(t)\,dt\right\}; & \frac{1}{2} < \alpha < \frac{2}{3} \\ \frac{\alpha}{2}\int_a^b g(t)\,dt; & \frac{2}{3} \leq \alpha \leq 1 \end{cases}$$

for all $x \in [c, \frac{c+d}{2}]$, where, $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$. Furthermore, the constant $\frac{1-\alpha}{4}$ for $0 \leq \alpha \leq \frac{1}{2}$ and the constant $\frac{\alpha}{2}$ for $\frac{2}{3} \leq \alpha \leq 1$ are the best possible.

In this paper, a weighted version of Alomari's result (1.5) is proved. Therefore, several weighted inequalities are deduced. Application to a composite quadrature rule is pointed out.

2. A weighted companion of Ostrowski–Midpoint inequality

Let us start with following result.

Theorem 2.1. *Under the assumptions of Theorem 1.4, we have*

$$\left|\left[\alpha\frac{f(x)+f(a+b-x)}{2}+(1-\alpha)f\left(\frac{a+b}{2}\right)\right]\int_a^b g(t)\,dt-\int_a^b f(t)g(t)\,dt\right|\le K\cdot\bigvee_a^b(f) \tag{2.1}$$

where,

$$K=\begin{cases}\max\left\{\frac{\alpha}{4}\int_a^b g(t)\,dt+\left|h(x)-\left[\left(1-\frac{\alpha}{4}\right)h(a)+\frac{\alpha}{4}h(b)\right]\right|,\right.\\ \left.\frac{\alpha}{4}\int_a^b g(t)\,dt+\left|\left[\frac{\alpha}{4}h(a)+\left(1-\frac{\alpha}{4}\right)h(b)\right]-h(a+b-x)\right|\right\}, & \frac{2}{3}\le\alpha\le 1\\ \\ \max\left\{\frac{\alpha}{4}\int_a^b g(t)\,dt+\left|h(x)-\left[\left(1-\frac{\alpha}{4}\right)h(a)+\frac{\alpha}{4}h(b)\right]\right|,\right.\\ \frac{\alpha}{4}\int_a^b g(t)\,dt+\left|\left[\frac{\alpha}{4}h(a)+\left(1-\frac{\alpha}{4}\right)h(b)\right]-h(a+b-x)\right|,\\ \left.\frac{(1-\alpha)}{2}\int_a^b g(t)\,dt+\left|\frac{h(a)+h(b)}{2}-h\left(\frac{a+b}{2}\right)\right|\right\}, & \frac{1}{2}<\alpha<\frac{2}{3}\\ \\ \frac{(1-\alpha)}{2}\int_a^b g(t)\,dt+\left|\frac{h(a)+h(b)}{2}-h\left(\frac{a+b}{2}\right)\right|, & 0\le\alpha\le\frac{1}{2}\end{cases}$$

for all $\alpha\in[0,1]$ and $a\le x\le h^{-1}\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right)$, where $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$. Furthermore, the constant $\frac{\alpha}{4}$ for $\frac{2}{3}\le\alpha\le 1$ is the best possible.

Proof. Define the mapping

$$S_h(t)=\begin{cases}h(t)-h(a), & t\in[a,x]\\ h(t)-\left[\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right], & t\in\left(x,\frac{a+b}{2}\right]\\ h(t)-\left[\frac{\alpha}{2}h(a)+\left(1-\frac{\alpha}{2}\right)h(b)\right], & t\in\left(\frac{a+b}{2},a+b-x\right]\\ h(t)-h(b), & t\in(a+b-x,b]\end{cases}$$

for all $\alpha\in[0,1]$ and $a\le x\le h^{-1}\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right)$.
Using integration by parts, we have the following identity:

