Konuralp Journal of Mathematics, 7 (2) (2019) 371-379

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath
e-ISSN: 2147-625X

New Perturbed Inequalities for Functions Whose Higher Degree Derivatives are Absolutely Continuous

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Abstract

We firstly derive inequalities for high order differentiable functions with the property (S) and mappings whose higher derivatives are convex by using the same equality. Also, it is obtained Hermite Hadamard type and Bullen type inequalities for higher order differentiable functions. Then, we establish inequalities for high degree Lipschitzian derivatives via an equality which was presented previous by Erden in [12]. We also examine connection in between inequalities obtained in earlier works and our results.

Keywords: *Absolutely continuous functions, Convex functions, Lipschitzian derivatives, Perturbed inequalities.*
2010 Mathematics Subject Classification: *26D15, 26D10, 26A46.*

1. Introduction

In 1938, Ostrowski [19] established an useful inequality as follows:
Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty:=\sup\limits_{t\in(a,b)}|f'(t)|<\infty$. Then, we have the inequality

$$\left|f(x)-\frac{1}{b-a}\int\limits_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\left\|f'\right\|_\infty, \tag{1.1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.
The classical Hermite-Hadamard inequality which was first published in [15] gives us an estimate of the mean value of a convex function $f:I\to\mathbb{R}$,

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}. \tag{1.2}$$

In [4], Bullen proved the following inequality which is known as Bullen's inequality for convex function:
Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$. The inequality

$$\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]. \tag{1.3}$$

The above inequalities have wide applications in numerical analysis and in the theory of some special means. What is more, these inequalities are found to have a number of uses. In particular, various generalizations and developments of (1.1) and (1.2) are deduced (for example [22]). Hence, these inequalities have attracted considerable attention and interest from mathematicans and researchers. Now, we give the equalities established to obtain some perturbed inequalities which were proved in recent years.
In [8], Dragomir established the following identity in order to obtain some perturbe inequalities of Ostrowski type.

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Theorem 1.1. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous on $[a,b]$ and $x\in[a,b]$. Then, for any $\lambda_1(x)$ and $\lambda_2(x)$ complex numbers, we have*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-\lambda_1(x)\right]dt+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-\lambda_2(x)\right]dt\\
&= f(x)+\frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x)-(x-a)^2\lambda_1(x)\right]-\frac{1}{b-a}\int_a^b f(t)dt
\end{aligned}$$

where the integrals in the left hand side are taken in the Lebesgue sense.

In [12], Erden obtained the following equality in order to give some perturbed inequalities of Ostrowski type for higher degree differentiable functions.

Lemma 1.2. *Let $f:[a,b]\to\mathbb{C}$ be an $n-$times differantiable function on (a,b) and $x\in[a,b]$. Then, for any $\lambda_1(x)$ and $\lambda_2(x)$ complex numbers, we have the identity*

$$\begin{aligned}
&\int_a^x \frac{(t-a)^n}{n!}\left[f^{(n)}(t)-\lambda_1(x)\right]dt+\int_x^b \frac{(t-b)^n}{n!}\left[f^{(n)}(t)-\lambda_2(x)\right]dt \\
&= \sum_{k=0}^{n-1}\frac{(-1)^{n+1}(b-x)^{k+1}+(-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!}f^{(k)}(x)\\
&\quad -\lambda_1(x)\frac{(x-a)^{n+1}}{(n+1)!}-(-1)^n\lambda_2(x)\frac{(b-x)^{n+1}}{(n+1)!}+(-1)^n\int_a^b f(t)dt
\end{aligned}\tag{1.4}$$

where the integrals in the left hand side of the equality (1.4) are taken in the Lebesgue sense.

