

Ruled Surfaces with Constant Slope Ruling According to Darboux Frame

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Abstract

The main scope of this paper is to investigate ruled surfaces with constant slope ruling with respect to the given surface according to Darboux Frame. We give necessary and sufficient conditions for these surfaces to be developable. Also, the investigations are observed under some special cases. Especially we obtained that the striction lines and Gaussian curvature of the surfaces.

Keywords: Ruled Surface; Surface with Constant Slope; Darboux Frame..

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1. Introduction

In differential geometry ruled surface is important surface which is generated by moving a straight line continuously in the space. This surface has many applications in the study of computer aided geometric design. In all studies on curves and surfaces, frame fields play an important role. The most commonly used frames are the Frenet and the Darboux frame. The Darboux frame is build by the velocity of the curve and the normal vector of the surface while the Frenet–Serret frame is constructed from the velocity and the acceleration of the curve. Indicating the derivatives of these frames' vector fields in terms of the vector fields themselves involve some real valued functions. These functions are called the curvature and torsion for the Frenet frame; normal curvature, geodesic curvature and the geodesic torsion for Darboux frame.

Striction curves plays an significant role in the differential geometry and are important for the study of singularities of constant slope surfaces, because it has some interesting specification. The singularities of a ruled surface are located on the striction curve.

2. Preliminary Results

In this section, we will give the necessary informations to understand the main subject of the study. A ruled surface is generated by a straight line acting along a curve. Let M be a surface and let consider a curve α on the surface M . In the Frenet frame $\{t, n, b\}$ at each points of the curve α , t is unit tangent vector, n is principal normal vector and b is binormal vector, respectively. The derivative of Frenet frame $\{t, n, b\}$ of the curve α is given as follows.

$$\begin{aligned} t' &= \kappa n \\ n' &= -\kappa t - \tau b \\ b' &= -\tau n \end{aligned}$$

The functions κ and τ , which are respectively determined by the first and third equations in Frenet formula, are called the first and second curvatures of $\alpha(s)$, or respectively the curvature and the torsion. The planes spanned by

$\{t, n\}$, $\{t, b\}$ and $\{n, b\}$ are called osculating plane, rectifying plane and normal plane, respectively. If the curve α lies on the surface M , then another frame is obtained of the curve α . This is called Darboux frame and formulized by $\{t, y, N\}$. In this frame t is the unit tangent of the curve, N is the unit normal of the surface M and y is a unit vector. Thus the relations between Frenet and Darboux frames obtained as follows

$$\begin{aligned}t' &= k_g y + k_n N, \\y' &= -k_g t + t_r N, \\N' &= -k_n t - t_r y,\end{aligned}$$

where, k_g is the geodesic curvature, defined by equality as follows

$$k_g(s) = \langle \alpha''(s), y(s) \rangle,$$

k_n is the normal curvature defined by following equality

$$k_n(s) = \langle \alpha''(s), N(s) \rangle,$$

and t_r is the geodesic torsion of $\alpha(s)$ with following equality

$$t_r(s) = -\langle N'(s), y(s) \rangle.$$

[4]

Ruled surface with base curve α and generating line X is parameterized as follows

$$M(u, v) = \alpha(u) + vX(u).$$

The equality of the striction curve on the ruled surface is dedicated as follows

$$\bar{\beta} = \alpha(s) - \frac{\langle \vec{t}, \vec{X} \rangle}{\langle \vec{X}', \vec{X}' \rangle} \cdot \vec{X}(s).$$

[5]

Ruled surface's Gaussian curvature is defined by

$$\begin{aligned}K &= \det S \\ &= -(\langle S(X), Y \rangle)^2,\end{aligned}$$

such that the vector fields X and Y are orthogonal [2].

