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Some Properties of the Riemannian Extensions

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Abstract

In this article, we construct an almost complex structure on the cotangent bundle. Then we investigate Nordenian properties of the Riemannian extension in the cotangent bundle.

Keywords: *Almost complex structure, cotangent bundle, Norden metric, Riemannian extension.*
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1. Introduction

Let T^*M^n be the cotangent bundle of n-dimensional differentiable manifold (M^n, g) and π the bundle projection $T^*M^n \to M^n$. The local coordinates $\left(U, x^j\right), j = 1, ..., n$ on M^n induces a system of local coordinates $\left(\pi^{-1}(U), x^j, x^{\bar{j}} = p_j\right)$, $\bar{j} = n+1, ..., 2n$, on T^*M^n, where $x^{\bar{j}} = p_j$ are the components of the covector p in each cotangent space $T^*_x M^n$, $x \in U$, with respect to the natural coframe $\left\{dx^j\right\}$. By $\Im^r_s(M^n)$ (resp. $\Im^r_s(T^*M^n)$) we denote the set of all tensor fields of type (r,s) on M^n(resp. T^*M^n). Manifolds, tensor fields and connections are always assumed to be differentiable and of class C^∞.
Suppose that a vector and covector (1-form) field $X \in \Im^1_0(M^n)$ and $\omega \in \Im^0_1(M^n)$ have the local expressions $X = X^j \frac{\partial}{\partial x^j}$ and $\omega = \omega_j dx^j$ in $U \subset M^n$, respectively. The horizontal and complete lifts ${}^H X, {}^C X \in \Im^1_0(T^*M^n)$ of $X \in \Im^1_0(M^n)$ and the vertical lift ${}^V\omega \in \Im^1_0(T^*M^n)$ of $\omega \in \Im^0_1(M^n)$ are given, respectively, by

$$ {}^H X = X^j \frac{\partial}{\partial x^j} + \sum_j p_h \Gamma^h_{ji} X^i \frac{\partial}{\partial x^{\bar{j}}}, \tag{1.1} $$

$$ {}^C X = X^j \frac{\partial}{\partial x^j} - \sum_j p_h \partial_j X^h \frac{\partial}{\partial x^{\bar{j}}}, \tag{1.2} $$

$$ {}^V \omega = \sum_j \omega_j \frac{\partial}{\partial x^{\bar{j}}} \tag{1.3} $$

with respect to the natural frame $\left\{\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^{\bar{j}}}\right\}$, where Γ^h_{ji} are the coefficients of the Levi-Civita connection ∇_g on M^n [12].
A pseudo-Riemannian metric ${}^R\nabla \in \Im^0_2(T^*M^n)$ is given by (see [12, p. 268])

$$ {}^R\nabla\left({}^C X, {}^C Y\right) = -\gamma(\nabla_X Y + \nabla_Y X) $$

for any $X, Y \in \Im^1_0(M^n)$, where $\gamma(\nabla_X Y + \nabla_Y X) = p_m\left(X^j \nabla_j Y^m + Y^j \nabla_j X^m\right)$. ${}^R\nabla$ is called the Riemannian extension of the symmetric connection ∇ to T^*M^n. Any tensor field of type (0,2) is entirely detected by its action of ${}^H X$ and ${}^V\omega$ on T^*M^n [12]. Then the Riemann extension ${}^R\nabla$ is defined by

$$ \begin{aligned} {}^R\nabla\left({}^V\omega, {}^V\theta\right) &= {}^R\nabla\left({}^H X, {}^H Y\right) = 0, \\ {}^R\nabla\left({}^V\omega, {}^H Y\right) &= {}^V(\omega(X)) = \omega(X) \circ \pi \end{aligned} \tag{1.4} $$

for any $X, Y \in \Im^1_0(M^n)$ and $\omega, \theta \in \Im^0_1(M^n)$ [1].
In this paper, in section 2, we recall the expressions of the Levi-Civita connection of the Riemanian extension from [1] and then we write their invariant forms by using the horizontal and vertical lifts. In section 3, we define an almost complex structure J. Then we get the conditions under which the triple $\left(T^*M^{2n}, {}^R\nabla, J\right)$ is a Kähler-Norden manifold and an anti-Kähler-Codazzi manifold.