In recent years, some researchers have studied some generalizations of Ostrowski and Hermite Hadamard type inequalities for differentiable, twice differentiable and $n-$times differentiable functions. For example, some Ostrowski type inequalities are obtained for twice differentiable functions in [6] and [7]. What is more, in [5],[23] and [24], some mathematicians established generalized inequalities of Ostrowski type for higher order differentiable functions on L_1, L_p and L_∞. In addition, in [14], it is established a generalization of the inequality (1.2) for twice differentiable functions by Farissi et. al. In [20] and [21], the researchers deduced midpoint and trapezoidal formula related to the inequality (1.2) for n-times differentiable mappings, respectively. In [16], [17] and [18], it is established Hermite-Hadamard type inequalities for n-times differentiable mappings by Latif and Dragomir. Furthermore, Ardic gave some new inequalities for $n-$times differentiable convex functions in [1]. On the other side, in [8]-[10], it is presented perturbed inequalities of Ostrowski type for absolutely continuous functions by Dragomir. Afterwards, some mathematicians studied perturbed Ostrowski type inequalities for twice differentiable functions in [3] and [11]. Recently, in [2], [12] and [13], it is given some perturbed inequalities of Ostrowski type for functions whose nth derivatives are of bounded variation, convex mappings and functions whose nth derivatives are absolutely continuous.
In this study, it is presented some perturbed inequalities of Ostrowski type inequalities for higher order differentiable convex functions and n times differentiable mappings with the property (S). Also, Hermite-Hadamard and Bullen type inequalities are obtained by using these inequalities. Next, some generalized inequalities are deduced for higher order Lipschitzian derivatives and these results give perturbed inequalities presented in earlier works.

2. Inequalities for n Time Differentiable Functions with the Property (S)

It is given some inequalities for functions which satisfying the property (S) in this section.
Let $f:I\to\mathbb{C}$ be an n times differentiable convex function on I° and $[a,b]\subset I^\circ$. Then $f^{(n)}$ is monotonic nondecreasing on $[a,b]$. In this case, if we take n as an odd number and also write $f^{(n)}(a)$ and $f^{(n)}(b)$ instead of $\lambda_1(x)$ and $\lambda_2(x)$ in the identity (1.4), respectively, then we have

$$\int_a^b f(t)dt\le\sum_{k=0}^{n-1}\frac{(b-x)^{k+1}+(-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!}f^{(k)}(x)-f^{(n)}(a)\frac{(x-a)^{n+1}}{(n+1)!}+f^{(n)}(b)\frac{(b-x)^{n+1}}{(n+1)!}\tag{2.1}$$

for any $x\in[a,b]$.
Let $f:I\to\mathbb{C}$ be an n times differentiable convex function on I° and $[a,b]\subset I^\circ$. It said that the function f satisfies the *property* (S) on $[a,b]$, if f provides the condition $f^{(n)}(a)\le f^{(n)}(t)\le f^{(n)}(b)$ for any $t\in[a,b]$. Therefore, we observe that the inequality (2.1) remains valid for functions f satisfy the *property* (S).

Theorem 2.1. *Let $f:I\to\mathbb{C}$ be an n times differentiable function on I° and $[a,b]\subset I^\circ$, and let n be odd number.*
(i) If $f^{(n-1)}$ satisfies the property (S) on the intervals $[a,x]$ and $[x,b]$ for any $x\in[a,b]$, then we have the inequality

$$\begin{aligned}
&\sum_{k=0}^{n-1}\frac{(b-x)^{k+1}+(-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!}f^{(k)}(x)-\int_a^b f(t)dt\\
&\le \frac{(x-a)^{n+1}-(b-x)^{n+1}}{(n+1)!}f^{(n)}(x)
\end{aligned}\tag{2.2}$$

(ii) If $f^{(n-1)}$ satisfies the property (S) on $[a,b]$, then we have

$$\begin{aligned} & \sum_{k=0}^{n-1} \frac{(b-x)^{k+1}+(-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) - \int_a^b f(t)dt \\ \leq \; & n\frac{(b-x)^{n+1} f^{(n)}(b) - (x-a)^{n+1} f^{(n)}(a)}{(n+1)!} \\ & + \frac{(x-a)^n + (b-x)^n}{n!} f^{(n-1)}(x) \\ & - \frac{(x-a)^n f^{(n-1)}(a) + (b-x)^n f^{(n-1)}(b)}{n!} \end{aligned} \tag{2.3}$$

for any $x \in [a,b]$.