The standard unit normal vector field n on a surface M can be defined by

$$n = \frac{\langle M_s, M_v \rangle}{\|\langle M_s, M_v \rangle\|}$$

The first fundamental form characterizes the interior geometry of the surface in a neighborhood of a given point. This means that measurements on the surface can be carried out by means of it. The first fundamental form is given by

$$I = E ds^2 + 2F ds dv + G dv^2$$

such that

$$E = \langle M_s, M_s \rangle, \quad F = \langle M_s, M_v \rangle \quad G = \langle M_v, M_v \rangle$$

At the same time the second fundamental form

$$II = eds^2 + 2fdsdv + gdv^2$$

where

$$e = \langle M_{ss}, n \rangle, \quad f = \langle M_{sv}, n \rangle, \quad g = \langle M_{vv}, n \rangle$$

The Gauss curvature is defined as Weingarten map

$$K = \det S.$$

The Gauss curvatures of a parametrized surface can be computed as

$$K = \frac{\det II}{\det I} = \frac{eg - f^2}{EG - F^2}$$

[1].

3. Ruled Surfaces with Constant Slope Ruling According to Darboux Frame

Let ϕ be a ruled surface whose generating lines are given by points on the curve α , while in all points they have the constant slope with respect to the tangent planes to the given surface [3]. This surfaces will be called ruled surfaces with constant slope ruling with respect to the given surface.

In this section, we investigate the developable condition for the ruled surfaces with constant slope ruling with respect to the given surface with $\{t, y, N\}$ and we obtain the singular locus of the this surface with Darboux Frame, at the same time we show that singular locus of the constant slope surface is collide with the striction line of the surface. Finally, give some relations and special cases about developable condition.

3.1 Ruled Surfaces with Constant Slope Ruling with Respect to the (t, y) Planes

Generating lines of the surface ϕ are given by points on the curve α and they have the constant slope σ with respect to the (t, y) planes to the curve at every point on the curve $X(s)$. The surface will be called the ruled surfaces with constant slope ruling with respect to the (t, y) planes to a curve.

Direction vector of generating lines of the surface is defined by

$$X(s) = \sin w(s)t(s) + \cos w(s)y(s) + \sigma N(s),$$

where the vector t is the direction vector of a tangent to the curve X , n is the direction vector of a normal to the surface and $y = N \times t$ is the direction vector. The surface with constant slope ruling is parametrized by

$$\phi(s, v) = \alpha(s) + vX(s).$$

Example 3.1. Let the curve be a cylindrical helix parametrized by the vector function

$$\alpha(s) = \left(4 \cos \frac{s}{5}, 4 \sin \frac{s}{5}, 3 \frac{s}{5} \right).$$

So the surface parametrization obtained as follows

$$\begin{aligned} \phi(s, v) = & \left(4 \cos \frac{s}{5} - v \left(\frac{4}{5} \sin \frac{4s}{25} \sin \frac{s}{5} + \cos \frac{4s}{25} \cos \frac{s}{5} - \frac{3}{40} \sin \frac{s}{5} \right), \right. \\ & \left. 4 \sin \frac{s}{5} - v \left(\frac{4}{5} \sin \frac{4s}{25} \cos \frac{s}{5} - \cos \frac{4s}{25} \sin \frac{s}{5} - \frac{3}{40} \cos \frac{s}{5} \right), \right. \\ & \left. 3 \frac{s}{5} + v \left(\frac{3}{5} \sin \frac{4s}{25} + \frac{1}{10} \right) \right) \end{aligned}$$

The surface for $s \in [0, 10]$, $v \in [-10, 10]$ and $w(s) = \frac{4}{25}s$ is shown in following figure

Figure 1

Definition 3.1. Surface can be developable ruled surface if the tangent planes are same along the generating lines of the ruled surface. Thus ruled surface is developable if and only if $P_X = 0$ [2].

Definition 3.2. Dral of the ruled surface is the ratio of the shortest distance between the two generating lines of ruled surface and the angle of between these two lines. So dral of the ruled surfaces is defined by

$$P_X = \frac{\det\left(\frac{d\alpha}{ds}, X, X'\right)}{\|X'\|^2}$$

[2].