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2. Levi-Civita connection of ${}^{R}\nabla$

In [12, p.238, p.277], the following formulas were given

$$\begin{array}{ll} \text{i) } \left[{}^{H}X,{}^{H}Y\right]={}^{H}[X,Y]+\gamma R(X,Y)={}^{H}[X,Y]+{}^{V}(pR(X,Y)), & \\ \text{ii) } \left[{}^{H}X,{}^{V}\omega\right]={}^{V}(\nabla_X\omega), & \text{iii) } \left[{}^{V}\omega,{}^{V}\theta\right]=0, \\ \text{iv) } {}^{V}\omega^{V}f=0, & \text{v) } {}^{H}X^{V}f={}^{V}(Xf) \end{array} \tag{2.1}$$

for any $X,Y\in\Im_0^1(M^n)$, $\omega,\theta\in\Im_1^0(M^n)$, R denoted the curvature tensor of ∇.
The adapted frame $\{\tilde{e}_{(\beta)}\}=\{\tilde{e}_{(j)},\tilde{e}_{(\bar{j})}\}=\{{}^{H}X_{(j)},{}^{V}\theta^{(j)}\}$ (see [12]) to the Levi-Civita connection ∇_g on T^*M^n is given by

$$\tilde{e}_{(j)}={}^{H}X_{(j)}=\frac{\partial}{\partial x^j}+\sum_h p_a\Gamma^a_{hj}\frac{\partial}{\partial x^{\bar{h}}}, \tag{2.2}$$

$$\tilde{e}_{(\bar{j})}={}^{V}\theta^{(j)}=\frac{\partial}{\partial x^{\bar{j}}}. \tag{2.3}$$

Then using (1.2), (1.3), (2.2) and (2.3), we obtain

$${}^{H}X=X^j\tilde{e}_{(j)},\quad {}^{H}X=({}^{H}X^{\alpha})=\begin{pmatrix} X^j \\ 0 \end{pmatrix}, \tag{2.4}$$

$${}^{V}\omega=\sum_j\omega_j\tilde{e}_{(\bar{j})},\quad {}^{V}\omega=({}^{V}\omega^{\alpha})=\begin{pmatrix} 0 \\ \omega_j \end{pmatrix} \tag{2.5}$$

with respect to the adapted frame $\{\tilde{e}_{(\beta)}\}$, where X^j and ω_j are the local components of $X\in\Im_0^1(M^n)$ and $\omega\in\Im_1^0(M^n)$, respectively.
Let ${}^{C}\nabla$ be the Levi-Civita connection of ${}^{R}\nabla$, i.e. ${}^{C}\nabla\left({}^{R}\nabla\right)=0$ (${}^{C}\nabla$ is called the complete lift of ∇ to T^*M^n). The components of ${}^{C}\nabla$ in $\pi^{-1}(U)\subset T^*M^n$, computed in [1], are given by

$$\begin{cases} {}^{C}\Gamma^{h}_{\bar{j}i}={}^{C}\Gamma^{h}_{j\bar{i}}={}^{C}\Gamma^{h}_{\bar{j}\bar{i}}={}^{C}\Gamma^{\bar{h}}_{\bar{j}i}={}^{C}\Gamma^{\bar{h}}_{\bar{j}\bar{i}}=0, \\ {}^{C}\Gamma^{h}_{ji}=\Gamma^{h}_{ji}, \qquad {}^{C}\Gamma^{\bar{h}}_{j\bar{i}}=-\Gamma^{i}_{jh}, \\ {}^{C}\Gamma^{\bar{h}}_{ji}=\frac{1}{2}p_m\left(R_{jih}{}^{m}-R_{ihj}{}^{m}+R_{hji}{}^{m}\right) \end{cases} \tag{2.6}$$

with respect to the adapted frame $\{\tilde{e}_{(\beta)}\}$, where $R_{jih}{}^{a}$ are the local components of the curvature tensor R of ∇_g. The curvature tensor satisfies $R_{jih}{}^{m}+R_{ihj}{}^{m}+R_{hji}{}^{m}=0$ so we can write

$${}^{C}\Gamma^{\bar{h}}_{ji}=\frac{1}{2}p_m\left(R_{jih}{}^{m}-R_{ihj}{}^{m}+R_{hji}{}^{m}\right)=\frac{1}{2}p_m\left(-2R_{ihj}{}^{m}\right)=p_mR_{hij}{}^{m} \tag{2.7}$$