$$\begin{aligned}\int_a^b S_h(t)\,df(t)&=[h(t)-h(a)]\cdot f(t)|_a^x-\int_a^x f(t)g(t)\,dt\\&\quad+\left[h(t)-\left[\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right]\right]\cdot f(t)\Big|_x^{\frac{a+b}{2}}-\int_x^{\frac{a+b}{2}}f(t)g(t)\,dt\\&\quad+\left[h(t)-\left[\frac{\alpha}{2}h(a)+\left(1-\frac{\alpha}{2}\right)h(b)\right]\right]\cdot f(t)\Big|_{\frac{a+b}{2}}^{a+b-x}-\int_{\frac{a+b}{2}}^{a+b-x}f(t)g(t)\,dt\\&\quad+[h(t)-h(b)]\cdot f(t)|_{a+b-x}^b-\int_{a+b-x}^b f(t)g(t)\,dt\\&=\left[\alpha\frac{f(x)+f(a+b-x)}{2}+(1-\alpha)f\left(\frac{a+b}{2}\right)\right][h(b)-h(a)]-\int_a^b f(t)g(t)\,dt\\&=\left[\alpha\frac{f(x)+f(a+b-x)}{2}+(1-\alpha)f\left(\frac{a+b}{2}\right)\right]\int_a^b g(t)\,dt-\int_a^b f(t)g(t)\,dt.\end{aligned}$$

Now, we use the fact that for a continuous function $p:[a,b]\to\mathbb{R}$ and a function $\nu:[a,b]\to\mathbb{R}$ of bounded variation, one has the inequality

$$\left|\int_a^b p(t)\,d\nu(t)\right|\le\sup_{t\in[a,b]}|p(t)|\bigvee_a^b(\nu). \tag{2.2}$$

Applying the inequality (2.2) for $p(t)=S_h(t)$, as above and $\nu(t)=f(t)$, $t\in[a,b]$, we get

$$\begin{aligned}&\left|\left[\alpha\frac{f(x)+f(a+b-x)}{2}+(1-\alpha)f\left(\frac{a+b}{2}\right)\right]\int_a^b g(t)\,dt-\int_a^b f(t)g(t)\,dt\right|\\&\le\left|\int_a^b S_h(t)\,df(t)\right|\le\sup_{t\in[a,b]}|S_h(t)|\bigvee_a^b(f),\end{aligned}$$

where,

$$\sup_{t\in[a,b]}|S_h(t)| = \max\left\{\frac{\alpha}{4}[h(b)-h(a)]+\left|h(x)-\left[\left(1-\frac{\alpha}{4}\right)h(a)+\frac{\alpha}{4}h(b)\right]\right|,\right.$$
$$\frac{\alpha}{4}[h(b)-h(a)]+\left|\left[\frac{\alpha}{4}h(a)+\left(1-\frac{\alpha}{4}\right)h(b)\right]-h(a+b-x)\right|$$
$$\left.\frac{(1-\alpha)}{2}[h(b)-h(a)]+\left|\frac{h(a)+h(b)}{2}-h\left(\frac{a+b}{2}\right)\right|\right\}$$
$$= \max\left\{\frac{\alpha}{4}\int_a^b g(t)\,dt+\left|h(x)-\left[\left(1-\frac{\alpha}{4}\right)h(a)+\frac{\alpha}{4}h(b)\right]\right|,\right.$$
$$\frac{\alpha}{4}\int_a^b g(t)\,dt+\left|\left[\frac{\alpha}{4}h(a)+\left(1-\frac{\alpha}{4}\right)h(b)\right]-h(a+b-x)\right|,$$
$$\left.\frac{(1-\alpha)}{2}\int_a^b g(t)\,dt+\left|\frac{h(a)+h(b)}{2}-h\left(\frac{a+b}{2}\right)\right|\right\}$$

and thus we obtain the desired result in (2.1).
To prove the sharpness of the constant $\frac{\alpha}{4}$, for $\frac{1}{2}\leq\alpha\leq 1$ assume that (2.1) holds with a constant $C_1>0$, i.e.,