Theorem 3.1. The necessary and sufficient condition for the ruled surface $\phi(s, v)$ to be developable is given as follows

$$\cos w(s) \cdot (k_n \sin w(s) + t_r \cos w(s)) - \sigma(k_g \sin w(s) - w' \sin w(s) - \sigma t_r) = 0.$$

Proof. The surface ϕ is developable if and only if $\det(t, X, X') = 0$. So derivative of the direction vector of generating lines of the surface is as follows

$$\begin{aligned} X'(s) &= \left((w'(s) - k_g) \cos w - \sigma k_n \right) t(s) \\ &+ \left(- (w'(s) - k_g) \sin w - \sigma t_r \right) y(s) \\ &+ (k_n \sin w(s) + t_r \cos w(s)) N(s). \end{aligned} \quad (3.1)$$

And so developable condition for surface $\phi(s, v)$ is given by

$$\begin{aligned} \det(t, X, X') &= \cos w(s) (k_n \sin w(s) + t_r \cos w(s)) - \sigma(k_g \sin w(s) - w' \sin w(s) - \sigma t_r), \\ \cos w(s) (k_n \sin w(s) + t_r \cos w(s)) &- \sigma(k_g \sin w(s) - w' \sin w(s) - \sigma t_r) = 0. \end{aligned}$$

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Corollary 3.1. We consider that $w = w_0 = \text{const}$. Then we get $\cos w_0 = x_1 = \text{const}$. and $\sin w_0 = x_2 = \text{const}$. So generating lines of the surface is given as follows

$$X(s) = x_1 t(s) + x_2 y(s) + \sigma N(s).$$

Let α be an asymptotic line. In that case $k_n = 0$. Since $\kappa^2 = k_n^2 + k_g^2$, we have $k_g = \kappa$ and $t_r = \tau$. If the surface developable then the main curve is general helix. In fact, from (3.1.1) we obtain as

$$\frac{\tau}{\kappa} = \frac{\sigma x_2}{x_1^2 + \sigma^2}.$$

Let α be a geodesic curve. In that case $k_g = 0$. Since $\kappa^2 = k_n^2 + k_g^2$, we have $k_n = \kappa$ and $t_r = \tau$. If the surface developable, then the main curve is general helix. In fact, from (3.1.1) we obtain as

$$\frac{\tau}{\kappa} = -\frac{x_1 x_2}{2x_1^2 + \sigma^2},$$

and α is a principal line, the surface is developable if and only if

$$x_2 (x_1 k_n - \sigma(k_g)) = 0.$$

By using above equation we can say that the surface is developable if and only if $w_0 = k\pi$, $k \in \mathbb{Z}$ or $x_1 k_n - \sigma k_g = 0$.

Corollary 3.2. If

$$w(s) = \sigma k_g ds,$$

such that $\sin w(s) \neq 0$ and α is a principal and asymptotic line, then surface ϕ is developable .

Theorem 3.2. The striction line on the surface $\phi(s, v)$ is defined by

$$\bar{\beta} = \alpha - \frac{-\sigma k_n + (w'(s) - k_g) \cos w(s)}{(w'(s) - k_g)((w'(s) - k_g) - 2\sigma k_n \cos w(s) + 2\sigma t_r \sin w(s)) + (k_n \sin w(s) + t_r \cos w(s))^2 + \sigma^2 (k_n^2 + t_r^2)} X(s).$$

Proof. By using (3.1.1), we calculate

$$\langle t, X' \rangle = -\sigma \kappa_n + (w'(s) - k_g) \cos w(s)$$

and

$$\begin{aligned} \langle X', X' \rangle &= (w'(s) - k_g)((w'(s) - k_g) - 2\sigma k_n \cos w(s) + 2\sigma t_r \sin w(s)) \\ &\quad + (k_n \sin w(s) + t_r \cos w(s))^2 + \sigma^2 (k_n^2 + t_r^2) \end{aligned}$$

It completes the proof.

