Konuralp Journal of Mathematics, 7 (2) (2019) 352-358

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath
e-ISSN: 2147-625X

Convexity and Hermite-Hadamard Type Inequality via Non-Newtonian Calculus

Erdal Ünlüyol[1*] and Seren Salaş[2]

[1]*Department of Mathematics, Faculty of Science and Arts, Ordu University, Ordu, Turkey*
[2]*Department of Mathematics, Science Institute, Ordu University, Ordu, Turkey*
[*]*Corresponding author E-mail: eunluyol@yahoo.com*

Abstract

In this paper, firstly we research basic definition of convexity in terms of non-Newtonian calculi, i.e. interval, convex set, convexity, etc. Secondly, we deal with the different classes of convexity and generalizations via non-Newtonian calculi. Finally, we reveal the new generalization of the definition of convexity that can reduce many order of convexity and constitute some new Hermite-Hadamard type inequalities for this calculi.

Keywords: *Non-Newtonian calculi, α-convexity, Hermite-Hadamard type inequality*
2010 Mathematics Subject Classification: *26A51, 26D07.*

Isaac Newton and Gottfried Wilhelm Leibnitz obtained the differential and integral calculus, applied in mathematical theory, independently in the second half of the 17^{th} century. Afterwards Leonard Euler deflected calculus by giving a pivotal position and so founded analysis.
Differentiation and integration are basic operations of analysis. Indeed, they are many versions of the subtraction and addition operations on numbers, respectively.
From 1967 till 1970 Michael Grossman and Robert Katz [1] gave definitions of a new kind of derivative and integral, converting the roles of subtraction and addition into division and multiplication, respectively and thus establish a new calculus, called Non-Newtonian Calculus.
Non-Newtonian calculus has a limited area of applications than the classical calculus. But Non-Newtonian calculus can particulary be useful for economics and finance [2].
In this area the reader can refer to a lot of authors, i.e. [1]-[9], and references therein. The reader can refer to the recent papers [10]-[17] related to the multiplicative calculi and related topics
Kadak and Gürefe [9] introduced some characteristic features of weighted means and convex functions in term of the non-Newtonian calculus which is a self-contained system independent of any other system of calculus, i.e. *-convex function.
Some new definitions, theorems and corollaries is obtained for Non-Newtonian Calculi by E. Unluyol et all. [18].

1. Preliminaries

Arithmetic is any system that satisfies the whole of the ordered field axiom whose domain is a subset of $\mathbb{R}$. There are many types arithmetic, all of which are isomorphic that is, structurally equivalent.
A generator α is a one-to-one function whose domain is $\mathbb{R}$ and whose range is a subset $\mathbb{R}_\alpha$ of $\mathbb{R}$ where $\mathbb{R}_\alpha = \{\alpha(x) : x \in \mathbb{R}\}$. Each generator generates exactly one arithmetic, and conversely each arithmetic is generated by exactly one generator.The inverse of the identity function defined by $I(x) = x$ for all $x \in \mathbb{R}$ is itself. In the special cases $\alpha = I$ and $\alpha = exp$, α generates the classical and geometric arithmetic, respectively. By α-arithmetic, we mean the arithmetic whose domain is $\mathbb{R}$ and whose operations are defined as follows: for $x, y \in \mathbb{R}_\alpha$ and generator α,

$$\begin{aligned}
\alpha-\texttt{addition},\ x\dot{+}y &= \alpha\{\alpha^{-1}(x)+\alpha^{-1}(y)\},\\
\alpha-\texttt{subtraction},\ x\dot{-}y &= \alpha\{\alpha^{-1}(x)-\alpha^{-1}(y)\},\\
\alpha-\texttt{multiplication},\ x\dot{\times}y &= \alpha\{\alpha^{-1}(x)\times\alpha^{-1}(y)\},\\
\alpha-\texttt{division},\ x\dot{/}y &= \alpha\{\alpha^{-1}(x)/\alpha^{-1}(y)\},\\
\alpha-\texttt{order},\ x\dot{<}y &\iff \alpha^{-1}(x)<\alpha^{-1}(y).
\end{aligned}$$

Email addresses: **, eunluyol@yahoo.com (Erdal Ünlüyol), serensalas@gmail.com (Seren Salaş)**

As a generator, we choose *exp* function acting from $\mathbb{R}$ into the set $\mathbb{R}_{exp} = (0,\infty)$ as follows:

$$\begin{array}{rcl} \alpha: & \mathbb{R} \longrightarrow & \mathbb{R}_{exp} \\ x & \longrightarrow & y = \alpha(x) = e^x \end{array}$$

