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Global Dynamics of Solutions of A New Class of Rational Difference Equations

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Abstract

The purpose of this paper is to investigate the global dynamics of solutions of the following delay nonlinear difference equation

$$x_{n+1} = a + \frac{bx_{n-k}}{x_{n-l}} + \frac{cx_{n-l}}{x_{n-k}}, n = 0, 1, ...$$

where the parameters a, b, c are non-zero real numbers, $k, l \in \mathbb{Z}^+$ and the initial values $x_{-\max\{k,l\}}, ..., x_{-1}, x_0 \in \mathbb{R} - \{0\}$. The results obtained here improve and generalize some known ones in the literature. Moreover, several numerical simulations are provided to support obtained results.

Keywords: *Dynamics, Two periodic solution, Boundedness, Attractivity.*
2010 Mathematics Subject Classification: *39A10, 39A11.*

1. Introduction

For the last twenty years, the usage of difference equations is an extremely important research and difference equations have been applied in various mathematical models that are biology, engineering, ecology, statistics sciences and so on. The investigation of difference equations is not only valuable in its own rights, but also it can provide insights into their continuous counterparts , namely, ordinary and partial differential equations (ODEs and PDEs). ODEs and PDEs are solved widely by using their approximate difference equation formulations. For instance, any applied mathematics researcher making an investigation of differential equations will recognise that even simple type equations can be so diffucult to obtain the solution. On the contrary, the related discrete type equation, namely, difference equation is quite easier to solve. These are only considered as the discrete analogs of ODEs and PDEs, which is unfortunately the common view. It should be noted that difference equations appeared much earlier than ODEs and PDEs. It is only recently that difference equations have started receiving the worth they merit. Especially, the researcher who endeavor the computer scientists must use discrete equations in his investigation if he does not want to be forced. If we give an example to this situation, we can say that dynamic equations usually show up while specifying the cost of an algorithm in big-O notation. The utulazition of difference equations enormously decrease the complication and labour force in many areas of science.

A class of difference equations, namely, non-linear difference equations of order greater than one are of paramount importance in applications. It is very interesting to investigate the behavior of solutions of higher-order rational difference equations. A lot of work has been concentrated on it [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

In [7], Elsayed has obtained the period two and three solution using a new method of the following rational difference equation

$$x_{n+1} = \alpha + \frac{\beta x_n}{x_{n-1}} + \frac{\gamma x_{n-1}}{x_n}, n = 0, 1, ... \tag{1.1}$$

with positive parameters and positive initial conditions. In that paper, Elsayed has gained very important and constructive results, but there exist some notes about the paper as follows;

-Note that Theorem 4.3 in [7] is failure. That is, the author claimed that the equilibrium point $\bar{x}$ of Eq.(1.1) is a global attractor without any condition. Also, the proof of this theorem is wrong. For example, for $\alpha = 2$, $\beta = 0.11$, $\gamma = 222$, the equilibrium point of Eq.(1.1) is not a global attractor. (See Figure 6). In main results of this paper, we will give the correct proof for genaralized equation.

Also, in [5] Moaaz has investigated the results of [7] and he has also revealed some important results.

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In [4], Moaaz et al. have studied the behavior of solutions of the following rational difference equation

$$x_{n+1}=\alpha+\frac{\beta x_{n-r}}{x_{n-s}}+\frac{\gamma x_{n-r}}{x_{n-t}},\ n=0,1,... \tag{1.2}$$

where the initial conditions are arbitrary positive real numbers and α,β,γ are positive constans. In that paper the authors obtained very important results, but there exist some notes about the paper as follows;

- Note that the proof Theorem 6 is failure. That is, they have claimed that Eq.(1.2) has no prime period two solutions if $\alpha\neq\beta+\gamma$. They made a calculate error. So, this Theorem is wrong. But, we must state that in [5] the author gave correct proof in Theorem 3.1 for a different type equation. For the correct proof of Theorem 6, the readers should look the proof of Theorem 3.1 in [5].