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Corollary 3.3. If α is an asymptotic line, then $k_n = 0$. So striction line on the surface defined by

$$\bar{\beta} = \alpha - \frac{(w'(s) - k_g) \cos w(s)}{(w'(s) - k_g)(w'(s) - k_g + 2\sigma t_r \sin w(s)) + (t_r \cos w(s))^2 + \sigma^2 t_r^2} X(s).$$

α is principal line , then $t_r = 0$ and so striction line is given by

$$\bar{\beta} = \alpha - \frac{-\sigma \kappa_n + (w'(s) - k_g) \cos w(s)}{(w'(s) - k_g)(w'(s) - k_g - 2\sigma k_n \cos w(s)) + k_n \sin^2 w(s) + \sigma^2 k_n^2} X(s).$$

If α is a geodesic curve, then

$$\bar{\beta} = \alpha - \frac{-\sigma k_n + w'(s) \cos w(s)}{w'(s)(w'(s) - 2\sigma k_n \cos w(s) + 2\sigma t_r \sin w(s)) + (k_n \sin w(s) + t_r \cos w(s))^2 + \sigma^2 (k_n^2 + t_r^2)} X(s).$$

Corollary 3.4. If α is a geodesic curve and

$$w(s) = \sigma k_n \sec w(s) ds,$$

then striction line of surface $\phi(s, v)$ is equal to base curve.

Theorem 3.3. The Gaussian curvature of the surface the $\phi(s, v)$ is given by

$$K = - \frac{(ABv(\sigma - 1) + C \cos w(s) - B.\sigma)^2}{((1 + \sigma^2) \left((1 + vA)^2 + v^2 (B^2 + C^2) \right) - ((1 + vA) \sin w(s) + B.v. \cos w(s) + C.v.\sigma)^2) .D},$$

where

$$\begin{aligned} A &= -\sigma k_n + (w'(s) - k_g) \cos w(s), \\ B &= -\sigma t_r - (w'(s) - k_g) \sin w(s), \\ C &= k_n \sin w(s) + t_r \cos w(s), \\ D &= (Bv\sigma - Cv \cos w(s))^2 - (\sigma.(1 + vA) - Cv \sin w(s))^2 \\ &\quad + ((1 + vA) \cos w(s) - Bv \sin w(s))^2. \end{aligned}$$

Proof. The coefficients E, F and G of the first fundamental form of the surface $\phi(s, v)$ are given as follows

$$E = (1 + vA)^2 + v^2 (B^2 + C^2),$$

$$F = (1 + vA) \cdot \sin w(s) + B \cdot v \cdot \cos w(s) + C \cdot v \cdot \sigma,$$

$$G = 1 + \sigma^2.$$

After simple computations, we can easily obtain coefficients f and g of the second fundamental form of $\phi(s, v)$ obtained as follows

$$f = \frac{A \cdot B \cdot v(\sigma - 1) + C \cdot \cos w(s) - B \cdot \sigma}{D}$$

$$g = 0.$$

The Gauss curvature of a parametrized surface can be computed as follows

$$K = \frac{eg - f^2}{EG - F^2}.$$

So we substitute values of coefficients of first and second fundamental form of the $\phi(s, v)$ in above equation, we can find Gauss curvature of $\phi(s, v)$ as follows

$$K = -\frac{(ABv(\sigma - 1) + C \cos w(s) - B \cdot \sigma)^2}{((1 + \sigma^2) \left((1 + vA)^2 + v^2 (B^2 + C^2) \right) - ((1 + vA) \sin w(s) + B \cdot v \cdot \cos w(s) + C \cdot v \cdot \sigma)^2) \cdot D}.$$

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3.2 Ruled Surfaces with Constant Slope Ruling with Respect to the (N, y) Planes