It is obvious that α arithmetic reduces to the geometric arithmetic as follows:

$$\begin{array}{rcl} \texttt{geometric addition}, \; x\dot{+}y & = & e^{\{\ln x + \ln y\}} = x.y, \\ \texttt{geometric subtraction}, \; x\dot{-}y & = & e^{\{\ln x \times \ln y\}} = x/y, \\ \texttt{geometric multiplication}, \; x\dot{\times}y & = & e^{\{\ln x \ln y\}} = x^{\ln y} = y^{\ln x}, \\ \texttt{geometric division}, \; x\dot{/}y & = & e^{\{\ln x / \ln y\}} = x^{\frac{1}{\ln y}}, \\ \texttt{geometric order}, \; x\dot{<}y & \Longleftrightarrow & ln(x) < ln(y). \end{array}$$

Definition 1.1. *[1] Let $\alpha(p) = \dot{p}$ for all $p \in \mathbb{Z}$. If for $y \in \mathbb{R}_\alpha$, $y\dot{+}\dot{0} = y$ and $y\dot{\times}\dot{1} = y$, then according to α-addition $\dot{0}(\alpha - zero)$ and $\dot{1}(\alpha - one)$ numbers are called identity and unit elements, respectively.*

Definition 1.2. *[1] Let $\dot{-}\dot{n} = \dot{0}\dot{-}\dot{n} = \alpha(-n)$ for all $n \in \mathbb{Z}$. The set $\mathbb{Z}_\alpha$ or $\mathbb{Z}(N)$ of α-integers is defined, as follows;*

$$\begin{array}{rcl} \mathbb{Z}_\alpha = \mathbb{Z}(N) & = & \{\ldots, \dot{-}\dot{2}, \dot{-}\dot{1}, \dot{0}, \dot{1}, \dot{2}\ldots\} \\ & = & \{\ldots, \alpha(-2), \alpha(-1), \alpha(0), \alpha(1), \alpha(2), \ldots\}. \end{array}$$

Namely, $\mathbb{Z}_\alpha = \mathbb{Z}(N) = \{\dot{n} : \dot{n} = \alpha(n), n \in \mathbb{Z}\}$. Similarly we can define α-real numbers as follows,

$$\mathbb{R}(N) = \mathbb{R}_\alpha = \{\dot{n} : \dot{n} = \alpha(n), n \in \mathbb{R}\}$$

Definition 1.3. *[1] Let $p \in \mathbb{R} \setminus \{0\}$. In this case, $q_p : \mathbb{R} \to \mathbb{R}_q \subseteq \mathbb{R}$ and q_p^{-1} are defined as follows,*

$$q_p(x) = \begin{cases} x^{\frac{1}{p}}, & x > 0; \\ 0, & x = 0; \\ -(-x)^{\frac{1}{p}}, & x < 0. \end{cases}$$

and

$$q_p^{-1}(x) = \begin{cases} x^p, & x > 0; \\ 0, & x = 0; \\ -(-x)^p, & x < 0. \end{cases}$$

Specially, if we take $p = 1$ in q_p-function, then q_p calculus is reduced to the classical calculus.

Definition 1.4. *Let I be an interval in $\mathbb{R}$, $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function. $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be Hermite-Hadamard Type Inequality, if*

$$f\left(\frac{a+b}{2}\right) \le \int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

for all $x \in I$ and $t \in [0,1]$. If the above inequality is reversed, then f is said to be Hermite-Hadamard Type Inequality concave function.

Definition 1.5. *[10] Let I be an interval, $\varphi : I \subset \mathbb{R} \to \mathbb{R}$ be a continuous and strictly monotonic function. $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be $M_\varphi A$ convex, if*

$$f\left(\varphi^{-1}\left(t\varphi(x) + (1-t)\varphi(y)\right)\right) \le tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0,1]$. If the above inequality is reversed, then f is said to be $M_\varphi A$-concave function.

2. Some New Definitions and Theorems About Inequalities via Non-Newtonian Calculus

In this section, we define the notions interval, convex set, convex function etc. in terms of Non-Newtonian Calculi.

Definition 2.1. *Let α is a generator. Then $I_\alpha \subseteq \mathbb{R}_\alpha$ is said to an α-interval on $\mathbb{R}_\alpha$ if for all $x, y \in I_\alpha$;*

1. $\dot{(}x,y\dot{)} := \{z \in I_\alpha : x\dot{<}z\dot{<}y\} \subseteq I_\alpha$,
2. $\dot{(}x,y\dot{]} := \{z \in I_\alpha : x\dot{<}z\dot{\le}y\} \subseteq I_\alpha$,
3. $\dot{[}x,y\dot{)} := \{z \in I_\alpha : x\dot{\le}z\dot{<}y\} \subseteq I_\alpha$,
4. $\dot{[}x,y\dot{]} := \{z \in I_\alpha : x\dot{\le}z\dot{\le}y\} \subseteq I_\alpha$,
5. $\dot{(}x,\dot{+}\dot{\infty}\dot{)} := \{z \in I_\alpha : x\dot{<}z\dot{<}\dot{+}\dot{\infty}\} \subseteq I_\alpha$,
6. $\dot{(}\dot{-}\dot{\infty},y\dot{)} := \{z \in I_\alpha : \dot{-}\dot{\infty}\dot{<}z\dot{<}y\} \subseteq I_\alpha$,
7. $\dot{[}x,\dot{+}\dot{\infty}\dot{)} := \{z \in I_\alpha : x\dot{\le}z\dot{<}\dot{+}\dot{\infty}\} \subseteq I_\alpha$,
8. $\dot{(}\dot{-}\dot{\infty},y\dot{]} := \{z \in I_\alpha : \dot{-}\dot{\infty}\dot{<}z\dot{\le}y\} \subseteq I_\alpha$.