$$\left|\left[\alpha\frac{f(x)+f(a+b-x)}{2}+(1-\alpha)f\left(\frac{a+b}{2}\right)\right]\int_a^b g(t)\,dt-\int_a^b f(t)g(t)\,dt\right|$$
$$\leq \max\left\{C_1\int_a^b g(t)\,dt+\left|h(x)-\left[\left(1-\frac{\alpha}{4}\right)h(a)+\frac{\alpha}{4}h(b)\right]\right|, C_1\int_a^b g(t)\,dt+\left|\left[\frac{\alpha}{4}h(a)+\left(1-\frac{\alpha}{4}\right)h(b)\right]-h(a+b-x)\right|\right\}\cdot\bigvee_a^b(f). \quad (2.3)$$

Without loss of generality, assume that the maximum of the right hand side is the first term i.e.,

$$\left|\left[\alpha\frac{f(x)+f(a+b-x)}{2}+(1-\alpha)f\left(\frac{a+b}{2}\right)\right]\int_a^b g(t)\,dt-\int_a^b f(t)g(t)\,dt\right|$$
$$\leq\left[C_1\int_a^b g(t)\,dt+\left|h(x)-\left[\left(1-\frac{\alpha}{4}\right)h(a)+\frac{\alpha}{4}h(b)\right]\right|\right]\cdot\bigvee_a^b(f) \quad (2.4)$$

Consider the mapping

$$f(t)=\begin{cases}0, & t\in[a,b]\setminus\left\{h^{-1}\left(\left(1-\frac{\alpha}{4}\right)h(a)+\frac{\alpha}{4}h(b)\right)\right\}\\[2mm] \frac{1}{2}, & t=h^{-1}\left(\left(1-\frac{\alpha}{4}\right)h(a)+\frac{\alpha}{4}h(b)\right)\end{cases}$$

Then f is with bounded variation on $[a,b]$, and $\int_a^b f(t)g(t)\,dt=0$, $\bigvee_a^b(f)=1$, and for $x=h^{-1}\left(\left(1-\frac{\alpha}{4}\right)h(a)+\frac{\alpha}{4}h(b)\right)$, making of use (2.4), we get

$$\frac{\alpha}{4}\leq C_1,$$

which implies that the constant $\frac{\alpha}{4}$ is the best possible.
Now, assume that the maximum of the right hand side is the second term i.e.,

$$\left|\left[\alpha\frac{f(x)+f(a+b-x)}{2}+(1-\alpha)f\left(\frac{a+b}{2}\right)\right]\int_a^b g(t)\,dt-\int_a^b f(t)g(t)\,dt\right|$$
$$\leq\left[C_1\int_a^b g(t)\,dt+\left|\left[\frac{\alpha}{4}h(a)+\left(1-\frac{\alpha}{4}\right)h(b)\right]-h(a+b-x)\right|\right]\cdot\bigvee_a^b(f) \quad (2.5)$$

Consider the mapping

$$f(t)=\begin{cases}0, & t\in[a,b]\setminus\left\{a+b-h^{-1}\left(\frac{\alpha}{4}h(a)+\left(1-\frac{\alpha}{4}\right)h(b)\right)\right\}\\[2mm] \frac{1}{2}, & t=a+b-h^{-1}\left(\frac{\alpha}{4}h(a)+\left(1-\frac{\alpha}{4}\right)h(b)\right)\end{cases}$$

Then f is with bounded variation on $[a,b]$, and $\int_a^b f(t)g(t)\,dt=0$, $\bigvee_a^b(f)=1$, and for $x=a+b-h^{-1}\left(\frac{\alpha}{4}h(a)+\left(1-\frac{\alpha}{4}\right)h(b)\right)$, making of use (2.5), we get

$$\frac{\alpha}{4}\leq C_1,$$

which implies that the constant $\frac{\alpha}{4}$ is the best possible. Therefore, $\frac{\alpha}{4}$ is the best possible for (2.3). Thus, the proof of (2.1) is completely established. □