Proof. If we take $\lambda_1(x) = f^{(n)}(a)$ and $\lambda_2(x) = f^{(n)}(x)$ in (1.4), because n is odd number, we have

$$\begin{aligned} & \sum_{k=0}^{n-1} \frac{(b-x)^{k+1}+(-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) \\ & -f^{(n)}(a)\frac{(x-a)^{n+1}}{(n+1)!} + f^{(n)}(b)\frac{(b-x)^{n+1}}{(n+1)!} - \int_a^b f(t)dt \\ = \; & \int_a^x \frac{(t-a)^n}{n!}\left[f^{(n)}(t) - f^{(n)}(a)\right]dt + \int_x^b \frac{(b-t)^n}{n!}\left[f^{(n)}(b) - f^{(n)}(t)\right]dt. \end{aligned} \tag{2.4}$$

(i) Since $f^{(n-1)}$ satisfies the *property* (S) on the intervals $[a,x]$ and $[x,b]$, we get

$$\begin{aligned} & \int_a^x \frac{(t-a)^n}{n!}\left[f^{(n)}(t) - f^{(n)}(a)\right]dt + \int_x^b \frac{(b-t)^n}{n!}\left[f^{(n)}(b) - f^{(n)}(t)\right]dt \\ \leq \; & \left[f^{(n)}(x) - f^{(n)}(a)\right]\int_a^x \frac{(t-a)^n}{n!}dt + \left[f^{(n)}(b) - f^{(n)}(x)\right]\int_x^b \frac{(b-t)^n}{n!}dt. \end{aligned} \tag{2.5}$$

If we combine the expressions (2.4) and (2.5) and also apply necessary operations, then we obtain the inequality (2.2).
(ii) Because $f^{(n-1)}$ satisfies the *property* (S) on $[a,b]$, we have

$$\begin{aligned} & \int_a^x \frac{(t-a)^n}{n!}\left[f^{(n)}(t) - f^{(n)}(a)\right]dt + \int_x^b \frac{(b-t)^n}{n!}\left[f^{(n)}(b) - f^{(n)}(t)\right]dt \\ \leq \; & \frac{(x-a)^n}{n!}\int_a^x \left[f^{(n)}(t) - f^{(n)}(a)\right]dt + \frac{(b-x)^n}{n!}\int_x^b \left[f^{(n)}(b) - f^{(n)}(t)\right]dt \end{aligned} \tag{2.6}$$

for any $x \in [a,b]$, and then using the elementary analysis operations, with the help of (2.4) and (2.6), the inequality (2.3) can be readily deduced.
The proof is thus completed. □

Remark 2.2. *Under the same assumptions of Theorem 2.1 with $n=1$, then the following inequalities hold:*

$$f(x) + \left(\frac{a+b}{2} - x\right) f'(x) \leq \frac{1}{b-a}\int_a^b f(t)dt$$

and

$$\frac{(x-a)f(a) + (b-x)f(b)}{1} - \int_a^b f(t)dt \leq \frac{(b-x)^2 f'(b) - (x-a)^2 f'(a)}{2}$$

which were given by Dragomir in [10].

Remark 2.3. *If we take $x = \frac{a+b}{2}$ in (2.2), then we have*

$$\sum_{k=0}^{n-1} \frac{(b-a)^{k+1}\left[1+(-1)^{n+1-k}\right]}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) \leq \int_a^b f(t)dt \tag{2.7}$$

which is higher order Hermite- Hadamard type inequality. Also, if we choose $n=1$ in (2.7), then we get the inequality in left side of the Hermite-Hadamard inequality.