Remark 2.2. *Alternatively, we can express these (1)-(8) α-intervals as follows respectively;*

$$(x,y)_\alpha, (x,y]_\alpha, [x,y)_\alpha, [x,y]_\alpha, (x,+\infty)_\alpha, (-\infty,y)_\alpha, [x,+\infty)_\alpha, (-\infty,y]_\alpha.$$

Remark 2.3. *Afterwards $[x,y]_\alpha,(x,y)_\alpha$ are said to be respectively α-closed interval, α-open interval.*

Definition 2.4. *Let L_α be an α-linear space and $A\subseteq L_\alpha$. A set is said to be an α-convex set, if for all $x,y\in A$*

$$B_\alpha=\{z\in L_\alpha : z=\theta_1\dot{\times}x\dot{+}\theta_2\dot{\times}y,\ \theta_1\dot{+}\theta_2=\dot{1},\ \dot{0}\dot{\leq}\theta_1,\theta_2\dot{\leq}\dot{1}\}\subseteq A.$$

It is immediate that $z=\theta_1\dot{\times}x\dot{+}\theta_2\dot{\times}y,\ \ \theta_1\dot{+}\theta_2=\dot{1},\ \theta_1,\theta_2\in[0,1]_\alpha$ for $x,y\in A, z\in B_\alpha$.

Lemma 2.5. *$I_\alpha:=[x,y]_\alpha$ interval on $\mathbb{R}_\alpha$ is an α-convex set.*

Proof. For $u\in\{\theta_1\dot{\times}x+\theta_2\dot{\times}y:\theta_1,\theta_2\in[\dot{0},\dot{1}],\theta_1\dot{+}\theta_2=\dot{1}\}$ then $u=\theta_1\dot{\times}x\dot{+}\theta_2\dot{\times}y$. In the circumstances, we can write the following,

$$\begin{aligned}
u &= \alpha\Big\{\alpha^{-1}(\theta_1)\alpha^{-1}(x)+\alpha^{-1}(\theta_2)\alpha^{-1}(y)\Big\}\\
\alpha^{-1}\{u\} &= \alpha^{-1}(\theta_1)\alpha^{-1}(x)+\alpha^{-1}(\theta_2)\alpha^{-1}(y)\\
&\leq \min\{\alpha^{-1}(x),\alpha^{-1}(y)\}=\alpha^{-1}(x)\leq\max\{\alpha^{-1}(x),\alpha^{-1}(y)\}=\alpha^{-1}(y)\\
&\Longleftrightarrow \alpha^{-1}(x)\leq\alpha^{-1}(u)\leq\alpha^{-1}(y)\\
&\Longleftrightarrow x\dot{\leq}u\dot{\leq}y\\
&\Longleftrightarrow u\in[x,y]_\alpha.
\end{aligned}$$

So the proof is completed. □

Lemma 2.6. *$A\subset\mathbb{R}_\alpha$ is $\alpha-$convex set if and only if $[\dot{x},\dot{y}]\subset A$ for all $x,y\in A$, such that $x\dot{\leq}y$.*

Theorem 2.7. *$A\subset\mathbb{R}_\alpha$ is α-convex set if and only if for all $a,b\in A,\ \theta_1,\theta_2\in[\dot{0},\dot{1}],\ \theta_1\dot{+}\theta_2=\dot{1}$*

$$\theta_1\dot{\times}a\dot{+}\theta_2\dot{\times}b\in A.$$

In other words, we can write the above condition, as follows:
Let $A\subset\mathbb{R}_\alpha$. Then, for all $a,b\in A$ and $\theta_1\dot{+}\theta_2=\dot{1},\ \theta_1,\theta_2\in[\dot{0},\dot{1}]$

$$\theta_1\dot{\times}a\dot{+}\theta_2\dot{\times}b\in A$$

if and only if for all $a,b\in A$ and $t\in[0,1]$

$$\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b\in A.$$

Proof. By using $\theta_1,\theta_2\in[\dot{0},\dot{1}]$ such that $\theta_1\dot{+}\theta_2=\dot{1}$, we can write the below equality:

$$\alpha(\alpha^{-1}(\theta_1)+\alpha^{-1}(\theta_2))=\alpha(1).$$

From injectivity of α , we have

$$\alpha^{-1}(\theta_1)+\alpha^{-1}(\theta_2)=1.$$

Thus, by choosing $\theta_1=\alpha(t),\ \theta_2=\alpha(1-t)$, then $\theta_1=\alpha(t),\ \theta_2=\alpha(1-t)$,

$$\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b\in A.$$

So we have proved the desired conclusion. □

Definition 2.8. *Let I_α be an $\alpha-$closed interval in $\mathbb{R}_\alpha$. Then the function $f:I_\alpha\longrightarrow\mathbb{R}$ is said to be α_*-convex if*

$$f(\lambda_1\dot{\times}x\dot{+}\lambda_2\dot{\times}y)\leq\theta_1 f(x)+\theta_2 f(y) \tag{2.1}$$

