Some Properties of the Riemannian Extensions

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Abstract

In this article, we construct an almost complex structure on the cotangent bundle. Then we investigate Nordenian properties of the Riemannian extension in the cotangent bundle.

Keywords: Almost complex structure, cotangent bundle, Norden metric, Riemannian extension.

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1. Introduction

Let $T^*M^n$ be the cotangent bundle of $n$-dimensional differentiable manifold $(M^n, g)$ and $\pi$ the bundle projection $T^*M^n \to M^n$. The local coordinates $\{U, x^i\}, i = 1, \ldots, n$ on $M^n$ induces a system of local coordinates $\{\pi^{-1}(U), x^i, y^i = p^i_j, j = n + 1, \ldots, 2n, \}$ on $T^*M^n$, where $x^i = p^i_j$ are the components of the covector $\rho$ in each cotangent space $T^*_xM^n, x \in U$, with respect to the natural coframe $\{dx^i\}$. By $\mathfrak{X}(M^n)$ (resp. $\mathfrak{X}(T^*M^n)$) we denote the set of all tensor fields of type $(r, s)$ on $M^n$ (resp. $T^*M^n$). Manifolds, tensor fields and connections are always assumed to be differentiable and of class $\mathcal{C}^\infty$.

Suppose that a vector and covector (1-form) field $X \in \mathfrak{X}_1(M^n)$ and $\omega \in \mathfrak{X}^0_1(M^n)$ have the local expressions $X = X^j \frac{\partial}{\partial x^j}$ and $\omega = \omega^i dx^i$ in $U \subset M^n$, respectively. The horizontal and complete lifts $H^X, C^X \in \mathfrak{X}_1(T^*M^n)$ of $X \in \mathfrak{X}_1(M^n)$ and the vertical lift $V^\omega \in \mathfrak{X}_1(T^*M^n)$ of $\omega \in \mathfrak{X}^0_1(M^n)$ are given, respectively, by

\begin{align}
H^X &= X^j \frac{\partial}{\partial x^j} + \sum_{j} p_{jk} x^k \frac{\partial}{\partial x^j}, \\
C^X &= X^j \frac{\partial}{\partial x^j} - \sum_{j} p_{jk} \partial_j x^k \frac{\partial}{\partial x^j}, \\
V^\omega &= \sum_{j} \omega^i \frac{\partial}{\partial x^j}
\end{align}

with respect to the natural frame $\left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right\}$, where $I^h_{ij}$ are the coefficients of the Levi-Civita connection $\nabla$ on $M^n$ [12].

A pseudo-Riemannian metric $R\nabla \in \mathfrak{X}^0_1(T^*M^n)$ is given by (see [12, p. 268])

$R\nabla \left( C^X, C^Y \right) = -\gamma(\nabla_X Y + \nabla_Y X)$

for any $X, Y \in \mathfrak{X}_1(M^n)$, where $\gamma(\nabla_X Y + \nabla_Y X) = p_{mn} (X^i \nabla_j Y^m + Y^i \nabla_j X^m)$. $R\nabla$ is called the Riemannian extension of the symmetric connection $\nabla$ to $T^*M^n$. Any tensor field of type $(0,2)$ is entirely detected by its action of $H^X$ and $V^\omega$ on $T^*M^n$ [12]. Then the Riemann extension $R\nabla$ is defined by

\begin{align}
R\nabla \left( V^\omega, Y \right) &= \nabla_H X \left( Y \right) = 0, \\
R\nabla \left( V^\omega, H^Y \right) = V^\omega \left( \nabla_H Y \right) = 0
\end{align}

for any $X, Y \in \mathfrak{X}_1(M^n)$ and $\omega, \theta \in \mathfrak{X}^0_1(M^n)$ [1].

In this paper, in section 2, we recall the expressions of the Levi-Civita connection of the Riemannian extension from [1] and then we write their invariant forms by using the horizontal and vertical lifts. In section 3, we define an almost complex structure $J$. Then we get the conditions under which the triple $(T^*M^{2n}, R\nabla, J)$ is a Kähler-Norden manifold and an anti-Kähler-Codazzi manifold.
2. Levi-Civita connection of $^R\nabla$

In [12, p.238, p.277], the following formulas were given

\begin{equation}
\begin{aligned}
i) \ [H_X; H_Y] & = H_X [Y, Y] + \gamma R(X, Y) = H_X [Y, Y] + V (pR(X, Y)), \\
ii) [H_X; V] \omega & = V (\nabla_X \omega), \quad \text{iii) } \nabla^V \omega \theta = 0,
iv) V H X f = V (X f)
\end{aligned}
\end{equation}

for any $X, Y \in \mathfrak{g}_0^0(M^n)$, $\omega, \theta \in \mathfrak{g}_0^0(M^n)$. $R$ denoted the curvature tensor of $\nabla$.