Remark 2.2. *If we choose $h(t)=t$ and $g(t)=1$, then the inequality (2.1) reduces to (1.7).*

Corollary 1. *In (2.1), choose $x = a$, then we get*

$$\left|\left[\alpha\frac{f(a)+f(b)}{2}+(1-\alpha)f\left(\frac{a+b}{2}\right)\right]\int_a^b g(t)\,dt-\int_a^b f(t)g(t)\,dt\right|$$
$$\leq \max\left\{\frac{\alpha}{2}\int_a^b g(t)\,dt, \frac{(1-\alpha)}{2}\int_a^b g(t)\,dt+\left|\frac{h(a)+h(b)}{2}-h\left(\frac{a+b}{2}\right)\right|\right\}\cdot\bigvee_a^b(f), \quad (2.6)$$

for all $x \in [a, \frac{a+b}{2}]$, which is the "weighted version of (1.6)".

Remark 2.3. *In Corollary 1, choose*

1. $\alpha = 0$, *then we get*

$$\left|f\left(\frac{a+b}{2}\right)\int_a^b g(t)\,dt-\int_a^b f(t)g(t)\,dt\right|\leq\left[\frac{1}{2}\int_a^b g(t)\,dt+\left|\frac{h(a)+h(b)}{2}-h\left(\frac{a+b}{2}\right)\right|\right]\cdot\bigvee_a^b(f), \quad (2.7)$$

which is the "weighted midpoint inequality".

2. $\alpha = \frac{1}{3}$, *then we get*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\int_a^b g(t)\,dt-\int_a^b f(t)g(t)\,dt\right|$$
$$\leq \max\left\{\frac{1}{6}\int_a^b g(t)\,dt, \frac{1}{3}\int_a^b g(t)\,dt+\left|\frac{h(a)+h(b)}{2}-h\left(\frac{a+b}{2}\right)\right|\right\}\cdot\bigvee_a^b(f), \quad (2.8)$$

which is the "weighted Simpson inequality".

3. $\alpha = 1$, *then we get*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt-\int_a^b f(t)g(t)\,dt\right|\leq\frac{1}{2}\int_a^b g(t)\,dt\cdot\bigvee_a^b(f), \quad (2.9)$$

which is the "weighted trapezoid inequality".

Corollary 2. *Let $0 \leq \alpha \leq 1$. Let $f \in C^{(1)}[a,b]$. Then we have the inequality*

$$\left|\left[\alpha\frac{f(x)+f(a+b-x)}{2}+(1-\alpha)f\left(\frac{a+b}{2}\right)\right]\int_a^b g(t)\,dt-\int_a^b f(t)g(t)\,dt\right|\leq K\cdot\left\|f'\right\|_1, \quad (2.10)$$

for all $a \leq x \leq h^{-1}\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right)$, where $\|\cdot\|_1$ is the L_1 norm, namely $\|f'\|_1 := \int_a^b |f'(t)|\,dt$.

Corollary 3. *Let $0 \leq \alpha \leq 1$. Let $f : [a,b] \to \mathbb{R}$ be a Lipschitzian mapping with the constant $L > 0$. Then we have the inequality*

$$\left|\left[\alpha\frac{f(x)+f(a+b-x)}{2}+(1-\alpha)f\left(\frac{a+b}{2}\right)\right]\int_a^b g(t)\,dt-\int_a^b f(t)g(t)\,dt\right|\leq KL(b-a), \quad (2.11)$$

for all $a \leq x \leq h^{-1}\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right)$.