Surface $\bar{\phi}$ are given by points on the curve α and they have the constant slope σ with respect to the (N, y) planes to the curve at every point on the curve $\bar{X}(s)$. The surface will be called the ruled surfaces with constant slope ruling with respect to the (N, y) planes to the a curve. Direction vector of generating lines of the surface is given by

$$\bar{X}(s) = \sin w(s)N(s) + \cos w(s)y(s) + \sigma t(s).$$

The surface with constant slope ruling $\bar{\phi}$ is parametrized by

$$\bar{\phi}(s, v) = \alpha(s) + v\bar{X}(s).$$

Example 3.2. Let the curve parametrized by the vector function

$$\alpha(s) = \left(\frac{4}{\sqrt{5}} \cos \frac{s\sqrt{5}}{4}, \frac{4}{\sqrt{5}} \sin \frac{s\sqrt{5}}{4}, \frac{r}{\sqrt{5}} \right).$$

and the surface parametrized by

$$\begin{aligned} \bar{\phi}(s, v) = & \left(\frac{4}{\sqrt{5}} \cos \frac{s\sqrt{5}}{4} + v \left(\sin \frac{s}{2\sqrt{5}} \cdot \frac{1}{\sqrt{5}} \cos \frac{s\sqrt{5}}{4} - \frac{1}{\sqrt{5}} \cos \frac{s\sqrt{5}}{4} \cos \frac{s}{2\sqrt{5}} - \frac{1}{4} \sin \frac{s\sqrt{5}}{4} \right) \right. \\ & \left. \frac{4}{\sqrt{5}} \sin \frac{s\sqrt{5}}{4} + v \left(\sin \frac{s}{2\sqrt{5}} \cdot \frac{1}{\sqrt{5}} \sin \frac{s\sqrt{5}}{4} - \frac{1}{\sqrt{5}} \sin \frac{s\sqrt{5}}{4} \cos \frac{s}{2\sqrt{5}} + \frac{1}{4} \cos \frac{s\sqrt{5}}{4} \right) \right. \\ & \left. \frac{4}{\sqrt{5}} + v \left(\sin \frac{s}{2\sqrt{5}} + \frac{1}{\sqrt{5}} \right) \right) \end{aligned}$$

The surface for $s \in [0, 3\sqrt{5}\pi]$, $v \in [-5, 5]$ and $w(s) = \frac{s}{2\sqrt{5}}$ is shown in following figure

Figure 2

Theorem 3.4. *The ruled surface $\bar{\phi}(s, v)$ is developable if and only if*

$$-w'(s) - t_r + \sigma(k_g \sin w(s) - k_n \cos w(s)) = 0$$

Proof. If we take derivative of the direction vector of generating lines of the surface \bar{X} with respect to the parameter s , then we get

$$\begin{aligned} \bar{X}'(s) &= (w'(s) + t_r) \cos w(s) + k_n \sigma N(s) + (- (w'(s) + t_r) \sin w(s) + k_g \sigma) y(s) \\ &\quad + (-k_n \sin w(s) - k_g \cos w(s)) t(s). \end{aligned}$$

We easily find that

$$\begin{aligned} \det(t, \bar{X}, \bar{X}') &= \sin w(s) (\sin w(s) (-t_r - w'(s)) + k_g \sigma) - \cos w(s) (\cos w(s) (w'(s) + t_r) + k_n \sigma) \\ &\quad - (w'(s) + t_r) + \sigma (k_g \sin w(s) - k_n \cos w(s)). \end{aligned}$$

So developable condition for surface $\bar{\phi}(s, v)$ is given by

$$-w'(s) - t_r + \sigma(k_g \sin w(s) - k_n \cos w(s)) = 0.$$

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Corollary 3.5. *We consider that $\cos w(s) = x_1 = \text{const}$ and $\sin w(s) = x_2 = \text{const}$.*

$$X(s) = x_1 N(s) + x_2 y(s) + \sigma t(s).$$

Let α is an asymptotic line, the surface is developable if and only if base curve is general helix by

$$\frac{\tau}{\kappa} = \sigma x_2.$$

Similarly, α is a geodesic curve, the surface is developable if and only if base curve is general helix by

$$\sigma x_1 = \frac{\tau}{\kappa}.$$

Theorem 3.5. *The striction line on the surface $\bar{\phi}(s, v)$ is given by*

$$\bar{\beta} = \alpha - \left(\frac{-k_n \sin w(s) - k_g \cos w(s)}{(w'(s) + t_r)^2 + 2\sigma(w'(s) + t_r)(k_n \cos w(s) - k_g \sin w(s)) + (k_n \sin w(s) + k_g \cos w(s))^2 + \sigma^2(k_n^2 + k_g^2)} \right) X(s).$$