The adapted frame $\{\tilde{e}_{(\beta)}\} = \{\tilde{e}_{(j)}, \tilde{\theta}^{(j)}\}$ (see [12]) to the Levi-Civita connection $\nabla_g$ on $T^*M^n$ is given by

\begin{equation}
\tilde{e}_{(j)} = H X_{(j)} = \frac{\partial}{\partial x^j} + \sum_i \rho_i \tilde{\Gamma} Y_i \frac{\partial}{\partial \theta^i},
\end{equation}

\begin{equation}
\tilde{\theta}^{(j)} = V \theta^{(j)} = \frac{\partial}{\partial x^j}.
\end{equation}

Then using (1.2), (1.3), (2.2) and (2.3), we obtain

\begin{equation}
H_X = X^l \tilde{e}_{(j)}, \quad H_X = (H X^a) = \left( \begin{array}{c} X^l \\ 0 \end{array} \right),
\end{equation}

\begin{equation}
\nabla \omega = \sum_j \nabla \omega \tilde{e}_{(j)}, \quad V \omega = (V \omega^a) = \left( \begin{array}{c} 0 \\ \omega \end{array} \right)
\end{equation}

with respect to the adapted frame $\{\tilde{e}_{(\beta)}\}$, where $X^l$ and $\omega_j$ are the local components of $X \in \mathfrak{g}_0^0(M^n)$ and $\omega \in \mathfrak{g}_0^0(M^n)$, respectively. Let $^C \nabla$ be the Levi-Civita connection of $^RV$, i.e. $^C \nabla (^R\nabla) = 0$ ( $^C \nabla$ is called the complete lift of $\nabla$ to $T^*M^n$). The components of $^C \nabla$ in $\pi^1(U) \subset T^*M^n$, computed in [1], are given by

\begin{equation}
\left\{ \begin{array}{c}
C \Gamma^{hi}_{jk} = C \Gamma^{hi}_{jk} = C \Gamma^{hi}_{jk} = C \Gamma^{hi}_{jk} = 0, \\
C \Gamma^{hj}_{jk} = \Gamma^{hj}_{jk}, \\
C \Gamma^{ji}_{jk} = \frac{1}{2} \rho_m (R_{hjm} - R_{jhm} + R_{hjm})
\end{array} \right.
\end{equation}

with respect to the adapted frame $\{\tilde{e}_{(\beta)}\}$, where $R_{ijk}^a$ are the local components of the curvature tensor $R$ of $\nabla_g$. The curvature tensor satisfies $R_{ijk}^m + R_{jkm}^m + R_{hjm}^m = 0$ so we can write

\begin{equation}
C \Gamma^{hj}_{jk} = \frac{1}{2} \rho_m (R_{hjm} - R_{jhm} + R_{hjm}) = \frac{1}{2} \rho_m (-2 R_{hjm}) = \rho_m R_{hjm}^m
\end{equation}

Let $\tilde{X}, \tilde{Y} \in \mathfrak{g}_0^0(T^*M^n)$ and $\tilde{X} = \tilde{X}^\beta \tilde{e}_{(\beta)}, \tilde{Y} = \tilde{Y}^\beta \tilde{e}_{(\beta)}$. Then the covariant derivative $^C \nabla_{\tilde{v}} \tilde{X}$ along $\tilde{Y}$ has components

\begin{equation}
^C \nabla_{\tilde{v}} \tilde{X} = \tilde{v}^\alpha \tilde{X}^\alpha + C \Gamma^{hi}_{jk} \tilde{e}_{(j)} \tilde{e}_{(k)},
\end{equation}

with respect to the adapted frame $\{\tilde{e}_{(\beta)}\}$.