Corollary 4. *Let $0 \leq \alpha \leq 1$. Let $f : [a,b] \to \mathbb{R}$ be a monotonic mapping. Then we have the inequality*

$$\left|\left[\alpha\frac{f(x)+f(a+b-x)}{2}+(1-\alpha)f\left(\frac{a+b}{2}\right)\right]\int_a^b g(t)\,dt-\int_a^b f(t)g(t)\,dt\right|\leq K\left|f(b)-f(a)\right|, \quad (2.12)$$

for all $a \leq x \leq h^{-1}\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right)$.

3. Application to a quadrature rule

Let $I_n : a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a,b]$ and $c_i = h^{-1}\left(\left(1-\frac{\alpha}{2}\right)h(x_i)+\frac{\alpha}{2}h(x_{i+1})\right)$, $\xi_i \in [x_i, c_i]$ $(i = 0, 1, \cdots, n-1)$. Put $L_i = h(x_{i+1}) - h(x_i) = \int_{x_i}^{x_{i+1}} g(t)dt$, and define the sum

$$A_\alpha(f,g,h,I_n,\xi)=\sum_{i=0}^{n-1}\left[(1-\alpha)\cdot f\left(\frac{x_i+x_{i+1}}{2}\right)+\alpha\cdot\frac{f(\xi_i)+f(x_i+x_{i+1}-\xi_i)}{2}\right]L_i \quad (3.1)$$

for all $\alpha \in [0,1]$. In the following we propose an approximation for the integral $\int_a^b f(t)g(t)\,dt$.

Theorem 3.1. *Let f,g,h be defined as in Theorem 2.1, then we have*

$$\int_a^b f(t)g(t)\,dt = A_\alpha(f,g,h,I_n,\xi)+R_\alpha(f,g,h,I_n,\xi). \tag{3.2}$$

where, $A_\alpha(f,g,h,I_n,\xi)$ is given in (3.1) and the remainder $R_\alpha(f,g,h,I_n,\xi)$ satisfies the bounds

$$R_\alpha(f,g,h,I_n,\xi) \le \sum_{i=0}^{n-1} K_{i,\alpha}\cdot\bigvee_{x_i}^{x_{i+1}}(f) \le M_{1,\alpha}\cdot\bigvee_a^b(f),$$

where,

$$K_{i,\alpha}=\begin{cases}\max\left\{\frac{\alpha}{4}L_i+\left|h(\xi_i)-\left[\left(1-\frac{\alpha}{4}\right)h(x_i)+\frac{\alpha}{4}h(x_{i+1})\right]\right|,\right. \\ \left.\frac{\alpha}{4}L_i+\left|\left[\frac{\alpha}{4}h(x_i)+\left(1-\frac{\alpha}{4}\right)h(x_{i+1})\right]-h(x_i+x_{i+1}-\xi_i)\right|\right\}, & \frac{2}{3}\le\alpha\le 1\\ \max\left\{\frac{\alpha}{4}L_i+\left|h(\xi_i)-\left[\left(1-\frac{\alpha}{4}\right)h(x_i)+\frac{\alpha}{4}h(x_{i+1})\right]\right|,\right. \\ \frac{\alpha}{4}L_i+\left|\left[\frac{\alpha}{4}h(x_i)+\left(1-\frac{\alpha}{4}\right)h(x_{i+1})\right]-h(x_i+x_{i+1}-\xi_i)\right|, \\ \left.\frac{(1-\alpha)}{2}L_i+\left|\frac{h(x_i)+h(x_{i+1})}{2}-h\left(\frac{x_i+x_{i+1}}{2}\right)\right|\right\}, & \frac{1}{2}<\alpha<\frac{2}{3}\\ \frac{(1-\alpha)}{2}L_i+\left|\frac{h(x_i)+h(x_{i+1})}{2}-h\left(\frac{x_i+x_{i+1}}{2}\right)\right|, & 0\le\alpha\le\frac{1}{2}\end{cases}$$