Let $\tilde{X},\tilde{Y}\in\Im_0^1(T^*M^n)$ and $\tilde{X}=\tilde{X}^{\beta}\tilde{e}_{\beta},\tilde{Y}=\tilde{Y}^{\gamma}\tilde{e}_{\gamma}$. Then the covariant derivative ${}^{C}\nabla_{\tilde{Y}}\tilde{X}$ along $\tilde{Y}$ has components

$${}^{C}\nabla_{\tilde{Y}}\tilde{X}^{\beta}=\tilde{Y}^{\varepsilon}\tilde{e}_{\varepsilon}\tilde{X}^{\beta}+{}^{C}\Gamma^{\beta}_{\varepsilon\alpha}\tilde{X}^{\alpha}\tilde{Y}^{\varepsilon}, \tag{2.8}$$

with respect to the adapted frame $\{\tilde{e}_{(\beta)}\}$.
Using (2.4-2.8) we have the next theorem:

Theorem 2.1. *Let (M^n,g) be a n-dimensional differentiable manifold and ${}^{C}\nabla$ be the Levi-Civita connection of the cotangent bundle T^*M^n equipped with the Riemann extension ${}^{R}\nabla$. Then ${}^{C}\nabla$ satisfies the following equations:*

$$\begin{array}{l} i)\ {}^{C}\nabla_{{}^{V}\omega}{}^{V}\theta={}^{C}\nabla_{{}^{V}\omega}{}^{H}Y=0, \\ ii)\ {}^{C}\nabla_{{}^{H}X}{}^{V}\omega={}^{V}(\nabla_X\omega), \\ iii)\ {}^{C}\nabla_{{}^{H}X}{}^{H}Y={}^{H}(\nabla_XY)+{}^{V}(pR(\ ,Y)X) \end{array} \tag{2.9}$$

for all $X,Y\in\Im_0^1(M^n)$ and $\omega,\theta\in\Im_1^0(M^n)$, R denoted the curvature tensor of ∇, where $(pR(\ ,Y)X)=p_aR_{jki}{}^{a}X^iY^k$.

3. The Nordenian structures on $\left(T^*M^{2n},{}^{R}\nabla\right)$

Let M^{2n} be an almost complex manifold with an almost complex structure J. We know the almost complex structure satisfies $J^2=-I$, where $J\in\Im_1^1\left(M^{2n}\right)$ is an affinor field and I is the identity transformation.
Let $\left(M^{2n},J\right)$ be an almost complex manifold. A semi-Riemannian metric $g\in\Im_2^0\left(M^{2n}\right)$ is a Norden metric [2] with respect to J if

$$g(JX,Y)=g(X,JY) \tag{3.1}$$

for any $X,Y\in\Im_0^1\left(M^{2n}\right)$. This metric was studied as pure, anti-Hermitian and B-metric [4], [5], [7], [8], [9], [11].
If $\left(M^{2n},J\right)$ is an almost complex manifold with Norden metric g, then we say that $\left(M^{2n},J,g\right)$ is an almost Norden manifold. If J is integrable,

then $\left(M^{2n},J,g\right)$ is a Norden manifold. When J satisfies $\nabla J=0$ where ∇ is Levi-Civita connection of g, $\left(M^{2n},J,g\right)$ is to be a Kähler-Norden manifold. Note that the condition $\nabla J=0$ is equivalent to $\phi_J g=0$, where ϕ_J is the Tachibana operator and

$$\begin{aligned}(\phi_J g)(X,Y,Z) &= (JX)(g(Y,Z))-X(g(JY,Z))+g((L_Y J)X,Z)\\ &\quad +g(Y,(L_Z J)X)\end{aligned} \tag{3.2}$$

for all $X,Y,Z\in\Im_0^1\left(M^{2n}\right)$, where L_Y denotes the Lie differentiation with respect to Y [7].
In the paper [1] the authors considered on T^*M^{2n} the almost complex structure given as the horizontal lift of the almost complex structure from M^{2n}. Here we construct another tensor field $J\in\Im_1^1\left(T^*M^{2n}\right)$ given by