The purpose of this paper is to extend the results due to [4, 5, 7] and investigate the global dynamics of solutions of the following difference equation

$$x_{n+1}=a+\frac{bx_{n-k}}{x_{n-l}}+\frac{cx_{n-l}}{x_{n-k}},\ n=0,1,... \tag{1.3}$$

where the parameters a,b,c are non-zero real numbers, $k,l\in\mathbb{Z}^+$ and the initial values $x_{-\max\{k,l\}},...,x_{-1},x_0\in\mathbb{R}-\{0\}$.
There exist many other papers related with Eq.(1.3) and on its extensions (see [1, 2, 3, 6, 8, 9, 12, 13]). Motivated by these papers, we study of the difference equation (1.3).
As far as we examine, there is no paper dealing with Eq.(1.3). Therefore, it is meaningful to study their deep results.

2. Preliminaries

For the completeness in the paper, we find useful to remind some basic concepts of the difference equations theory as follows:
Let I be an interval of real numbers and let $f:I^{k+3}\to I$ be a continuously differentiable function. Then for any condition $x_{-(k+2)}$, $x_{-(k+1)},...,x_{-1}$, $x_0\in I$, the difference equation

$$x_{n+1}=f(x_n,x_{n-1},...,x_{n-(k+1)},x_{n-(k+2)}),\ n\in\mathbb{N} \tag{2.1}$$

has a unique positive solution $\{x_n\}_{n=-(k+2)}^{\infty}$.

Definition 2.1. *An equilibrium point of Eq.(2.1) is a point $\bar{x}$ that satisfies*

$$\bar{x}=f(\bar{x},\bar{x},...,\bar{x}).$$

The point $\bar{x}$ is also said to a fixed point of the function f.

Definition 2.2. *Let $\bar{x}$ be a positive equilibrium of (2.1).*
(a) $\bar{x}$ is stable if for every $\varepsilon>0$, there is $\delta>0$ such that for every positive solution $\{x_n\}_{n=-(k+2)}^{\infty}$ of (2.1) with $\sum\limits_{i=-(k+2)}^{0}|x_i-\bar{x}|<\delta$, $|x_n-\bar{x}|<\varepsilon$, holds for $n\in\mathbb{N}$.
(b) $\bar{x}$ is locally asymptotically stable if $\bar{x}$ is stable and there is $\gamma>0$ such that $\lim x_n=\bar{x}$ holds for every positive solution $\{x_n\}_{n=-(k+2)}^{\infty}$ of (2.1) with $\sum\limits_{i=-(k+2)}^{0}|x_i-\bar{x}|<\gamma$.
(c) $\bar{x}$ is a global attractor if $\lim x_n=\bar{x}$ holds for every positive solution $\{x_n\}_{n=-(k+2)}^{\infty}$ of (2.1).

(d) $\bar{x}$ is globally asymptotically stable if $\bar{x}$ is both stable and global attractor.

Definition 2.3. *The linearized equation of (2.1) about the equilibrium point $\bar{x}$ is*

$$y_{n+1}=\zeta_0y_n+\zeta_1y_{n-1}+...+\zeta_{k+1}y_{n-(k+1)}+\zeta_{k+2}y_{n-(k+2)},\ n\in\mathbb{N} \tag{2.2}$$

where

$$\zeta_0=\frac{\partial f}{\partial x_n}(\bar{x},\bar{x},...,\bar{x}),\ \zeta_1=\frac{\partial f}{\partial x_{n-1}}(\bar{x},\bar{x},...,\bar{x}),...,\ \zeta_{k+2}=\frac{\partial f}{\partial x_{n-(k+2)}}(\bar{x},\bar{x},...,\bar{x}).$$

The characteristic equation of (2.2) is

$$\lambda^{k+3}-\zeta_0\lambda^{k+2}-\zeta_1\lambda^{k+1}-...-\zeta_{k+1}\lambda-\zeta_{k+2}=0. \tag{2.3}$$

Theorem 2.4. *(Linearized Stability Theorem) Consider Eq.(2.1) such that $\bar{x}$ is a fixed point of f. If all roots of the function f about $\bar{x}$ lie inside the open unit disk $|\lambda|<1$, then $\bar{x}$ is locally asymtotically stable. If one of them has a modulus greater than one, then $\bar{x}$ is unstable. The fixed point $\bar{x}$ of f is called a saddle point if f has roots both inside and outside the unit disk. If any root of f has absolute value equal to one, then the fixed point $\bar{x}$ of f is called a non-hyperbolic point.*

The following comparison result will be useful in estimating the values of a solution of Eq.(2.1).