$$M_{1,\alpha}:=\begin{cases}\max\left\{\frac{\alpha}{4}\nu(L)+\max\limits_{i=0,1,\dots,n-1}\left|h(\xi_i)-\left[\left(1-\frac{\alpha}{4}\right)h(x_i)+\frac{\alpha}{4}h(x_{i+1})\right]\right|,\right. & \frac{2}{3}\le\alpha\le 1\\ \left.\frac{\alpha}{4}\nu(L)+\max\limits_{i=0,1,\dots,n-1}\left|\left[\frac{\alpha}{4}h(x_i)+\left(1-\frac{\alpha}{4}\right)h(x_{i+1})\right]-h(x_i+x_{i+1}-\xi_i)\right|\right\}, & \\ \max\left\{\frac{\alpha}{4}\nu(L)+\max\limits_{i=0,1,\dots,n-1}\left|h(\xi_i)-\left[\left(1-\frac{\alpha}{4}\right)h(x_i)+\frac{\alpha}{4}h(x_{i+1})\right]\right|,\right. & \frac{1}{2}<\alpha<\frac{2}{3}\\ \left.\frac{\alpha}{4}\nu(L)+\max\limits_{i=0,1,\dots,n-1}\left|\left[\frac{\alpha}{4}h(x_i)+\left(1-\frac{\alpha}{4}\right)h(x_{i+1})\right]-h(x_i+x_{i+1}-\xi_i)\right|,\frac{1-\alpha}{2}\nu(L)\right\}, & \\ \frac{1-\alpha}{2}\nu(L)+\max\limits_{i=0,1,\dots,n-1}\left|\frac{h(x_i)+h(x_{i+1})}{2}-h\left(\frac{x_i+x_{i+1}}{2}\right)\right|, & 0\le\alpha\le\frac{1}{2}\end{cases}$$

and $\nu(L):=\max\{L_i : i=0,1,\dots,n-1\}$. In the last inequality the constant $\frac{\alpha}{4}$ for $\frac{2}{3}\le\alpha\le 1$ is the best possible.

Proof. Applying Theorem 2.1 on the intervals $[x_i,x_{i+1}]$, we may state that

$$\left|\left[(1-\alpha)\cdot f\left(\frac{x_i+x_{i+1}}{2}\right)+\alpha\cdot\frac{f(\xi_i)+f(x_i+x_{i+1}-\xi_i)}{2}\right]L_i-\int_{x_i}^{x_{i+1}} f(t)g(t)\,dt\right| \le K_{i,\alpha}\cdot\bigvee_{x_i}^{x_{i+1}}(f)$$

for all $i=0,1,\cdots,n-1$.
Using this and the generalized triangle inequality, we have

$$\begin{aligned}&R_\alpha(f,g,h,I_n,\xi)\\ &\le\sum_{i=0}^{k-1}\left|\left[(1-\alpha)\cdot f\left(\frac{x_i+x_{i+1}}{2}\right)+\alpha\cdot\frac{f(\xi_i)+f(x_i+x_{i+1}-\xi_i)}{2}\right]L_i-\int_{x_i}^{x_{i+1}} f(t)g(t)\,dt\right|\\ &\le\sum_{i=0}^{k-1}K_{i,\alpha}\cdot\bigvee_{x_i}^{x_{i+1}}(f)\le\max_{i=0,1,\dots,n-1}\{K_{i,\alpha}\}\cdot\sum_{i=0}^{k-1}\bigvee_{x_i}^{x_{i+1}}(f)\le M_{1,\alpha}\cdot\bigvee_a^b(f)\end{aligned}$$

□

Corollary 5. *In Theorem 3.1, choose*

1. *$\alpha=0$, then we get*

$$\int_a^b f(t)g(t)\,dt = A_0(f,g,h,I_n,\xi)+R_0(f,g,h,I_n,\xi). \tag{3.3}$$

where, $A_0(f,g,h,I_n,\xi)$ is given in (3.1) and the remainder $R_0(f,g,h,I_n,\xi)$ satisfies the bounds

$$R_0(f,g,h,I_n,\xi)\le\sum_{i=0}^{n-1}K_{i,0}\cdot\bigvee_{x_i}^{x_{i+1}}(f)\le M_{1,0}\cdot\bigvee_a^b(f),$$