$$\begin{cases} J^H X=-^V\tilde{X},\\ J^V\omega={}^H\tilde{\omega}\end{cases} \tag{3.3}$$

for any $X\in\Im_0^1\left(M^{2n}\right)$ and $\omega\in\Im_1^0\left(M^{2n}\right)$, where $\tilde{X}=g\circ X\in\Im_1^0\left(M^{2n}\right)$, $\tilde{\omega}=g^{-1}\circ\omega\in\Im_0^1\left(M^{2n}\right)$ (the musical isomorphisms $\flat$ and $\sharp$ can be used instead of the notations $g\circ X$ and $g^{-1}\circ\omega$, respectively (see e.g. [3])). Then we see

$$\begin{aligned} J^2\left({}^H X\right)&=J\left(J^H X\right)=J\left(-^V\tilde{X}\right)=-^H\tilde{\tilde{X}}=-^H X,\\ J^2\left({}^V\omega\right)&=J\left(J^V\omega\right)=J\left({}^H\tilde{\omega}\right)=-^V\tilde{\tilde{\omega}}=-^V\omega\end{aligned}$$

for any $X\in\Im_0^1\left(M^{2n}\right)$ and $\omega\in\Im_1^0\left(M^{2n}\right)$, i.e. $J^2=-I$. Hence we have that J is an almost complex structure.

Theorem 3.1. *The triple* $\left(T^*M^{2n},{}^R\nabla,J\right)$ *is an almost Norden manifold.*

Proof. Using (3.1) we write

$$Q\left(\tilde{X},\tilde{Y}\right)={}^R\nabla\left(J\tilde{X},\tilde{Y}\right)-{}^R\nabla\left(\tilde{X},J\tilde{Y}\right)$$

for any $\tilde{X},\tilde{Y}\in\Im_0^1\left(T^*M^{2n}\right)$. From (1.4) and (3.3), we have

$$\begin{aligned} Q\left({}^H X,{}^H Y\right)&={}^R\nabla\left(J^H X,{}^H Y\right)-{}^R\nabla\left({}^H X,J^H Y\right)\\ &={}^R\nabla\left(-^V\tilde{X},{}^H Y\right)-{}^R\nabla\left({}^H X,-^V\tilde{Y}\right)\\ &=-^V\left(\tilde{X}(Y)\right)+\left(\tilde{Y}(X)\right)=-\tilde{X}_iY^i+\tilde{Y}_iX^i\\ &=-g_{ki}X^kY^i+g_{ki}Y^kX^i=0\\ Q\left({}^H X,{}^V\omega\right)&={}^R\nabla\left(J^H X,{}^V\omega\right)-{}^R\nabla\left({}^H X,J^V\omega\right)\\ &={}^R\nabla\left(-^V\tilde{X},{}^V\omega\right)-{}^R\nabla\left({}^H X,{}^H\tilde{\omega}\right)=0,\\ Q\left({}^V\omega,{}^H Y\right)&=-Q\left({}^H Y,{}^V\omega\right)=0,\\ Q\left({}^V\omega,{}^V\theta\right)&={}^R\nabla\left(J^V\omega,{}^V\theta\right)-{}^R\nabla\left({}^V\omega,J^V\theta\right)\\ &={}^R\nabla\left({}^H\tilde{\omega},{}^V\theta\right)-{}^R\nabla\left({}^V\omega,{}^H\tilde{\theta}\right)=0\end{aligned}$$

i.e. ${}^R\nabla$ is pure with respect to J. Thus Theorem 3.1 is proved. □

From (1.4), (2.1), (3.2) and (3.3) we find the following equations:

$$\begin{aligned}\left(\phi_J{}^R\nabla\right)\left({}^V\omega,{}^H Y,{}^H Z\right)&={}^V\left(pR(Y,\tilde{\omega})Z+pR(Z,\tilde{\omega})Y\right),\\ \left(\phi_J{}^R\nabla\right)\left({}^H X,{}^H Y,{}^V\omega\right)&=-^V\left(g^{-1}\left(\omega,pR(Y,X)\right)\right),\\ \left(\phi_J{}^R\nabla\right)\left({}^H X,{}^V\omega,{}^H Y\right)&=-^V\left(g^{-1}\left(\omega,pR(Y,X)\right)\right)\end{aligned} \tag{3.4}$$

and the others are zero. Therefore we have

Theorem 3.2. *The triple* $\left(T^*M^{2n},J,{}^R\nabla\right)$ *is a Kähler-Norden manifold if and only if* M^{2n} *is flat.*

