Çankaya University Journal of Science and Engineering
Volume 16, No. 2 (2019) 010–015
Date Received: July 30, 2019
Date Accepted: October 13, 2019

Copure Submodules and Related Results

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Abstract: Let M be a module over a commutative ring R with identity. A submodule K of M is copure provided that $(K :_M I) = K + (0 :_M I)$ for each ideal I of R. In this paper, we investigate some results about copure submodules of M.

Keywords: Pure submodule, copure submodule, copure sum property

1. Introduction

Throughout this work, R denotes a commutative ring with identity and $\mathbb{Z}$ denotes the ring of integers.

A submodule N of an R-module M is called a *pure submodule* of M if $JN = N \cap JM$ for every ideal J of R [2].

H. Ansari-Toroghy and F. Farshadifar introduced the dual notion of pure submodules (that is copure submodules) of an R-module and discussed some properties of this class of modules, see [4]. A submodule K of an R-module M is called *copure* if $(K :_M I) = K + (0 :_M I)$ for each ideal I of R [4].

The aim of this note is to explore more information about this class of R-modules. Furthermore, we investigate the properties of R-modules that the sum of any two copure submodules is a copure submodule.

2. Main Results

Theorem 2.1. Let M be a distributive R-module. Then the following hold.

(a) A submodule N of M is copure if and only if for each $a \in R$ we have

$$(N :_M a) = N + (0 :_M a).$$

(b) A submodule N of M is pure if and only if for each $a \in R$ we have

$$aN = N \cap aM.$$

(c) A submodule N of M is a pure submodule if and only if it is a copure submodule.

Proof. (a) First assume that for each $a \in R$ we have $(N :_M a) = N + (0 :_M a)$. Suppose that I is an ideal of R. Then we have

$$(N :_M I) = (N :_M \sum_{a \in I} Ra) = \cap_{a \in I}(N :_M a) = \cap_{a \in I}(N + (0 :_M a)).$$

Now as M is distributive, we have

$$\cap_{a \in I}(N + (0 :_M a)) = N + \cap_{a \in I}(0 :_M a) = N + (0 :_M I).$$

Therefore, N is a copure submodule of M. The reverse implication is clear.

(b) First assume that for each $a \in R$ we have $aN = N \cap aM$. Suppose that I is an ideal of R. Then as M is a distributive R-module, we have

$$IN = (\sum_{a \in I} Ra)N = \sum_{a \in I}(RaM \cap N) = (\sum_{a \in I} Ra)M \cap N = IM \cap N.$$

Hence, N is a pure submodule of M. The reverse implication is clear.

(c) This follows from parts (a), (b) and [4, Theorem 2.12]. ■

Let R_i be a commutative ring with identity and M_i be an R_i-module, for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an R-module. Clearly, every submodule of M is in the form of $N = N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2.

Proposition 2.2. Let $R = R_1 \times R_2$ be a ring and let $M = M_1 \times M_2$ be an R-module, where M_1 is an R_1-module and M_2 is an R_2-module. Then $N = N_1 \times N_2$ is a pure (resp. copure) submodule of M if and only if N_i is a pure (resp. copure) submodule of M_i for $i = 1, 2$.

Proof. This is straightforward. ■

Proposition 2.3. Let R be a Noetherian ring and let M be an R-module. Then the following hold.

(a) If N is a copure submodule of M, then for each prime ideal P of R, N_P is a copure submodule of M_P as an R_P-module.

(b) If N_P is a copure submodule of an R_P-module M_P for each maximal ideal P of R, then N is a copure submodule of M.

Proof. (a) This follows from the fact that by [9, 9.13], if I is a finitely generated ideal of R, then $((N :_M I))_P = (N_P :_{M_P} I_P)$.

(b) Suppose that I is an ideal of R. As R is a Noetherian ring, I is finitely generated ideal of R. Hence by [9, 9.13], for any maximal ideal P of R, $((N :_M I))_P = (N_P :_{M_P} I_P)$. Thus by assumption, for any maximal ideal P of R,

$$((N :_M I))_P = N_P + (0 :_{M_P} I_P) = (N + (0 :_M I))_P.$$

It follows that

$$(N :_M I) = N + (0 :_M I),$$

as needed. ■

Proposition 2.4. Let M be an R-module and let $f : M \to M$ be an endomorphism such that $f = f^2$. Then $Ker(f)$ is a copure submodule of M.