Theorem 2.4. *Let $f : I \to \mathbb{C}$ be an n times differentiable function on I° and $[a,b] \subset I^{\circ}$, and let n be odd number. If $f^{(n-1)}$ is convex on the interval $[a,x]$ and $[x,b]$, then we have*

$$\begin{aligned} &\sum_{k=0}^{n-1} \frac{(b-x)^{k+1} + (-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) \\ &+ \frac{(b-x)^n f^{(n-1)}(b) + (x-a)^n f^{(n-1)}(a)}{(n+1)!} \\ \geq\; & \frac{(b-x)^n + (x-a)^n}{(n+1)!} f^{(n-1)}(x) + \int_a^b f(t)dt \end{aligned} \tag{2.8}$$

for any $x \in [a,b]$.

Proof. Now, we use Chebyshev inequality for synchronous functions (functions with same monotonicity), namely

$$\frac{1}{d-c}\int_c^d g(t)h(t)dt \geq \frac{1}{d-c}\int_c^d g(t)dt \frac{1}{d-c}\int_c^d h(t)dt, \tag{2.9}$$

for two integrals given in the right side of the equality (2.4).
Due to the fact that $f^{(n-1)}$ is monotonic nondecreasing on $[a,x]$ and $[x,b]$, from (2.9), we have the inequalities

$$\int_a^x \frac{(t-a)^n}{n!}\left[f^{(n)}(t) - f^{(n)}(a)\right]dt \geq \frac{1}{x-a}\int_a^x \frac{(t-a)^n}{n!}dt \int_a^x \left[f^{(n)}(t) - f^{(n)}(a)\right]dt$$

and

$$\int_x^b \frac{(b-t)^n}{n!}\left[f^{(n)}(b) - f^{(n)}(t)\right]dt \geq \frac{1}{b-x}\int_x^b \frac{(b-t)^n}{n!}dt \int_x^b \left[f^{(n)}(b) - f^{(n)}(t)\right]dt.$$

If we sum these two inequalities, after we calculate integrals in the right hand side of the inequalities, then we easily obtain the desired result. Hence, the proof is completed. □

Remark 2.5. *If we take $x = \frac{a+b}{2}$ in (2.8), then we have*

$$\begin{aligned} &\sum_{k=0}^{n-1} \frac{(b-a)^{k+1}\left[1 + (-1)^{n+1-k}\right]}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) \\ &+ \frac{(b-a)^n \left[f^{(n-1)}(b) + f^{(n-1)}(a)\right]}{2^n (n+1)!} \\ \geq\; & \frac{(b-a)^n}{2^{n-1}(n+1)!} f^{(n-1)}\left(\frac{a+b}{2}\right) + \int_a^b f(t)dt \end{aligned} \tag{2.10}$$

which is higher order Bullen type inequalities. Also, if we choose $n = 1$ in (2.10), then the inequality (2.10) reduce to (1.3) that is Bullen inequality.

Remark 2.6. *Under the same assumptions of Theorem 2.4 with $n = 1$, then the following inequality holds:*

$$\frac{1}{2}\left[f(x) + \frac{(b-x)f(b) + (x-a)f(a)}{b-a}\right] \geq \frac{1}{b-a}\int_a^b f(t)dt$$

which was proved by Dragomir in [10].

3. Inequalities for High Order Lipschitzian Derivatives

Now, we establish some perturbed inequalities of Ostrowski type for higher order Lipschitzian derivatives. In addition, we present some special results of these inequalities. We firstly give the following definition.
$u : [a,b] \to \mathbb{C}$ is said to be *Lipschitzian* with the constant $L > 0$, if it satisfies the condition

$$|u(t) - u(s)| \leq L|t-s| \qquad \text{for any } t,s \in [a,b].$$

Theorem 3.1. *Let $f : I \to \mathbb{C}$ be an n times differentiable function on I° and $[a,b] \subset I^\circ$. Also, let $x \in (a,b)$. If the nth derivative $f^{(n)} : I^\circ \to \mathbb{C}$ is Lipschitzian with the constant $K_1(x)$ on $[a,x]$ and constant $K_2(x)$ on $[x,b]$, then we have*