Corollary 3.6. If $(w'(s) + t_r) = 0$, then

$$\bar{\beta} = \alpha + (-k_n \sin w(s) - k_g \cos w(s)(k_n \sin w(s) + k_g \cos w(s))^2 + \sigma^2(k_n^2 + k_g^2)).X(s).$$

If α is an asymptotic curve, then $k_n = 0$. So striction line on the surface defined by

$$\bar{\beta} = \alpha - \left(\frac{-k_g \cos w(s)}{(w'(s) + t_r)^2 + 2\sigma(w'(s) + t_r) - k_g \sin w(s) + (k_g \cos w(s))^2 + \sigma^2 k_g^2} \right) X(s).$$

α be a principal line, then $t_r = 0$ and so striction line is

$$\bar{\beta} = \alpha(s) - \left(\frac{-k_n \sin w(s) - k_g \cos w(s)}{w'(s)^2 + 2\sigma w'(s)[k_n \cos w(s) - k_g \sin w(s)] + [k_n \sin w(s) + k_g \cos w(s)]^2 + \sigma^2(k_n^2 + k_g^2)} \right) X(s).$$

If α is a geodesic curve, then

$$\bar{\beta} = \alpha(s) - \left(\frac{-k_n \sin w(s)}{(w'(s) + t_r)^2 + 2\sigma(w'(s) + t_r)k_n \cos w(s) + [k_n \sin w(s)]^2 + \sigma^2 k_n^2} \right) X(s).$$

Theorem 3.6. The Gaussian curvature of the surface $\bar{\phi}(s, v)$ is given by

$$K = - \frac{((w'(s) + t_r)(\sigma(b \cos^2 w(s) + a \sin^2 w(s)) + c) - c(1 + vc))^2}{(v^2(a^2 + b^2 + c^2) + 1 - 2vc)(1 + \sigma^2) - \sigma^2 \cdot (c(1 - v) + 1)^2} \cdot d,$$

where

$$\begin{aligned} a &= (w'(s) + t_r) \cos w(s) + k_n \sigma, \\ b &= -(w'(s) + t_r) \sin w(s) + k_g \sigma, \\ c &= (k_n \sin w(s) + k_g \cos w(s)), \\ d &= (\sigma b - c(1 + vc) \cos w(s))^2 + (\sigma a - c(1 + vc) \sin w(s))^2 + (w'(s) + t_r)^2. \end{aligned}$$

3.3 Ruled Surfaces with Constant Slope Ruling with Respect to the (t, N) Planes

Generating lines of the surface $\tilde{\phi}$ are given by points on the curve α and they have the constant slope σ with respect to the (t, N) planes to the curve at every point on the curve $\tilde{X}(s)$. The surface will be called the ruled surfaces with constant slope ruling with respect to the (t, N) planes to the a curve. Direction vector of generating lines of the surface is obtained as follows

$$\tilde{X}(s) = \sin w(s)t(s) + \cos w(s)N(s) + \sigma y(s).$$

The surface with constant slope ruling $\tilde{\phi}$ is parametrized by

$$\tilde{\phi}(s, v) = \alpha(s) + v(\sin w(s)t(s) + \cos w(s)N(s) + \sigma y(s)).$$

Theorem 3.7. The surface $\tilde{\phi}(s, v)$ is developable if and only if

$$\cos w(s)(\sin w(s)k_g - \cos w(s)t_r) + \sigma(-w'(s) + k_n \sin w(s) + \sigma t_r) = 0.$$