Theorem 2.5. *[10] (Comparison Theorem)*
Let m be a non-negative integer, let $\alpha_0,\alpha_1,...,\alpha_m$ be non-negative real numbers and let β be a real number. Suppose that $\{x_n\}_{n=-m}^{\infty}$, $\{y_n\}_{n=-m}^{\infty}$ and $\{z_n\}_{n=-m}^{\infty}$ are sequences of real numbers such that

$x_n \le y_n \le z_n$, *for all* $-m \le n \le 0$

and such that

$$\begin{array}{rcl} x_{n+1} & \le & \alpha_0 x_n + \alpha_1 x_{n-1} + ... + \alpha_m x_{n-m} + \beta \\ y_{n+1} & = & \alpha_0 y_n + \alpha_1 y_{n-1} + ... + \alpha_m y_{n-m} + \beta \\ z_{n+1} & \ge & \alpha_0 z_n + \alpha_1 z_{n-1} + ... + \alpha_m z_{n-m} + \beta \end{array}$$

for all $n = 0,1,...,$ then

$x_n \le y_n \le z_n$, *for all* $-m \le n$.

Theorem 2.6. *[10] (Clark theorem) Assume that $a,b \in \mathbb{R}$ and $k \in \{0,1,\ldots\}$. Then,*

$|p| + |q| < 1$

is a sufficient condition for the asymptotic stability of the delay difference equation

$z_{n+1} + pz_{n-k} + qz_{n-l} = 0, n = 0,1,\ldots.$

Theorem 2.7. *Assume that $b > 0$ and k is even. Then, the delay difference equation*

$z_{n+1} + bz_n - bz_{n-k} = 0,\ n = 0,1,\ldots$

is asymptotically stable if and only if

$$0 < b < \frac{1}{2\cos\left(\frac{\pi}{k+2}\right)}.$$

For other basic knowledge about difference equations, the reader is referred to [10, 11].

3. Dynamics of Eq.(1.3)

In this section we will give the global dynamics of Eq.(1.3).
It is clear that the unique positive equilibrium point of Eq.(1.3) is

$\bar{x} = a + b + c.$

If we define the function $h : A \subset (0,\infty)^2 \to B \subset (0,\infty)$,

$$h(x,y) = a + \frac{bx}{y} + \frac{cy}{x},$$

then we can obtain

$$\frac{\partial h}{\partial x}(x,y) = \frac{b}{y} - \frac{cy}{x^2}$$

and

$$\frac{\partial h}{\partial y}(x,y) = -\frac{bx}{y^2} + \frac{c}{x}.$$

Theorem 3.1. *If $2 < \frac{a+b+c}{|b-c|}$, then the positive equilibrium point $\bar{x} = a + b + c$ of Eq.(1.3) is locally asymptotically stable.*

Proof. The linearized equation of Eq.(1.3) about the equilibrium point $\bar{x}$ is

$y_{n+1} = py_{n-k} + qy_{n-l}$

where

$$p = \frac{\partial f}{\partial x}(\bar{x},\bar{x}) = \frac{b}{a+b+c} - \frac{c}{a+b+c}$$

and

$$q = -\frac{b}{a+b+c} + \frac{c}{a+b+c}.$$

From here, we can obtain $|p| + |q| = \frac{2|b-c|}{a+b+c}$. Using Theorem 2.6, one can conclude that Eq.(1.3) is asymptotically stable if $|p| + |q| < 1$. Thus, it must be $2 < \frac{a+b+c}{|b-c|}$ which is our assumption. The proof is completed. □