(2.1) holds, where $\lambda_1\dot{+}\lambda_2=\dot{1}$ and $\theta_1+\theta_2=1$ for all $\lambda_1,\lambda_2\in[0,1]_\alpha$ and $\theta_1,\theta_2\in[0,1]$. If we take $\theta_1=\alpha^{-1}(\lambda_1)$ and $\theta_2=\alpha^{-1}(\lambda_2)$ in (2.1), then we obtain

$$f(\lambda_1\dot{\times}x\dot{+}\lambda_2\dot{\times}y)\leq\alpha^{-1}(\lambda_1)\times f(x)+\alpha^{-1}(\lambda_2)\times f(y). \tag{2.2}$$

Therefore, by combining this with the generator α, we deduce that

$$f\Big(\alpha\Big\{\alpha^{-1}(\lambda_1).\alpha^{-1}(x)+\alpha^{-1}(\lambda_2).\alpha^{-1}(y)\Big\}\Big)\leq\theta_1.f(x)+\theta_2.f(y) \tag{2.3}$$

If (2.1) is strict for all $x\neq y$, then f said to be strictly α_*-convex. If the inequality in (2.1) is reversed, then fis said to be α_*-concave. Depending on the choice of generator functions, the definition of α_*-convex in (2.1) can be interpreted as follows.

Theorem 2.9. *The function $f:I_\alpha\subset\mathbb{R}_\alpha\to\mathbb{R}$ is an α_*-convex if and only if for $\alpha:\mathbb{R}\longrightarrow\mathbb{R}_\alpha$,*

$$f\circ\alpha:\alpha^{-1}(I_\alpha)\subset\mathbb{R}\to\mathbb{R}$$

is a convex function.

Proof. Firstly we assume that f is an α_*-convex function, then we write for all $\theta_1,\theta_2 \in [\dot{0},\dot{1}]$, such that $\theta_1 \dot{+} \theta_2 = \dot{1}$ and for all $a,b\in I_\alpha$, we have the following inequality

$$f(\theta_1 \dot{\times} a \dot{+} \theta_2 \dot{\times} b) \leq \alpha^{-1}(\theta_1) f(a) + \alpha^{-1}(\theta_2) f(b). \tag{2.4}$$

In (2.4), if we choose $a,b\in I_\alpha$, $t\in[0,1]$, $\alpha^{-1}(\theta_1)=t$ and $\alpha^{-1}(\theta_1)=(1-t)$, then we obtain following inequality

$$f(\alpha(t)\dot{\times} a \dot{+} \alpha(1-t)\dot{\times} b) \leq t f(a) + (1-t) f(b)$$

Then, definition of α-addition and α-multiplication , we get the following inequality

$$\begin{aligned} f\Big(\alpha(t\alpha^{-1}(a)+(1-t)\alpha^{-1}(b))\Big) &\leq tf(a)+(1-t)f(b) \\ (f\circ\alpha)(t\alpha^{-1}(a)+(1-t)\alpha^{-1}(b)) &\leq t(f\circ\alpha)(\alpha^{-1}(a))+(1-t)(f\circ\alpha)(\alpha^{-1}(b)). \end{aligned}$$

Hence, $(f\circ\alpha)$ is convex in $\alpha^{-1}(I_\alpha)$. Secondly vice versa. □

Definition 2.10. *If we choose $\beta = I$ in definitions of $*$-differential and $*$-derivative in [1], then we say, these definitions, respectively, α_*-differential and α_*-derivative.*

Remark 2.11. *Let $f : I_\alpha \subset \mathbb{R}_\alpha \to \mathbb{R}$ is a second order α_*-differentiable function and $\alpha : \mathbb{R}\to\mathbb{R}_\alpha$ is a generator. Then f is α_*-convex if and only if for all $x\in\alpha^{-1}(I_\alpha)$,*

$$(f\circ\alpha)''(x)\geq 0.$$

Remark 2.12. *If we take $\alpha(x) = I(x)$ and $\alpha^{-1}(\lambda_1) = t,\ \alpha^{-1}(\lambda_2) = (1-t)$ in (2.1), then we obtain the definition of usual convex function, namely for all $t\in[0,1]$,*

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y).$$

Remark 2.13. *If we take $\alpha(x) = I(x)$, $\mathbb{R}=\mathbb{R}_\alpha$ in (2.2), then we obtain α-convex function [18], that is for all $t\in[0,1]$*

$$f(tx+(1-t)y)\leq \theta_1 \dot{\times} f(x) \dot{+} \theta_2 \dot{\times} f(y).$$