2. $\alpha = 1$, *then we get*

$$\int_a^b f(t)\,g(t)\,dt = A_1(f,g,h,I_n,\xi) + R_1(f,g,h,I_n,\xi). \tag{3.4}$$

where, $A_1(f,g,h,I_n,\xi)$ *is given in (3.1) and the remainder* $R_1(f,g,h,I_n,\xi)$ *satisfies the bounds*

$$R_1(f,g,h,I_n,\xi) \le \sum_{i=0}^{n-1} K_{i,1} \cdot \bigvee_{x_i}^{x_{i+1}}(f) \le M_{1,1} \cdot \bigvee_a^b (f),$$

References

[1] M.W. Alomari and S.S. Dragomir, Various error estimations for several Newton–Cotes quadrature formulae in terms of at most first derivative and applications in numericaal integration, Jordan J. Math. & Stat., **7** (2) 2014, 89–108.

[2] M.W. Alomari, A generalization of weighted companion of Ostrowski integral inequality for mappings of bounded variation, *Preprint, RGMIA Res. Rep. Coll.*, 14 (2011), article87. [http://ajmaa.org/RGMIA/v14.php]

[3] M.W. Alomari, A companion of Dragomir's generalization of Ostrowski's inequality and applications in numerical integration, Ukrainian Math. J., **64** (4) (2012), 491–510

[4] H. Budak, F. Usta and M.Z. Sarikaya, New upper bounds of ostrowski type integral inequalities utilizing Taylor expansion, Hacettepe Journal of Mathematics and Statistics, 47 (3) (2018), 567–578.

[5] H. Budak, F. Usta, M.Z. Sarikaya and M.E. Ozdemir, On generalization of midpoint type inequalities with generalized fractional integral operators, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales - Serie A: Matematicas, 113(2) (2019), 769–790.

[6] A. Ostrowski, Über die absolutabweichung einer differentiiebaren funktion vonihrem integralmittelwert, Comment. Math. Helv., 10 (1938) 226–227.

[7] Ostrowski integral inequality for mappings of bounded variation, Bull. Austral. Math. Soc., 60 (1999) 495–508.

[8] S.S. Dragomir, A companion of Ostrowski's inequality for functions of bounded variation and applications, *RGMIA Preprint*, Vol. 5 Supp. (2002) article No. 28. [http://ajmaa.org/RGMIA/papers/v5e/COIFBVApp.pdf]

[9] A. Guessab and G. Schmeisser, Sharp integral inequalities of the Hermite-Hadamard type, J. Approx. Th., 115 (2002), 260–288.

[10] K.L. Tseng, S.R. Hwang, S.S. Dragomir, Generalizations of weighted Ostrowski type inequalities for mappings of bounded variation and their applications, Computers and Mathematics with Applications, 55 (2008) 1785–1793.

[11] K.L. Tseng, Improvements of some inequalites of Ostrowski type and their applications, Taiwanese J. Math., 12(9) (2008), 2427–2441.

[12] K.L. Tseng, S.R. Hwang, G.S. Yang and Y.M. Chou, Improvements of the Ostrowski integral inequality for mappings of bounded variation I, Appl. Math. Comp., 217 (2010) 2348–2355.

[13] F. Usta, H. Budak, M.Z. Sarikaya and E. Set, On generalization of trapezoid type inequalities for s-convex functions with generalized fractional integral operators, Filomat 32 (6), 2153–2171

[14] F. Usta, H. Budak and M.Z. Sarikaya, Montgomery identities and Ostrowski type inequalities for fractional integral operators, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales - Serie A: Matematicas, 113 (2) (2019), 1059–1080.