Corollary 3.2. *Assume $c > b$, k is even and $l = 0$ in Eq.(1.3). Then, the positive equilibrium point $\bar{x} = a+b+c$ of Eq.(1.3) is asymptotically stable if and only if*

$$c < b + \frac{a+b+c}{2\cos(\frac{\pi}{k+2})}.$$

Corollary 3.3. *Assume $b > c$, l is even and $k = 0$ in Eq.(1.3). Then, the positive equilibrium point $\bar{x} = a+b+c$ of Eq.(1.3) is asymptotically stable if and only if*

$$b < c + \frac{a+b+c}{2\cos(\frac{\pi}{l+2})}.$$

Theorem 3.4. *If $b+c < a$, then every solution of Eq.(1.3) is bounded and persists.*

Proof. Let $\{x_n\}_{n=-\max\{k,l\}}^{\infty}$ be any solution of Eq.(1.3). From Eq.(1.3), we can obtain that

$$a \le x_{n+1} = a + \frac{bx_{n-k}}{x_{n-l}} + \frac{cx_{n-l}}{x_{n-k}}, \text{ for all } n \ge 0.$$

Thus, $\{x_n\}_{n=-\max\{k,l\}}^{\infty}$ persists.
Also, from Eq.(1.3) and the above inequlity, we can obtain

$$x_{n+1} \le a + \frac{bx_{n-k}}{a} + \frac{cx_{n-l}}{a}.$$

By using *Comparison Theorem, it follows that*

$$\lim_{n\to\infty} \sup x_n \le \frac{a^2}{a-b-c}.$$

Therefore, for all $n \ge 0$

$$x_n \in \left[a, \frac{a^2}{a-b-c}\right].$$

Thus, the solution is bounded and this completes the proof. □

Theorem 3.5. *If*

$$a \ge b+c, \tag{3.1}$$

then the positive equilibrium point $\bar{x} = a+b+c$ of Eq.(1.3) is globally asymptotically stable.

Proof. We know by Theorem 3.1 that the positive equilibrium point $\bar{x}$ of Eq.(1.3) is locally asymptotically stable. So, it suffices to show that every solution of Eq.(1.3) converges to the positive equilibrium point $\bar{x}$ of Eq.(1.3). From (3.1) and Theorem 3.4, we have that every solution of Eq.(1.3) is bounded. So we have

$$0 < m = \liminf x_n,\ M = \limsup x_n < \infty. \tag{3.2}$$

We want to obtain that $m = M$. It is clear that $M \ge m$. So, it is enough to show that $m \ge M$. Then from Eq.(1.3) and (3.2) we get

$$M \le a + b\frac{M}{m} + c\frac{M}{m} \text{ and } m \ge a + b\frac{m}{M} + c\frac{m}{M}$$

and so

$$Mm \le am + (b+c)M \text{ and } Mm \ge aM + (b+c)m.$$

Hence,

$$aM + (b+c)m \le am + (b+c)M.$$

This implies that

$$(b+c)(m-M) \le a(m-M). \tag{3.3}$$

From (3.1) and (3.3) we obtain $m \ge M$. So, $m = M$ which implies that $\{x_n\}_{n=-\max\{k,l\}}^{\infty}$ converges to the positive equilibrium point $\bar{x}$ of Eq.(1.3). The proof is completed. □

Corollary 3.6. *If $\alpha \ge \beta + \gamma$, then the equilibrium point of Eq.(1.1) is a global attractor.*

4. Two periodic solutions using New Method

In this section, we will study the existence of two periodic solutions using the new method which was introduced by E. M. Elsayed in [7]. Thanks to this method, demonstration of the existence of two periodic solutions is quite easier than the method commonly used in the literature. It also provides short and easy proof for periodic solutions of Eq.(1.3).