$$\begin{aligned} &\left| \sum_{k=0}^{n-1} \frac{(-1)^{n+1}(b-x)^{k+1} + (-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) + (-1)^n \int_a^b f(t)dt \right. \\ &\left. - \frac{f^{(n)}(a)+f^{(n)}(x)}{2} \frac{(x-a)^{n+1}}{(n+1)!} - (-1)^n \frac{f^{(n)}(x)+f^{(n)}(b)}{2} \frac{(b-x)^{n+1}}{(n+1)!} \right| \\ \leq\ & \frac{1}{2} K_1(x) \frac{(x-a)^{n+2}}{(n+1)!} + \frac{1}{2} K_2(x) \frac{(b-x)^{n+2}}{(n+1)!}. \end{aligned} \tag{3.1}$$

Proof. We take absolute value of the equality (1.4) for $\lambda_1(x) = \frac{f^{(n)}(a)+f^{(n)}(x)}{2}$ and $\lambda_2(x) = \frac{f^{(n)}(x)+f^{(n)}(b)}{2}$, we find that

$$\begin{aligned} &\left| \sum_{k=0}^{n-1} \frac{(-1)^{n+1}(b-x)^{k+1} + (-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) + (-1)^n \int_a^b f(t)dt \right. \\ &\left. - \frac{f^{(n)}(a)+f^{(n)}(x)}{2} \frac{(x-a)^{n+1}}{(n+1)!} - (-1)^n \frac{f^{(n)}(x)+f^{(n)}(b)}{2} \frac{(b-x)^{n+1}}{(n+1)!} \right| \\ \leq\ & \int_a^x \frac{(t-a)^n}{n!} \left| f^{(n)}(t) - \frac{f^{(n)}(a)+f^{(n)}(x)}{2} \right| dt \\ &+ \int_x^b \frac{(b-t)^n}{n!} \left| f^{(n)}(t) - \frac{f^{(n)}(x)+f^{(n)}(b)}{2} \right| dt. \end{aligned} \tag{3.2}$$

On the grounds that $f^{(n)} : I^\circ \to \mathbb{C}$ is Lipschitzian with the constant $K_1(x)$ on $[a,x]$, we get

$$\begin{aligned} \left| f^{(n)}(t) - \frac{f^{(n)}(a)+f^{(n)}(x)}{2} \right| &= \left| \frac{f^{(n)}(t) - f^{(n)}(a) + f^{(n)}(t) - f^{(n)}(x)}{2} \right| \\ &\leq \frac{\left| f^{(n)}(t) - f^{(n)}(a) \right| + \left| f^{(n)}(t) - f^{(n)}(x) \right|}{2} \\ &\leq \frac{1}{2} K_1(x) \left[|t-a| + |x-t| \right] \\ &= \frac{1}{2} K_1(x)(x-a). \end{aligned} \tag{3.3}$$

Similarly, Since $f^{(n)} : I^\circ \to \mathbb{C}$ is Lipschitzian with the constant $K_2(x)$ on $[x,b]$, we have

$$\left| f^{(n)}(t) - \frac{f^{(n)}(x)+f^{(n)}(b)}{2} \right| \leq \frac{1}{2} K_2(x)(b-x). \tag{3.4}$$

If we substitute the inequalities (3.3) and (3.4) in (3.2) and then calculate the integrals given in right-hand side of the inequality (3.2), we can easily obtain desired inequality (3.1) which completes the proof. □