In [12, p.277], we know that the Lie bracket for complete, horizontal and vertical lifts of vector fields on the cotangent bundle T^*M^n of M^n satisfies the following:

$$\begin{cases}\left[{}^C X,{}^H Y\right]={}^H[X,Y]+{}^V\left(p\left(L_X\nabla\right)Y\right),\\ \left[{}^C X,{}^V\omega\right]={}^V\left(L_X\omega\right)\end{cases} \tag{3.5}$$

for any $X,Y\in\Im_0^1(M^n)$ and $\omega\in\Im_1^0(M^n)$, where $(L_X\nabla)Y=\nabla_Y\nabla X+R(X,Y)$.
It is well known that if a vector field $X\in\Im_0^1\left(M^{2n}\right)$ satisfies $L_Xg=0$ and $L_X\nabla_g=0$, then X is called Killing vector field (or infinitesimal isometry) and infinitesimal affine transformation, respectively. A Killing vector field is necessarily an infinitesimal affine transformation, i.e. we have $L_X\nabla_g=0$ as a consequence of $L_Xg=0$. If for a vector field $\tilde{X}\in\Im_0^1\left(T^*M^{2n}\right)$ the Lie derivative $(L_{\tilde{X}}J=0)$ with respect to almost Nordenian structure J vanishes, then $\tilde{X}$ is an almost holomorphic vector field [6].
Considering the Lie derivative of J with respect to the complete lift ${}^C X$. Using (3.3) and (3.5), we get the followings:

$$\begin{aligned}\left(L_{{}^C X}J\right){}^V\theta&=L_{{}^C X}J^V\theta-J\left(L_{{}^C X}{}^V\theta\right)=L_{{}^C X}{}^H\tilde{\theta}-J\left({}^V\left(L_X\theta\right)\right)\\ &=L_{{}^C X}{}^H\tilde{\theta}-{}^H\left(g^{-1}\circ\left(L_X\theta\right)\right)\\ &={}^H\left[X,\tilde{\theta}\right]+{}^V\left(p\left(L_X\nabla\right)\tilde{\theta}\right)-{}^H\left(g^{-1}\circ\left(L_X\theta\right)\right)\\ &={}^H\left(L_X\left(g^{-1}\circ\theta\right)-g^{-1}\circ\left(L_X\theta\right)\right)+{}^V\left(p\left(L_X\nabla\right)\tilde{\theta}\right),\end{aligned} \tag{3.6}$$

$$\begin{aligned}\left(L_{{}^C X}J\right){}^H Y&=L_{{}^C X}J^H Y-J\left(L_{{}^C X}{}^H Y\right)\\ &=L_{{}^C X}{}^V\tilde{Y}-J\left({}^H[X,Y]+{}^V\left(p\left(L_X\nabla\right)Y\right)\right)\\ &={}^V\left(L_X\left(g\circ Y\right)-g\circ L_XY\right)-{}^H\left(g^{-1}\circ p\left(L_X\nabla\right)Y\right).\end{aligned} \tag{3.7}$$

By using the relations (3.6) and (3.7) we prove the following result:

Theorem 3.3. *An infinitesimal transformation X of the Riemannian manifold $\left(M^{2n},g\right)$ is a Killing vector field if and only if its complete lift ^{C}X to the cotangent bundle $T^{*}M^{2n}$ is an almost holomorphic vector field with respect to the almost Nordenian structure $\left(J,{}^{R}\nabla\right)$.*