Theorem 4.1. *Assume that $n \in \mathbb{R} - \{0, \pm 1\}$, the parameters a, b, c are non-zero real numbers, $k, l \in \mathbb{Z}^+$ and the initial values $x_{-\max\{k,l\}}, \ldots, x_{-1}, x_0 \in \mathbb{R} - \{0\}$.*
(i) Let k be odd and l be odd, then Eq.(1.3) has no eventual prime period two solutions.
(ii) Let k be even and l be even, then Eq.(1.3) has no eventual prime period two solutions.
(iii) Let k be odd and l be even, then Eq.(1.3) possesses eventual prime period two solutions if and only if

$$a = b - c\left(n + \frac{1}{n} + 1\right). \tag{4.1}$$

(iv) Let k be even and l be odd, then Eq.(1.3) possesses eventual prime period two solutions if and only if

$$a = c - b\left(n + \frac{1}{n} + 1\right).$$

Proof. (i) Let

$$\ldots, x, y, x, y, \ldots$$

be the eventual prime period two solution of Eq.(1.3) such that $x \neq y$. From Eq.(1.3) we obtain

$$x = y = a + b + c$$

which contradicts the assumption $x \neq y$. So, Eq.(1.3) has no eventual prime period two solutions.
(ii) The proof of this case is proven the same way of the proof (i).
(iii) First, we assume that Eq.(1.3) has eventual prime period two solutions in the following form

$$\ldots, x, y, x, y, \ldots.$$

We shall show that Condition (4.1) holds. By using Elsayed's new method, from (1.3) we get

$$x = a + b\frac{x}{y} + c\frac{y}{x}$$

and

$$y = a + b\frac{y}{x} + c\frac{x}{y}.$$

We assume that $n = \frac{x}{y}$. So, we rewrite the equilities above

$$x = a + bn + \frac{c}{n} \tag{4.2}$$

and

$$y = a + \frac{b}{n} + cn.$$

From the second equility, we have

$$ny = na + b + cn^2. \tag{4.3}$$

Subtracting (4.3) from (4.2) gives

$$x - ny = 0 = a(1-n) + b(n-1) + c(\frac{1}{n} - n^2)$$

and so

$$a(n-1) + c(n^2 - \frac{1}{n}) = b(n-1).$$

Since $n \in \mathbb{R} - \{0, \pm 1\}$, we can obtain that

$$a = b - c\left(n + \frac{1}{n} + 1\right).$$

Thus, Condition (4.1) holds.

Also, from (4.2) and (4.3) we obtain that

$$\begin{aligned} x &= b-c\left(n+\frac{1}{n}+1\right)+bn+\frac{c}{n} \\ &= (b-c)(n+1), \end{aligned}$$

and

$$\begin{aligned} y &= b-c\left(n+\frac{1}{n}+1\right)+\frac{b}{n}+cn \\ &= \frac{(b-c)(n+1)}{n}. \end{aligned}$$

Secondly, assume that Condition (4.1) holds. We shall show that Eq.(1.3) has eventual prime period two solutions. Let

$x=(b-c)(n+1)$

and

$y=\frac{(b-c)(n+1)}{n}$.

From this, it is clear that $x \neq y$ with $n \in \mathbb{R}\backslash\{0,\pm1\}$. We assume in Eq.(1.3) that $k>l$. We choose the initial conditions as $x_{-(k+1)}=x$, $x_{-k}=y,...,x_{-l}=x$, $x_{-l+1}=y,\dots,x_{-1}=y$, $x_0=x$. We shall show that $x_1=y$, $x_2=x$. From Eq.(1.3)

$$\begin{aligned} x_1 &= a+b\frac{y}{x}+c\frac{x}{y}=b-c\left(n+\frac{1}{n}+1\right)+\frac{b}{n}+nc \\ &= \frac{(b-c)(n+1)}{n} \\ &= y \end{aligned}$$

and

$$\begin{aligned} x_2 &= a+b\frac{x}{y}+c\frac{y}{x}=b-c\left(n+\frac{1}{n}+1\right)+nb+\frac{c}{n} \\ &= (b-c)(n+1) \\ &= x. \end{aligned}$$

By induction, we can obtain $x_{2n}=x$ and $x_{2n+1}=y$ for all $n \geq -k$. Therefore, Eq.(1.3) has a prime period two solution of the following form

$y,x,y,x,\dots,$

where $x \neq y$. This completes the proof.
(iv) The proof of this case is proven the same way of the proof (iii). □