Proof. By taking derivative of the direction vector of generating lines of the surface \tilde{X} with respect to parameter s , then we get

$$\begin{aligned} \tilde{X}'(s) &= t(s)[(w'(s) - k_n) \cos w(s) - \sigma k_g] \\ &\quad + N(s)[-(w'(s) - k_n) \sin w(s) + \sigma t_r] \\ &\quad + y(s)[\sin w(s) \cdot k_g - t_r \cos w(s)] \end{aligned}$$

$$\det(t, \tilde{X}, \tilde{X}') = \cos w(s) [\sin w(s)k_g - \cos w(s)t_r] + \sigma [-(w'(s) - k_n) \sin w(s) + \sigma t_r].$$

So developable condition for surface $\bar{\phi}(s, v)$ is given by

$$\cos w(s) (\sin w(s)k_g - \cos w(s)t_r) + \sigma (-(w'(s) - k_n) \sin w(s) + \sigma t_r) = 0$$

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Corollary 3.7. We consider that $\cos w(s) = x_1 = \text{const}$ and $\sin w(s) = x_2 = \text{const}$.

$$X(s) = x_1 N(s) + x_2 y(s) + \sigma t(s).$$

Let α is an asymptotic line, the surface is developable if and only if base curve is general helix by

$$\frac{\tau}{\kappa} = \frac{x_1 x_2}{x_1^2 + \sigma^2}.$$

Similarly, α is a geodesic curve, the surface is developable if and only if the base curve is general helix by

$$\frac{\tau}{\kappa} = \frac{\sigma x_2}{x_1^2 + \sigma^2},$$

and α is a principal line, the surface is developable if and only if

$$x_2(x_1 k_g + \sigma k_n) = 0$$

so

$$\frac{k_n}{k_g} = \frac{-x_1}{\sigma}.$$

Corollary 3.8. If

$$w(s) = k_n d_s,$$

such that $\sin w(s) \neq 0$ and α is a principal and geodesic line, then surface $\tilde{\phi}$ is developable.

Theorem 3.8. The striction line on the surface $\tilde{\phi}$ is obtained as follows

$$\bar{\beta} = \alpha - (w'(s) - k_n) \cos w(s) - \sigma k_g \frac{(w'(s) - k_n)^2 + \sigma^2(k_g^2 + t_r^2) - 2\sigma(w'(s) - k_n)(k_g \cos w(s) + t_r \sin w(s))}{((\sin w(s)k_g - t_r \cos w(s))^2)} \cdot \tilde{X}(s).$$

Corollary 3.9. At the same time both $w'(s) = k_n$, striction line on the surface is given by

$$\bar{\beta} = \alpha - (\sigma k_g \sigma^2(k_g^2 + t_r^2) + ((\sin w(s)k_g - t_r \cos w(s))^2) \cdot \tilde{X}(s).$$

Corollary 3.10. Let $\cos w(s) = x_1 = \text{const}$ and $\sin w(s) = x_2 = \text{const}$. If

$$\frac{k_g}{k_n} = \frac{-x_1}{\sigma},$$

then striction line on the surface $\tilde{\phi}$ equal to base curve of the surface the $\tilde{\phi}$.

Corollary 3.11. It has been shown in [1], singular locus of the constant slope surfaces $\phi(s, v)$, $\tilde{\phi}(s, v)$, $\bar{\phi}(s, v)$ are striction lines of these surfaces.

Theorem 3.9. The Gaussian curvature of the surface $\tilde{\phi}(s, v)$ is given by

$$K = -\frac{((1 + vP)(R\sigma - S \cos w(s) + \sigma Rv + v.S.\sigma) - 2vRS \sin w(s))^2}{((1 + vP)^2 + v^2(R^2 + S^2))(1 + \sigma^2)}.T$$

where

$$\begin{aligned} P &= (w'(s) - k_n) \cos w(s) - \sigma k_g, \\ R &= -(w'(s) - k_n) \sin w(s) + \sigma t_r, \\ S &= \sin w(s).k_g - t_r \cos w(s), \\ T &= (R\sigma - S \cos w(s))^2 + (\sigma(1 + vP) - S \sin w(s))^2 + ((1 + vP)\sigma - R \sin w(s))^2. \end{aligned}$$

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