Proof. Let I be an ideal of R. Clearly $Ker(f) + (0 :_M I) \subseteq (Ker(f) :_M I)$. To see the reverse inclusion, suppose that $x \in (Ker(f) :_M I)$. Then $xI \subseteq Ker(f)$. It follows that $f(x) \in (0 :_M I)$. As $f = f^2$, we have $x - f(x) \in Ker(f)$. Therefore $x = x - f(x) + f(x) \in Ker(f) + (0 :_M I)$, as required. ■

Definition 2.5. We say that an R-module M is *copure simple* if M and (0) are the only copure submodules of M.

Example 2.6. The $\mathbb{Z}$-module $\mathbb{Z}_4$ is copure simple.

Definition 2.7. We say that an R-module M has *the copure sum property* if the sum of any two copure submodules is again copure.

Recall that an R-module M is called *fully copure* if each submodule of M is a copure submodule of M [5].

Example 2.8. (a) Every fully copure R-module has the copure sum property.

(b) Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_4 \oplus \mathbb{Z}_2$. Let $N_1 = 0 \oplus \mathbb{Z}_2$ and $N_2 = \mathbb{Z}(2,1)$, the submodule generated by $(2,1)$. It is easy to see that N_1 and N_2 are copure submodules of M. But $N_1 + N_2 = \{(0,0),(0,1),(2,1),(2,0)\}$ is not a copure submodule of M. Thus M does not have the copure sum property.

(c) Every copure simple R-module has the copure sum property.

(d) Since the submodules of the $\mathbb{Z}_{p^\infty}$ (as $\mathbb{Z}$-module) are comparable, the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ has the copure sum property.

Proposition 2.9. Suppose that M is an R-module. Then the following hold.

(a) If M has the copure sum property and N is a copure submodule of M, then N (resp. M/N) is also has the copure sum property.

(b) If M has the copure sum property, then

$$(N+K:_M I) = (N:_M I) + (K:_M I)$$

for every ideal I of R and for every copure submodules N and K of M.

(c) M as an R-module has the copure sum property if and only if M has the copure sum property as an $R/Ann_R(M)$-module.

Proof. (a) It follows from [4, 2.9].

(b) Let H and T be two copure submodules of M and J be an ideal of R. By assumption, $H+T$ is a copure submodule of M. Thus

$$(H+T:_M J) = H+T+(0:_M J) = H+T+(0:_M J)+(0:_M J)$$
$$= (H:_M J)+(T:_M J).$$

(c) This is clear. ■

Proposition 2.10. Suppose that R is a Noetherian ring and M is an R-module. If the R_m-module M_m has copure sum property for each maximal ideal m of R, then M has the copure sum property as R-module.

Proof. Let H and T be two copure submodules of M. Then H_m and T_m are copure submodules of M_m as R_m-module by Proposition 2.3. Since M_m has copure sum property, $H_m + T_m = (H+T)_m$ is copure in M_m for every maximal ideal m of R. Thus $H+T$ is a copure submodule of M by Proposition 2.3. ■

Remark 2.11. If an R-module M has the copure sum property, then the R-module $M \oplus M$ may not have the copure sum property. For example, consider $\mathbb{Z}_4$ as $\mathbb{Z}$-module. Then $\mathbb{Z}_4$ has the copure sum property. But the $\mathbb{Z}$-module $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ does not have the copure sum property.

Recall that a submodule K of an R-module H is called a *fully invariant submodule* if for every endomorphism $f: H \to H$, we have $f(K) \subseteq K$ [10].

Theorem 2.12. Let $M = \oplus_{i \in I} M_i$ be an R-module, where each M_i is a submodule of M. If M has copure sum property, then each M_i has the copure sum property. The converse is true if each copure submodule of M is fully invariant.