Remark 3.2. *Let $f : I \to \mathbb{C}$ be an n times differentiable function on I° and $[a,b] \subset I^\circ$. Also, let $x \in (a,b)$. If the nth derivative $f^{(n)} : I^\circ \to \mathbb{C}$ is Lipschitzian with the constant K on $[a,b]$, then we have*

$$\begin{aligned} &\left| \sum_{k=0}^{n-1} \frac{(-1)^{n+1}(b-x)^{k+1} + (-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) + (-1)^n \int_a^b f(t)dt \right. \\ &\left. - \frac{f^{(n)}(a)+f^{(n)}(x)}{2} \frac{(x-a)^{n+1}}{(n+1)!} - (-1)^n \frac{f^{(n)}(x)+f^{(n)}(b)}{2} \frac{(b-x)^{n+1}}{(n+1)!} \right| \\ \leq\ & \frac{K}{2} \left[\frac{(x-a)^{n+2}}{(n+1)!} + \frac{(b-x)^{n+2}}{(n+1)!} \right]. \end{aligned}$$

Remark 3.3. *Under the same assumptions of Theorem 3.1 with $n = 1$, then the following inequality holds:*

$$\begin{aligned} &\left| f(x) + \frac{1}{2}\left(\frac{a+b}{2} - x \right) f'(x) - \frac{1}{b-a} \int_a^b f(t)dt \right. \\ &\left. + \frac{1}{4(b-a)} \left[f'(b)(b-x)^2 - f'(a)(x-a)^2 \right] \right| \\ \leq\ & \frac{1}{4(b-a)} \left[K_1(x)(x-a)^3 + K_2(x)(b-x)^3 \right] \end{aligned}$$

which was given by Dragomir in [8].

Remark 3.4. *If we take $n=2$ in (3.1), then we have the inequality*

$$\begin{aligned} & \left|(b-a)\left(x-\frac{a+b}{2}\right)f'(x)-(b-a)f(x)+\int_a^b f(t)dt\right. \\ & \left. -\frac{f''(a)(x-a)^3+f''(b)(b-x)^3+f''(x)\left[(x-a)^3+(b-x)^3\right]}{12}\right| \\ \leq & \frac{1}{12}\left[K_1(x)(x-a)^4+K_2(x)(b-x)^4\right]. \end{aligned}$$

which was established by Budak et al. in [3].

Theorem 3.5. *Let $f:I\to\mathbb{C}$ be an n time differentiable function on I° and $[a,b]\subset I^\circ$. In addition, let $x\in(a,b)$. If the inequalities*

$$\left|f^{(n)}(t)-f^{(n)}(x)\right|\leq L_\alpha(x-t)^\alpha \quad \text{for any } t\in[a,x) \tag{3.5}$$

and

$$\left|f^{(n)}(t)-f^{(n)}(x)\right|\leq L_\beta(t-x)^\beta \quad \text{for any } t\in(x,b] \tag{3.6}$$

are satisfied for $\alpha,\beta>-1$ and $L_\alpha,L_\beta>0$, then we have

$$\begin{aligned} & \left|\sum_{k=0}^{n-1}\frac{(-1)^{n+1}(b-x)^{k+1}+(-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!}f^{(k)}(x)+(-1)^n\int_a^b f(t)dt\right. \\ & \left.-f^{(n)}(x)\left[\frac{(x-a)^{n+1}}{(n+1)!}+(-1)^n\frac{(b-x)^{n+1}}{(n+1)!}\right]\right| \\ \leq & \frac{L_\alpha}{n!}(x-a)^{n+\alpha+1}B(n+1,\alpha+1)+\frac{L_\beta}{n!}(b-x)^{n+\beta+1}B(n+1,\beta+1) \end{aligned} \tag{3.7}$$

where $B(n+1,\alpha+1)$ is beta function.