Remark 2.14. *Let $I_\alpha \subset (\dot{0},\dot{\infty})$ be an α-interval. In this case, if we take $\alpha(x) = q_p(x)$ and $\alpha^{-1}(\lambda_1)=t,\ \alpha^{-1}(\lambda_2)=(1-t)$ in (2.2), then we obtain p-convex function[19], so for all $t\in[0,1]$,*

$$f\Big(((tx)^p+((1-t)y)^p))^{\frac{1}{p}}\Big)\leq tf(x)+(1-t)f(y).$$

Remark 2.15. *If we take $\alpha(x) = exp(x)$ and $\alpha^{-1}(\lambda_1)=t,\ \alpha^{-1}(\lambda_2)=(1-t)$ in (2.2), then we get*

$$f(x^{lnt}.y^{ln(1-t)})\leq tf(x)+(1-t)f(y),\ (t\in[1,e]),$$

$f : I_{\exp} \longrightarrow \mathbb{R}$ is geometric convex function [20].

Remark 2.16. *If we take $\alpha(x) = \varphi^{-1}(x)$ and $\alpha^{-1}(\lambda_1)=t,\ \alpha^{-1}(\lambda_2)=(1-t)$ in (2.2), then we obtain*

$$f\big(\varphi^{-1}\big(t\varphi(x)+(1-t)\varphi(y)\big)\leq tf(x)+(1-t)f(y),\ (t\in[0,1]),$$

$M_\varphi A$ convex function [21].

Remark 2.17. *Let $I_\alpha \subset (\dot{0},\dot{\infty})$ is an α-interval. Thus if we take $\alpha(x)=I(x)=\frac{1}{x}$, and $\alpha^{-1}(\lambda_1)=t,\ \alpha^{-1}(\lambda_2)=(1-t)$ in (2.2), then we obtain*

$$f\Big(\frac{xy}{x(1-t)+yt}\Big)\leq tf(x)+(1-t)f(y),\ (t\in[0,1]),$$

Harmonically convex function [22].

3. Hermite-Hadamard Type Inequality in terms of Non-Newtonian Calculus and its Some Properties

Theorem 3.1. *Let I_α be a closed interval in $\mathbb{R}_\alpha$, and $f : I_\alpha\to\mathbb{R}$ also be any α_*-convex function. Then the following double inequality holds for all $a,b\in I_\alpha$,*

$$f\left(\alpha\left(\frac{1}{2}\right)\dot{\times}(a\dot{+}b)\right)\leq\int_0^1 f(\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b)dt\leq\frac{f(a)+f(b)}{2} \tag{3.1}$$

Proof. Since f is an α_*-convex function, we can write the following inequality

$$f(\lambda_1\dot{\times}a\dot{+}\lambda_2\dot{\times}b)\leq\alpha^{-1}(\lambda_1)f(a)+\alpha^{-1}(\lambda_2)f(b). \tag{3.2}$$

If we take $\alpha^{-1}(\lambda_1)=t$, $\alpha^{-1}(\lambda_2)=1-t$, then we have

$$f(\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b)\leq tf(a)+(1-t)f(b). \tag{3.3}$$

and integrating with respect to t over $\cdot$ $[0,1]$, then we have the following inequality, for $a,b\in I_\alpha$

$$\begin{aligned} \int_0^1 f\big(\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b\big)dt &\leq \int_0^1 \big(tf(a)+(1-t)f(b)\big)dt \\ \int_0^1 f\big(\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b\big)dt &\leq \frac{f(a)+f(b)}{2} \end{aligned} \tag{3.4}$$

If we choose $t=\frac{1}{2}$, $a=\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b$ and $b=\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a$ in (3.3), then we get the followings,

$$\begin{aligned} f\big(\alpha(\tfrac{1}{2})\dot{\times}a\dot{+}\alpha(\tfrac{1}{2})\dot{\times}b\big) &= f(\alpha(\tfrac{1}{2})\dot{\times}(a\dot{+}b)) \\ &= f\big(\alpha(\tfrac{1}{2})\dot{\times}(\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b)\dot{+}\alpha(\tfrac{1}{2})\dot{\times}(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}b)\big) \\ &\leq \frac{1}{2}[f(\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b)+f(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}b)]. \end{aligned} \tag{3.5}$$

If we apply integration to (3.5) over the interval $[0,1]$, taking into account that

$$\int_0^1 f\big(\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b\big)dt=\int_0^1 f\big(\alpha(1-t)\dot{\times}a\dot{+}\alpha(t)\dot{\times}b\big)dt,$$

and use (3.4), then we have the following inequality

$$f\big(\alpha(\tfrac{1}{2})\dot{\times}a\dot{+}\alpha(\tfrac{1}{2})\dot{\times}b\big)\leq\int_0^1 f\big(\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b\big)dt=\int_0^1 f(\lambda_1\dot{\times}a\dot{+}\lambda_2\dot{\times}b)dt\leq\frac{f(a)+f(b)}{2}.$$