5. Numerical Simulations

In order to verify our theoretical results we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions of Eq.(1.3). All plots in this section are drawn with Mathematica.
Example (1) Consider the following non-linear difference equations

$$x_{n+1}=a+b\frac{x_{n-1}}{x_{n-4}}+c\frac{x_{n-4}}{x_{n-1}},\ n=0,1,.... \tag{5.1}$$

with $a=3.5$, $b=1.1$, $c=2.2$ $(a \geq b+c)$ and the initial values $x_{-4}=1.6$, $x_{-3}=11.6$, $x_{-2}=0.6$, $x_{-1}=3.6$, $x_0=0.8$. Then the positive equilibrium point $\bar{x}=6.8$ of Eq.(5.1) is globally asymptotically stable (see Figure 1).
Example (2) Consider the following non-linear difference equations

$$x_{n+1}=a+b\frac{x_{n-3}}{x_{n-4}}+c\frac{x_{n-4}}{x_{n-3}},\ n=0,1,.... \tag{5.2}$$

with $a=1$, $b=4.85$, $c=1.1$ $\left(a=b-c\left(n+\frac{1}{n}+1\right)\right)$ and the initial values $x_{-4}=11.6$, $x_{-3}=1.6$, $x_{-2}=0.6$, $x_{-1}=1$, $x_0=0.8$. Eq.(5.2) has a prime period two solution of the following form

$\dots,11.25,5.625,11.25,5.625,\dots,$

(see Figure 2).
Example (3) Consider the following non-linear difference equations

$$x_{n+1}=a+b\frac{x_{n-4}}{x_{n-5}}+c\frac{x_{n-5}}{x_{n-4}},\ n=0,1,.... \tag{5.3}$$

with $a=1$, $b=2$, $c=8$ $\left(a=c-b\left(n+\frac{1}{n}+1\right)\right)$ and the initial values $x_{-5}=1.3$, $x_{-4}=1.6$, $x_{-3}=0.4$, $x_{-2}=2.6$, $x_{-1}=1$, $x_0=0.8$. Eq.(5.3) has a prime period two solution of the following form

$\dots,18,9,18,9,\dots,$

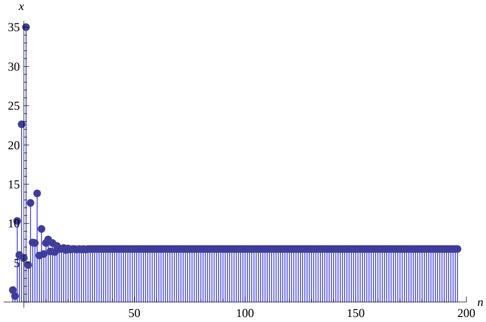

Figure 5.1: The plot of Eq.(5.1)

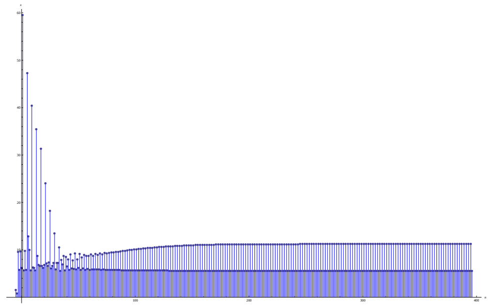

Figure 5.2: The plot of Eq.(5.2)

(see Figure 3).
Example (4) Consider the following non-linear difference equations

$$x_{n+1} = a + b\frac{x_{n-4}}{x_n} + c\frac{x_n}{x_{n-4}}, n = 0,1,.... \tag{5.4}$$

with $a = 3$, $b = 2$, $c = 10$ $(c < b + \frac{a+b+c}{2\cos(\frac{\pi}{k+2})})$ and the initial values $x_{-4} = 12.1$, $x_{-3} = 14.4$, $x_{-2} = 23.3$, $x_{-1} = 12.2$, $x_0 = 10$. the positive equilibrium point $\bar{x} = 15$ of Eq.(5.4) is locally asymptotically stable (see Figure 4).