Proof. Taking modulus in (1.4) for $\lambda_1(x)=\lambda_2(x)=f^{(n)}(x)$, and then utilizing the properties (3.5) and (3.6), we get the inequality

$$\begin{aligned} & \left|\sum_{k=0}^{n-1}\frac{(-1)^{n+1}(b-x)^{k+1}+(-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!}f^{(k)}(x)+(-1)^n\int_a^b f(t)dt\right. \\ & \left.-f^{(n)}(x)\left[\frac{(x-a)^{n+1}}{(n+1)!}+(-1)^n\frac{(b-x)^{n+1}}{(n+1)!}\right]\right| \\ \leq & \frac{L_\alpha}{n!}\int_a^x(t-a)^n(x-t)^\alpha dt+\frac{L_\beta}{n!}\int_x^b(b-t)^n(t-x)^\beta dt. \end{aligned} \tag{3.8}$$

Now, we calculate the integrals in right hand side of (3.8). If we use change of the variable

$$u=\frac{x-t}{x-a} \qquad du=\frac{-dt}{(x-a)}$$

for the first integral, then we get

$$\begin{aligned} \int_a^x(t-a)^n(x-t)^\alpha dt &= \int_a^x[(x-a)-(x-t)]^n(x-t)^\alpha dt \\ &= (x-a)^{n+\alpha+1}\int_0^1(1-u)^n u^\alpha du \\ &= (x-a)^{n+\alpha+1}B(n+1,\alpha+1). \end{aligned}$$

Similarly, we have

$$\int_x^b(b-t)^n(t-x)^\beta dt=(b-x)^{n+\beta+1}B(n+1,\beta+1).$$

Hence, the proof is completed. □

Remark 3.6. *Let $f: I \to \mathbb{C}$ be an n time differentiable function on I° and $[a,b] \subset I^\circ$. If the nth derivative $f^{(n)}$ is r−Hölder type on $[a,b]$, then we have the inequality*

$$\left| f^{(n)}(t) - f^{(n)}(s) \right| \leq H |t-s|^r \tag{3.9}$$

for any $t,s \in [a,b]$, where $r \in (0,1]$ and $H > 0$ are given. In this case, the following inequality holds:

$$\begin{aligned} & \left| \sum_{k=0}^{n-1} \frac{(-1)^{n+1}(b-x)^{k+1} + (-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) + (-1)^n \int_a^b f(t)dt \right. \\ & \left. - f^{(n)}(x) \left[\frac{(x-a)^{n+1}}{(n+1)!} + (-1)^n \frac{(b-x)^{n+1}}{(n+1)!} \right] \right| \\ \leq \quad & H \left[(b-x)^{n+r+1} + (x-a)^{n+r+1} \right] \frac{\Gamma(r+1)}{\Gamma(n+r+1)} \end{aligned}$$

for any $x \in [a,b]$. Especially, if $f^{(n)}$ is Lipschitzian with the constant $L > 0$, then we have

$$\begin{aligned} & \left| \sum_{k=0}^{n-1} \frac{(-1)^{n+1}(b-x)^{k+1} + (-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) + (-1)^n \int_a^b f(t)dt \right. \\ & \left. - f^{(n)}(x) \left[\frac{(x-a)^{n+1}}{(n+1)!} + (-1)^n \frac{(b-x)^{n+1}}{(n+1)!} \right] \right| \\ \leq \quad & \frac{L}{(n+2)!} \left[(b-x)^{n+2} + (x-a)^{n+2} \right] \end{aligned}$$

for any $x \in [a,b]$.

Remark 3.7. *Under the same assumptions of Theorem 3.5 with $n = 1$, then the following inequality holds:*

$$\begin{aligned} & \left| (b-a) f(x) + \frac{(b-x)^2 - (x-a)^2}{2} f'(x) - \int_a^b f(t)dt \right| \\ \leq \quad & \frac{L_\alpha}{(\alpha+1)(\alpha+2)} (x-a)^{\alpha+2} + \frac{L_\beta}{(\beta+1)(\beta+2)} (b-x)^{\beta+2} \end{aligned}$$

which was given by Dragomir in [9].

Remark 3.8. *If we take $n = 2$ in (3.7), then we have*

$$\begin{aligned} & \left| (b-a)\left(x - \frac{a+b}{2} \right) f'(x) - (b-a) f(x) + \int_a^b f(t)dt \right. \\ & \left. - \frac{1}{2} f''(x) \left[\frac{(b-x)^3 + (x-a)^3}{3} \right] \right| \\ \leq \quad & \frac{L_\alpha (x-a)^{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)} + \frac{L_\beta (b-x)^{\beta+3}}{(\beta+1)(\beta+2)(\beta+3)} \end{aligned}$$

which was proved by Erden et al. in [11].