Thus the proof is completed. □

Definition 3.2. *Here and after, we call the inequality given by (3.1) inequality Hermite-Hadamard Type Inequality in terms of Non-Newtonian Calculus or α_*-Hermite-Hadamard Type Inequality.*

Theorem 3.3. *(α-partial integration) Let I_α be an $\alpha-$interval, $v^*,u^*:I_\alpha\longrightarrow\mathbb{R}_\alpha$ also be two functions whose first $\alpha-$derivatives are $\alpha-$continuous and a,b are $\alpha-$points of I_α. Then we get*

$$\int_a^b u(x)\dot{\times}v^*(x)^{dx}=u(x)\dot{\times}v(x)|_a^b\dot{-}\int_a^b v(x)\dot{\times}u^*(x)^{dx} \tag{3.6}$$

or (Due to $u^*(x)^{dx}={}^{du}$, and, $v^*(x)^{dx}={}^{dv}$)

$$\int u^{dv}=u\dot{\times}v\dot{-}\int v^{du} \tag{3.7}$$

Proof. Suppose that $f,g:I_\alpha\longrightarrow\mathbb{R}_\alpha$ are any two functions with $f(x):=u,\ g(x):=v$. Since the functions f,g are $\alpha-$derivatable, we obtain

$$f^*(x)^{dx}={}^{du},\ g^*(x)^{dx}={}^{dv}.$$

If we taking into account that

$$[f(x)\dot{\times}g(x)]^*=f^*(x)\dot{\times}g(x)\dot{+}f(x)\dot{\times}g^*(x),$$

then we have

$$f^*(x)\dot{\times}g(x)=[f(x)\dot{\times}g(x)]^*\dot{-}f(x)\dot{\times}g^*(x). \tag{3.8}$$

Later, if we apply integration to (3.8)

$$\int f^*(x)\dot{\times}g(x)^{dx}=\int[f(x)\dot{\times}g(x)]^{*dx}\dot{-}\int f(x)\dot{\times}g^*(x)^{dx},$$

then we have the following equality

$$\int f(x)\dot{\times}g^*(x)^{dx}=[f(x)\dot{\times}g(x)]\dot{-}\int f^*(x)\dot{\times}g(x)^{dx}.$$

Since $f(x)=u$, $g(x)=v$, $f^*(x)^{dx}={}^{du}$ and $g*(x)^{dx}={}^{dv}$, we obtain

$$\int u^{dv}=u\dot{\times}v\dot{-}\int v^{du}$$

Hence the proof is completed. □

Lemma 3.4. *Let I_α be an α-closed interval in $\mathbb{R}_\alpha$ and $f:I_\alpha\to\mathbb{R}_\alpha$ be any function. Then the following equality holds for all $a,b\in I_\alpha$ and $t\in[0,1]_\alpha$*

$$\frac{b\dot{-}a}{\dot{2}}.\dot{\times}\int_{\dot{0}}^{\dot{1}}(\dot{1}\dot{-}\dot{2}\dot{\times}t)\dot{\times}f^*(t\dot{\times}a\dot{+}(\dot{1}\dot{-}t)\dot{\times}b)^{dt}=\frac{f(a)\dot{+}f(b)}{\dot{2}}.\dot{-}\frac{\dot{1}}{b\dot{-}a}.\dot{\times}\int_a^b f(x)^{dx}. \tag{3.9}$$

Proof. For all $a,b \in I_\alpha$ and $t \in [0,1]_\alpha$, we get

$$J := \int_{\dot{0}}^{\dot{1}} (\dot{1}\dot{-}\dot{2}\dot{\times}t)\dot{\times}f(t\dot{\times}a\dot{+}(\dot{1}\dot{-}t)\dot{\times}b)^{dt}$$

Now by applying α-partly integration to J, if we choose

$$u := (\dot{1}\dot{-}\dot{2}\dot{\times}t) \text{ and } {}^{dv} := f^*(t\dot{\times}a\dot{+}(\dot{1}\dot{-}t)\dot{\times}b)^{dt},$$

then we obtain

$$^{du} := \dot{-}\dot{2}^{dt} \text{ and } v := f(t\dot{\times}a\dot{+}(\dot{1}\dot{-}t)\dot{\times}b)\dot{\times}\frac{\dot{1}}{a\dot{-}b}..$$

Later on, we get by α-partly integration

$$\begin{aligned} \int_{\dot{0}}^{\dot{1}} (\dot{1}\dot{-}\dot{2}\dot{\times}t)\dot{\times}f^*(t\dot{\times}a\dot{+}(\dot{1}\dot{-}t)\dot{\times}b)^{dt} &= (\dot{1}\dot{-}\dot{2}t)\dot{\times}f(t\dot{\times}a\dot{+}(\dot{1}\dot{-}t)\dot{\times}b)\dot{\times}\frac{\dot{1}}{a\dot{-}b}.|_{\dot{0}}^{\dot{1}} \\ &\dot{+}\dot{2}\dot{\times}\frac{\dot{1}}{a\dot{-}b}.\dot{\times}\int_{\dot{0}}^{\dot{1}} f(t\dot{\times}a\dot{+}(\dot{1}\dot{-}t)\dot{\times}b)^{dt}. \end{aligned} \tag{3.10}$$