Using (2.4-2.8) we have the next theorem:

**Theorem 2.1.** Let $(M^n, g)$ be a n-dimensional differentiable manifold and $^C \nabla$ be the Levi-Civita connection of the cotangent bundle $T^*M^n$ equipped with the Riemann extension $^RV$. Then $^C \nabla$ satisfies the following equations:

\begin{equation}
\begin{aligned}
i) ^C \nabla \omega \theta & = ^C \nabla \omega \omega Y = 0, \\
ii) ^C \nabla \omega Y & = V (\nabla_X \omega), \\
iii) ^C \nabla \omega H Y & = H (\nabla_X Y) + V (pR(J, Y) X)
\end{aligned}
\end{equation}

for all $X, Y \in \mathfrak{g}_0^0(M^n)$ and $\omega, \theta \in \mathfrak{g}_0^0(M^n)$. $R$ denoted the curvature tensor of $\nabla$, where $\langle pR(J, Y) X \rangle = \rho_m R_{jkm} X^i Y^k$.

3. The Nordenian structures on $(T^*M^{2n}, ^R\nabla)$

Let $M^{2n}$ be an almost complex manifold with an almost complex structure $J$. We know the almost complex structure satisfies $J^2 = -I$, where $J \in \mathfrak{g}_0^0(M^{2n})$ is an affinor field and $I$ is the identity transformation.

Let $(M^{2n}, J)$ be an almost complex manifold. A semi-Riemannian metric $g \in \mathfrak{g}_0^0(M^{2n})$ is a Norden metric [2] with respect to $J$ if

\begin{equation}
g(JX, Y) = g (X, JY)
\end{equation}

for any $X, Y \in \mathfrak{g}_0^0(M^{2n})$. This metric was studied as pure, anti-Hermitian and B-metric [4], [5], [7], [8], [9], [11]. If $(M^{2n}, J)$ is an almost complex manifold with Norden metric $g$, then we say that $(M^{2n}, J, g)$ is an almost Norden manifold. If $J$ is integrable,
then $(M^{2n}, J, g)$ is a Norden manifold. When $J$ satisfies $\nabla J = 0$ where $\nabla$ is Levi-Civita connection of $g$, $(M^{2n}, J, g)$ is to be a Kähler-Norden manifold. Note that the condition $\nabla J = 0$ is equivalent to $\phi_J g = 0$, where $\phi_J$ is the Tachibana operator and

$$\phi_J (g(Y,Z)) = (L_Y g)(Z) + g([L_Y J]X, Z)$$

for all $X, Y, Z \in \mathfrak{X}(M^{2n})$, where $L_Y$ denotes the Lie differentiation with respect to $Y$ [7]. In the paper [1] the authors considered on $T^*M^{2n}$ the almost complex structure given as the horizontal lift of the almost complex structure from $M^{2n}$. Here we construct another tensor field $J \in \mathfrak{X}(T^*M^{2n})$ given by

$$J^H X = -V X, \quad J^V \omega = H \omega$$

for any $X \in \mathfrak{X}(M^{2n})$ and $\omega \in \Omega^1(M^{2n})$, where $V = g \circ X \in \mathfrak{X}(M^{2n})$, $H = g^{-1} \circ \omega \in \Omega^1(M^{2n})$ (the musical isomorphisms $\flat$ and $\sharp$ can be used instead of the notations $g \circ X$ and $g^{-1} \circ \omega$, respectively (see e.g. [3])). Then we see

$$J^2 (J^H X) = J(J^H X) = J (-V X) = -H X, \quad J^2 (J^V \omega) = J(J^V \omega) = J (-V \omega) = -V \omega$$

for any $X \in \mathfrak{X}(M^{2n})$ and $\omega \in \Omega^1(M^{2n})$, i.e. $J^2 = -I$. Hence we have that $J$ is an almost complex structure.

**Theorem 3.1.** The triple $(T^*M^{2n}, J, ^R \nabla, J)$ is an almost Norden manifold.

**Proof.** Using (3.1) we write

$$Q(JX,Y) = ^R \nabla(JX,Y) = -^R \nabla(X,JY)$$

for any $X, Y \in \mathfrak{X}(T^*M^{2n})$. From (1.4) and (3.3), we have

$$Q(H^X, H^Y) = ^R \nabla(p^R Y, H^X) - ^R \nabla(p^R Z, H^Y)$$

$$= ^R \nabla(g^{-1} \omega, H^X) - ^R \nabla(g^{-1} \omega, H^Y)$$

$$= -V (\tilde{\nabla} Y) = -V (\tilde{\nabla} X) = -V (\tilde{\nabla} X + \tilde{\nabla} Y - \tilde{\nabla} Z)$$

for any $X, Y, Z \in \mathfrak{X}(M^{2n})$ and $\omega \in \Omega^1(M^{2n})$. Thus Theorem 3.1 is proved.