Theorem 3.9. *Suppose that all the assumptions of the Theorem 3.5. If the conditions (3.5) and (3.6) are satisfied, then we have the inequality*

$$\begin{aligned} & \left| \sum_{k=0}^{n-1} \frac{(-1)^{n+1}(b-x)^{k+1} + (-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) + (-1)^n \int_a^b f(t)dt \right. \\ & \left. - \frac{(x-a)^{n+1} f^{(n)}(a) + (-1)^n (b-x)^{n+1} f^{(n)}(b)}{(n+1)!} \right| \\ \leq \quad & \frac{L_\alpha}{n!(n+\alpha+1)} (x-a)^{n+\alpha+1} + \frac{L_\beta}{n!(n+\beta+1)} (b-x)^{n+\beta+1} \end{aligned} \tag{3.10}$$

for any $x \in [a,b]$.

Proof. If we choose $\lambda_1(x) = f^{(n)}(a)$ and $\lambda_2(x) = f^{(n)}(b)$ and then we take modulus in (1.4), because of the properties (3.5) and (3.6), we get

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+1}(b-x)^{k+1} + (-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) + (-1)^n \int_a^b f(t)dt \right. \\
&\left. - \frac{(x-a)^{n+1} f^{(n)}(a) + (-1)^n (b-x)^{n+1} f^{(n)}(b)}{(n+1)!} \right| \\
\leq\ & \int_a^x \frac{(t-a)^n}{n!} \left| f^{(n)}(t) - f^{(n)}(a) \right| dt + \int_x^b \frac{(b-t)^n}{n!} \left| f^{(n)}(b) - f^{(n)}(t) \right| dt \\
\leq\ & L_\alpha \int_a^x \frac{(t-a)^n}{n!} (t-a)^\alpha dt + L_\beta \int_x^b \frac{(b-t)^n}{n!} (b-t)^\beta dt.
\end{aligned}$$

If we calculate the above integrals, we obtain desired inequality (3.10) which completes the proof. □

Remark 3.10. *Suppose that all the assumptions of the Corollary 3.6. If $r-$Hölder type inequality (3.9) is valid, then we have*

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+1}(b-x)^{k+1} + (-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) + (-1)^n \int_a^b f(t)dt \right. \\
&\left. - \frac{(x-a)^{n+1} f^{(n)}(a) + (-1)^n (b-x)^{n+1} f^{(n)}(b)}{(n+1)!} \right| \\
\leq\ & \frac{H}{n!(n+r+1)} \left[(x-a)^{n+r+1} + (b-x)^{n+r+1} \right]
\end{aligned}$$

for any $x \in [a,b]$. In particular, if $f^{(n)}$ is Lipschitzian with the constant $L > 0$, then we have

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+1}(b-x)^{k+1} + (-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) + (-1)^n \int_a^b f(t)dt \right. \\
&\left. - \frac{(x-a)^{n+1} f^{(n)}(a) + (-1)^n (b-x)^{n+1} f^{(n)}(b)}{(n+1)!} \right| \\
\leq\ & \frac{L}{n!(n+2)} \left[(x-a)^{n+2} + (b-x)^{n+2} \right]
\end{aligned}$$

for any $x \in [a,b]$.

Remark 3.11. *If we take $n = 1$ in (3.10), then we have the inequality*

$$\left| (b-a) f(x) + \frac{(b-x)^2 f'(b) - (x-a)^2 f'(a)}{2} - \int_a^b f(t)dt \right| \leq \frac{L_\alpha}{\alpha+2} (x-a)^{\alpha+2} + \frac{L_\beta}{\beta+2} (b-x)^{n+\beta+1}$$

which was given by Dragomir in [10].

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