We can easily see that

$$\int_{\dot{0}}^{\dot{1}} f(t\dot{\times}a\dot{+}(\dot{1}\dot{-}t)\dot{\times}b)^{dt} = \frac{\dot{1}}{b\dot{-}a}.\dot{\times}\int_a^b f(x)^{dx}.$$

and we get by straightforward calculations in (3.10), we can find the desired result. □

Remark 3.5. *Let I_α be an interval in $\mathbb{R}_\alpha$, and $f: I_\alpha \to \mathbb{R}$ be any α_*-convex function. Then for all $a,b \in I_\alpha$ and $t \in [0,1]$*

$$\int_0^1 f(\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b)dt = \frac{-1}{\alpha^{-1}(b)-\alpha^{-1}(a)}\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} f(\alpha(u))du \tag{3.11}$$

equality holds.

Proof. Let we denote for all $a,b \in I_\alpha$ and $t \in [0,1]$,

$$K := \int_0^1 f\big(\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b\big)dt.$$

On the other hand we can write

$$K = \int_0^1 f\Big(\alpha\big\{\alpha^{-1}(\alpha(t))\alpha^{-1}(a)+\alpha^{-1}(\alpha(1-t))\alpha^{-1}(b)\big\}\Big)dt.$$

If we choose $\alpha^{-1}(\alpha(t))\alpha^{-1}(a))+\alpha^{-1}(\alpha(1-t))\alpha^{-1}(b)) = u$, then

$$K = \int_0^1 f(\alpha(u))dt.$$

We can rewrite $u = t\alpha^{-1}(a)+(1-t)\alpha^{-1}(b)$, in this case

$$du = [\alpha^{-1}(a)-\alpha^{-1}(b)]dt \Rightarrow dt = \frac{-1}{\alpha^{-1}(b)-\alpha^{-1}(a)}du.$$

Thus we get

$$K = \frac{1}{\alpha^{-1}(b)-\alpha^{-1}(a)}\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} f(\alpha(u))du,$$

namely,

$$\int_0^1 f(\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b)dt = \frac{1}{\alpha^{-1}(b)-\alpha^{-1}(a)}\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} f(\alpha(u))du$$

Consequently, the proof is completed. □

Remark 3.6. *Let $I_\alpha \subset (\dot{0},\dot{\infty})$ be an α-interval. If we take $\alpha(x) = q_p$ and $\alpha^{-1}(\lambda_1) = t$, $\alpha^{-1}(\lambda_2) = (1-t)$ in α_* Hermite-Hadamard Type Inequality (Definition 3.2), then we obtain*

$$f\Big(\Big(\frac{(x)^p+(y)^p}{2}\Big)^{\frac{1}{p}}\Big) \le \int_0^1 f\Big(\Big(\frac{(x)^p+(y)^p}{2}\Big)^{\frac{1}{p}}\Big)dt \le \frac{f(a)+f(b)}{2},\ (t\in[0,1])$$

p-convex function [19].

Remark 3.7. *If we take $\alpha(x) = exp(x)$ and $\alpha^{-1}(\lambda_1) = t$, $\alpha^{-1}(\lambda_2) = (1-t)$ in α_*-Hermite-Hadamard Type Inequality (Definition 3.2), then we obtain*

$$f\Big(\Big(\frac{a.b}{2}\Big)\Big) \le \int_0^1 f(x^{lnt}.y^{ln(1-t)})dt \le \frac{f(a)+f(b)}{2},\ (t\in[1,e]),$$

$f: I_{exp} \to \mathbb{R}$ geometric convex function [20].

Remark 3.8. *If we take $\alpha(x) = \varphi^{-1}(x)$ and $\alpha^{-1}(\lambda_1) = t$, $\alpha^{-1}(\lambda_2) = (1-t)$ in α_*-Hermite-Hadamard Type Inequality (Definition 3.2), then for $t \in [0,1]$ we obtain*

$$f\Big(\varphi^{-1}\Big(\frac{\varphi(x)+\varphi(y)}{2}\Big)\Big) \le \int_0^1 f(\varphi^{-1}(t\varphi(x)+(1-t)\varphi(y)))dt \le \frac{f(a)+f(b)}{2}$$

$M_\varphi A$ convex function [21].