Proof. Suppose that M has copure sum property. Since each M_i is a summand of M, then each M_i is a copure submodule of M by [5, 3.11]. Thus by Proposition 2.9, each M_i has the copure sum property. For the converse, let N and K be two copure submodules of M. Then N and K are fully invariant by assumption. Thus $N = \oplus_{i \in I}(N \cap M_i)$ and $K = \oplus_{i \in I}(K \cap M_i)$ by [10, 8.11]. So

$$N+K = \oplus_{i \in I}((N \cap M_i) + (K \cap M_i)).$$

One can see that $N \cap M_i$ and $K \cap M_i$ are copure submodules of M_i and M_i has the copure sum property, thus $(N \cap M_i) + (K \cap M_i)$ is a copure submodule of M_i. Therefore, $N + K$ is a copure submodule of M by [4, 2.11]. ■

Proposition 2.13. Let M_1 and M_2 be R-modules with copure sum property such that $Ann_R(M_1) + Ann_R(M_2) = R$. Then the R-module $M_1 \oplus M_2$ has the copure sum property.

Proof. Let T and H be two copure submodules of $M_1 \oplus M_2$. Since $Ann_R(M_1) + Ann_R(M_2) = R$, then $T = T_1 \oplus T_2$ and $H = H_1 \oplus H_2$, where T_1, H_1 are submodules of M_1 and T_2 , H_2 are submodules of M_2 by [1]. Now by assumption, $T_1 + H_1$ is a copure submodule of M_1 and $T_2 + H_2$ is a copure submodule of M_2. Hence by [4, 2.11], $(T_1 + H_1) \oplus (T_2 + H_2)$ is a copure submodule of $M_1 \oplus M_2$. So $T + H$ is a copure submodule of $M_1 \oplus M_2$, as desired. ■

Theorem 2.14. Suppose that $R = R_1 \times R_2$ is a commutative ring and $M = M_1 \times M_2$ is an R-module, where M_1 is an R_1-module and M_2 is an R_2-module. Then M has the copure sum property if and only if M_i has the copure sum property for $i = 1, 2$.

Proof. This is straightforward by using Proposition 2.2. ■

An R-module M satisfies the *double annihilator conditions* if, for every ideal J of R, we have $J = Ann_R((0 :_M J))$ [7].

An R-module H is called a *comultiplication R-module* if for each submodule K of H there exists an ideal J of R such that $K = (0 :_H J)$ [3].

An R-module S is a *strong comultiplication R-module* if S is a comultiplication R-module and satisfies the double annihilator conditions [4].

Theorem 2.15. Let M be a strong comultiplication R-module. Then M has the copure sum property.

Proof. Let N_1 and N_2 be two copure submodules of M. Since M is a comultiplication R-module, $N_1 = (0 :_M I_1)$ and $N_2 = (0 :_M I_2)$ for some ideals I_1 and I_2 of R. Now since M is a strong comultiplication module, we have

$$(0 :_M I_1) + (0 :_M I_2) = (0 :_M I_1 \cap I_2).$$

By [4, Theorem 2.13], I_1 and I_2 are pure submodules of R. Clearly, $I_1 \cap I_2$ is a pure submodule of R. Now let I_3 be an ideal of R. Then we have

$$I_3(I_1 \cap I_2) = I_3 \cap (I_1 \cap I_2).$$

Therefore,

$$(N_1 + N_2 :_M I_3) = ((0 :_M I_1) + (0 :_M I_2) :_M I_3) = ((0 :_M I_1 \cap I_2) :_M I_3) =$$

$$(0:_M I_1 \cap I_2 \cap I_3) = N_1 + N_2 + (0:_M I_3).$$

■

An R-module M has *the pure sum property* if the sum of any two pure submodules is again pure [8].

Proposition 2.16. Let R be a PID and M be an R-module. Then M has the pure sum property if and only if M has the copure sum property.

Proof. This follows from the fact that every submodule N of M is a pure submodule of M if and only if it is a copure submodule of M by [4, Theorem 2.12]. ■

An R-module H is called a *multiplication module* if for each submodule K of H there exists an ideal J of R such that $K = JH$ [6].

Corollary 2.17. Let R be a PID and let M be a locally cyclic R-module (in particular, M be a multiplication R-module). Then M has the copure sum property.

Proof. By Proposition 2.16 and [8, Corollary 3.5]. ■

Example 2.18. The $\mathbb{Z}$-module $\mathbb{Z}$ has the copure sum property.

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