From (1.4), (2.1), (3.2) and (3.3) we find the following equations:

$$\phi_J ^R \nabla (\omega, H^X) = V (\omega, p^R Z) = V (\omega, p^R X) = 0$$

$$\phi_J ^R \nabla (\omega, J^V \omega) = V (\omega, p^R (J^H X)) = V (\omega, p^R (J^H Y)) = 0$$

and the others are zero. Therefore we have

**Theorem 3.2.** The triple $(T^*M^{2n}, J, ^R \nabla, J)$ is a Kähler-Norden manifold if and only if $M^{2n}$ is flat.

In [12, p.277], we know that the Lie bracket for complete, horizontal and vertical lifts of vector fields on the cotangent bundle $T^*M^n$ of $M^n$ satisfies the following:

$$\left\{ \begin{array}{l}
\phi_J ^R \nabla (C^X, H^Y) = ^R \nabla (C^X, J^H Y)
\phi_J ^R \nabla (C^X, J^V \omega) = V (C^X, p^R (\omega, J^H Y))
\phi_J ^R \nabla (C^X, H^V \omega) = V (C^X, p^R (\omega, J^H Y))
\phi_J ^R \nabla (C^X, J^V \omega) = V (C^X, p^R (\omega, J^H Y))
\end{array} \right.$$
Theorem 3.3. An infinitesimal transformation $X$ of the Riemannian manifold $(M^{2n}, g)$ is a Killing vector field if and only if its complete lift $\hat{C}X$ to the cotangent bundle $T^*M^{2n}$ is an almost holomorphic vector field with respect to the almost Nordenian structure $(J, R^g)$. 

Proof. Let $X$ be a Killing vector field, i.e. $L_X g = 0$. Then by virtue of $L_X \nabla g = 0$, (from (3.6) and (3.7) we have $L_X J = 0$ ($C X$ is holomorphic with respect to $J$). Conversely, if we assume that $L_X J = 0$ and compute the right-hand side of (3.7) at $(x^i, 0)$, $p_j = 0$, then we get $L_X (g \circ Y) = g \circ L_X Y$. Thus it follows that $L_X g = 0$.

Let $(M^{2n}, J, g)$ be an almost Norden manifold. The twin Norden metric defined by $G(X, Y) = (g \circ J)(X, Y)$ for any $X, Y \in \mathfrak{X}(M^{2n})$ [5]. If the twin Norden metric $G$ satisfies the Codazzi equation

$$\nabla_Y G(X, Z) = \nabla_X G(Y, Z)$$

for any $X, Y, Z \in \mathfrak{X}(M^{2n})$, then the triple $(M^{2n}, J, g)$ is called an anti-Kähler-Codazzi manifold [10].

Let $G$ be the twin Norden metric with respect to the Riemann extension $R^g$ and the almost complex structure $J$. Using $G(\hat{X}, \hat{Y}) = (R^g \circ J)(\hat{X}, \hat{Y})$, we have

$$G(\hat{H}X, \hat{H}Y) = \hat{R}^g (\hat{J} \hat{H}X, \hat{H}Y) = \hat{R}^g (\hat{J} \hat{X}, \hat{Y})$$

for any $X, Y \in \mathfrak{X}(M^{2n})$ and $\omega, \theta \in \mathfrak{Y}(M^{2n})$.

Now we use the equation $(\hat{C} \nabla_X G)(\hat{Y}, \hat{Z}) - (\hat{C} \nabla_Y G)(\hat{X}, \hat{Z}) = 0$, we find the following:

$$\nabla_Y G(X, Z) = \nabla_X G(Y, Z)$$

and the others are zero. Then we obtain the following theorem:

**Theorem 3.4.** The triple $(T^*M^{2n}, J, R^g)$ is an anti-Kähler-Codazzi manifold if and only if $M^{2n}$ is flat.

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**References**