Remark 3.9. *If we take $\alpha(x) = I(x)$ and $\alpha^{-1}(\lambda_1) = t$, $\alpha^{-1}(\lambda_2) = (1-t)$ in (3.1), then we obtain Hermite-Hadamard Type inequality (Definition 1.4), namely*

$$f\Big(\frac{a+b}{2}\Big) \le \int_0^1 f(ta+(1-t)b)dt \le \frac{f(a)+f(b)}{2}.$$

Remark 3.10. *If we take $\alpha(x) = I(x) = \frac{1}{x}$ and $\alpha^{-1}(\lambda_1) = t$, $\alpha^{-1}(\lambda_2) = (1-t)$ in α_*-Hermite-Hadamard Type Inequality (Definition 3.2), then we obtain Harmonically convex function [22], that is,*

$$f\Big(\Big(\frac{2xy}{x+y}\Big)\Big) \le \int_0^1 f(\frac{xy}{x(1-t)+yt})dt \le \frac{f(a)+f(b)}{2},\ (t\in[0,1]).$$

Acknowledgment

We thank to Assoc. Prof. Dr. İmdat İŞCAN to valuable contributions.

References

[1] M. Grossman, R. Katz, *Non-Newtonian Calculus*, Pigeon Cove (1972), Massachusetts, USA: Lee Press.
[2] A. E. Bashirov, E. M. Kurpınar, A. Özyapıcı, *Multiplicative calculus and its applications*, J. Math. Anal. Appl. 337(2008), 36–48.
[3] A.E. Bashirov, S. Norozpour, *On an alternative view to complex calculus*, Math. Methods. Appl. Sci. 41(2018), 7313–7324.
[4] C. Türkmen, F. Başar, *Some Basic Results on the Sets of Squences with Geometric Calculus*, Commun. Fac. Univ. Ank. Series A1 61(2012), 17–34.
[5] A. F. Çakmak, F. Başar, *Some New Results on Squence Spaces with respect to Non-Newtonian Calculus*, J. Inequal. Appl. 228(2012), 1–17.
[6] Z. Çakır, *Spaces of Continuous and Bounded Functions over the Field of Geometric Complex Numbers* J. Inequal. Appl. 363(2013), 1–8.
[7] U. Kadak, H. Efe, *The Contruction of Hilbert Spaces over the Non-Newtonian Field*, Int. J. Anal. 2014(2014), 1–10.
[8] Y. Gürefe, U. Kadak, E. Misirli, A. Kurdi, *A New Look at the Classical Squence Spaces by Using Multiplicative Calculus*, U. P. B. Sci. Bull. Series A 78(2016), 9–20.
[9] U. Kadak, Y. Gürefe, *A Generalization on Weighted Means and Convex Functions with respect to the Non-Newtonian Calculus*, Int. J. Anal. 2016(2016), 1–9.
[10] A. F. Çakmak, F. Başar, *Some sequence spaces and matrix transformations in multiplicative sense*, TWMS J. Pure Appl. Math. 6 (1) (2015), 27–37.
[11] A. F. Çakmak, F. Başar, *Certain spaces of functions over the field of non-Newtonian complex numbers*, Abstr. Appl. Anal. 2014, Article ID 236124, 12 pages, 2014. doi:10.1155/2014/236124.
[12] A. F. Çakmak, F. Başar, *On line and double integrals in the non-Newtonian sense*, AIP Conference Proceedings, 1611 (2014), 415–423.
[13] S. Tekin, F. Başar, *Certain sequence spaces over the non-Newtonian complex field*, Abstr. Appl. Anal. 2013, Article ID 739319, 11 pages, doi: 10.1155/2013/ 739319.
[14] C. Türkmen, F. Başar, *Some basic results on the sets of sequences with geometric calculus*, AIP Conference Proceedings 1470 (2012), 95–98.
[15] K. Boruah, B. Hazarika, *Application of Geometric Calculus in Numerical Analysis and difference sequence spaces*, J. Math. Anal. Appl. 449(2)(2017), 1265–1285.
[16] K. Boruah, B. Hazarika, *G-Calculus*, TWMS J. Appl. Eng. Math. 8(1)(2018), 94–105.
[17] K. Boruah, B. Hazarika, *Bigeometric integral calculus*, TWMS J. Appl. Eng. Math. 8(2)(2018), 374–385.
[18] E. Unluyol, S. Salaş, İ. İşcan, *Convex functions and some inequalities in terms of the Non-Newtonian Calculus*, AIP Publishing: AIP Conf Proc 1833 020043(2017), 1–4.
[19] S. S. Dragomir, J. Pečarić, L. E. Persson, *Some inequalities of Hadamard type*, Soochow J. Math., 21(1995), 335–341.
[20] T. Y. Zhang, A. P. Ji, F. Qi, *On Integral Inequalities of Hermite-Hadamard Type Inequalities for s-Geometrically Convex Functions*, Abstr. Appl. Anal., doi:10.11 55/2012/560586.
[21] S. Turhan, İ. İşcan, M. Kunt, *Hermite-Hadamard Type Inequalities for $M_\varphi A$ convex functions*, https://doi.org/10.13140/rg.2.2.14526.28486
[22] İ. İşcan, *Hermite-Hadamard Type Inequalities for harmonically convex functions*, Hacet. J. Math. Stat., 43(2014), 6, 935–942.