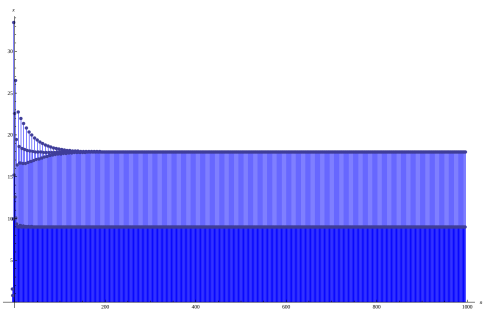

Figure 5.3: The plot of Eq.(5.3)

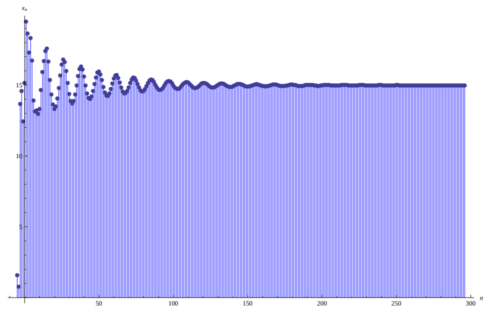

Figure 5.4: The plot of Eq.(5.4)

Example (5) Consider the following non-linear difference equations

$$x_{n+1} = a + b\frac{x_{n-2}}{x_{n-4}} + c\frac{x_{n-4}}{x_{n-2}}, n = 0,1,.... \tag{5.5}$$

with $a = 3$, $b = 0.11$, $c = 222$ $(a < b + c)$ and the initial values $x_{-4} = 1.6$, $x_{-3} = 11.6$, $x_{-2} = 7.6$, $x_{-1} = 6$, $x_0 = 8$. the positive equilibrium point $\bar{x} = 15$ of Eq.(1.3) is not globally asymptotically stable. Also, Eq.(5.5) has unbounded solution (see Figure 5).
Example (6) Consider Eq.(1.1) with $a = 2$, $b = 0.11$, $c = 222$ and the initial values $x_{-1} = 68$, $x_0 = 81$. the positive equilibrium point $\bar{x} = 15$ of Eq.(1.1) is not a global attractor (see Figure 6).

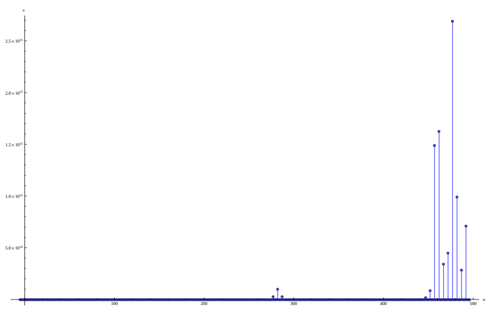

Figure 5.5: The plot of Eq.(5.5)

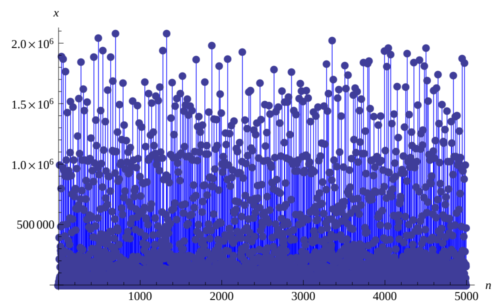

Figure 5.6: The plot of Eq.(1.1)

6. Conclusions

It is well known that using difference equations in problems involving time-dependent fluid flows, neutron diffusion and convection, radiation flow, thermonuclear reactions and the solution of several partial differential equations at the same time provides great convenience. Differently the utilization of difference equations as approximations to ODEs and PDEs, they also avail a powerful method for the analysis of electrical, mechanical, thermal, and other systems in which there is a recurrence of identic parts. By usage the difference equations, the investigation of

the behavior of electric-wave filters, multistage booster, magnetic amplifiers, insulator strings, continuous beams of equal span, crankshafts of multicylinder engines, acoustical filters, etc., is hugely simplified. The standard methods for solving such systems are in general very long when the number of elements related is grand.
In the paper, we completed the picture as regards the global behavior of positive solutions of Eq.(1.3). The main aim of dynamical systems theory is to approach the global behavior of solutions. So, we here give the asymptotic behavior of solutions for a class of non-linear difference equations. The results obtained here improve and generalize [4, 5, 7]. Also, we present some results about the general behavior of solutions of Eq.(1.3) and some numerical effective examples are provided to support our theoretical results.

7. Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this manuscript.

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