Proof. Let X be a Killing vector field, i.e. $L_X g=0$. Then by virtue of $L_X\nabla_g=0$, from (3.6) and (3.7) we have $L_{^CX}J=0$ (^CX is holomorphic with respect to J). Conversely, if we assume that $L_{^CX}J=0$ and compute the righthand side of (3.7) at $(x^i,0)$, $p_i=0$, then we get $L_X\left(g\circ Y\right)=g\circ L_XY$. Thus it follows that $L_Xg=0$. □

Let $\left(M^{2n},J,g\right)$ be an almost Norden manifold. The twin Norden metric defined by

$G\left(X,Y\right)=\left(g\circ J\right)\left(X,Y\right)=g\left(JX,Y\right)$

for any $X,Y\in\Im_0^1\left(M^{2n}\right)$ [5]. If the twin Norden metric G satisfies the Codazzi equation

$\left(\nabla_XG\right)\left(Y,Z\right)-\left(\nabla_YG\right)\left(X,Z\right)=0$

for any $X,Y\in\Im_0^1\left(M^{2n}\right)$, then the triple $\left(M^{2n},J,g\right)$ is called an anti-Kähler-Codazzi manifold [10].
Let G be the twin Norden metric with respect to the Riemann extension $^R\nabla$ and the almost complex structure J. Using $G\left(\tilde{X},\tilde{Y}\right)=\left({}^R\nabla\circ J\right)\left(\tilde{X},\tilde{Y}\right)={}^R\nabla\left(J\tilde{X},\tilde{Y}\right)$, we have

$$\begin{aligned}G\left({}^HX,{}^HY\right)&={}^R\nabla\left(J^HX,{}^HY\right)={}^R\nabla\left(-{}^V\tilde{X},{}^HY\right)\\&=-{}^V\left(\tilde{X}\left(Y\right)\right)=-g_{ij}X^iY^j=-{}^V\left(g\left(X,Y\right)\right),\\G\left({}^V\omega,{}^V\theta\right)&={}^R\nabla\left(J^V\omega,{}^V\theta\right)={}^R\nabla\left({}^H\tilde{\omega},{}^V\theta\right)\\&={}^V\left(\theta\left(\tilde{\omega}\right)\right)=g^{ij}\omega_i\theta_j={}^V\left(g^{-1}\left(\omega,\theta\right)\right),\\G\left({}^HX,{}^V\theta\right)&={}^R\nabla\left(J^HX,{}^V\theta\right)={}^R\nabla\left(-{}^V\tilde{X},{}^V\theta\right)=0,\\G\left({}^V\omega,{}^HY\right)&={}^R\nabla\left(J^V\omega,{}^HY\right)={}^R\nabla\left({}^H\tilde{\omega},{}^HY\right)=0\end{aligned}$$

for any $X,Y\in\Im_0^1\left(M^{2n}\right)$ and $\omega,\theta\in\Im_1^0\left(M^{2n}\right)$.
Now we use the equation $\left({}^C\nabla_{\tilde{X}}G\right)\left(\tilde{Y},\tilde{Z}\right)-\left({}^C\nabla_{\tilde{Y}}G\right)\left(\tilde{X},\tilde{Z}\right)=0$, we find the following:

$$\begin{aligned}\left({}^C\nabla_{^HX}G\right)\left({}^HY,{}^V\omega\right)-\left({}^C\nabla_{^HY}G\right)\left({}^HX,{}^V\omega\right)&={}^V\left(g^{-1}\left(pR\left(Y,X\right),\omega\right)\right),\\\left({}^C\nabla_{^HX}G\right)\left({}^V\omega,{}^HZ\right)-\left({}^C\nabla_{^V\omega}G\right)\left({}^HX,{}^HZ\right)&={}^V\left(g^{-1}\left(\omega,pR\left(\ ,Z\right)X\right)\right),\\\left({}^C\nabla_{^V\omega}G\right)\left({}^HY,{}^HZ\right)-\left({}^C\nabla_{^HY}G\right)\left({}^V\omega,{}^HZ\right)&={}^V\left(g^{-1}\left(\omega,pR\left(\ ,Z\right)Y\right)\right)\end{aligned}$$

and the others are zero. Then we obtain the following theorem:

Theorem 3.4. *The triple $\left(T^{*}M^{2n},J,{}^R\nabla\right)$ is an anti- Kähler-Codazzi manifold if and only if M^{2n} is flat.*

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