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Fuzzy Sets in UP-algebras with Respect to A Triangular Norm

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Abstract

In this paper, we apply the notion of fuzzy sets with respect to a triangular norm to UP-algebras. We introduce the notions of T-fuzzy UP-subalgebras, T-fuzzy near UP-filters, T-fuzzy UP-filters, T-fuzzy UP-ideals, and T-fuzzy strongly UP-ideals, their properties are investigated and some useful examples are discussed. We discuss the relations between T-fuzzy UP-subalgebras (resp., T-fuzzy near UP-filters, T-fuzzy UP-filters, T-fuzzy UP-ideals, and T-fuzzy strongly UP-ideals) and a notion of UP-subalgebras (resp., near UP-filters, UP-filters, UP-ideals, strongly UP-ideals), and their level subsets and UP-homomorphisms are studied. Moreover, we have introduced the notion of fuzzy sets with respect to a triangular norm of anti-type in UP-algebras, and studied the properties as well as previous notions.

Keywords: *UP-algebra; T-fuzzy UP-subalgebra; T-fuzzy near UP-filter; T-fuzzy UP-filter; T-fuzzy UP-ideal; T-fuzzy strongly UP-ideal*
2010 Mathematics Subject Classification: *03G25; 08A72*

1. Introduction

The branch of the logical algebra, a UP-algebra was introduced by Iampan [5] in 2017, and it is known that the class of KU-algebras [17] is a proper subclass of the class of UP-algebras. It have been examined by several researchers, for example, Somjanta et al. [31] introduced the notion of fuzzy sets in UP-algebras, the notion of intuitionistic fuzzy sets in UP-algebras was introduced by Kesorn et al. [9], Kaijae et al. [8] introduced the notions of anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras of UP-algebras, the notion of Q-fuzzy sets in UP-algebras was introduced by Tanamoon et al. [34], Sripaeng et al. [33] introduced the notion anti-Q-fuzzy UP-ideals and anti Q-fuzzy UP-subalgebras of UP-algebras, the notion of $\mathcal{N}$-fuzzy sets in UP-algebras was introduced by Songsaeng and Iampan [32], Senapati et al. [27, 28] applied cubic set and interval-valued intuitionistic fuzzy structure in UP-algebras, Mosrijai et al. [15] proved UP-isomorphism theorems for UP-algebras in the meaning of the congruence determined by a UP-homomorphism. Romano [18, 19] studied UP-ideals and proper UP-filters of UP-algebras, etc.
A fuzzy subset f of a set S is a function from S to a closed interval $[0,1]$. The notion of a fuzzy subset of a set was first considered by Zadeh [35] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere.
A triangular norm (t-norm for short) was introduced by Schweizer and Sklar in [26, 24, 25, 23], following some ideas of Menger in the context of probabilistic metric spaces [13] (as statistical metric spaces were called after 1964). With the development of t-norms in statistical metric spaces, they also play an important role in decision making, in statistics as well as in the theories of cooperative games. In particular, in fuzzy set theory, t-norms have been widely used for fuzzy operations, fuzzy logic and fuzzy relation equations [30]. In recent years, a systematic study concerning the properties and related matters of t-norms have been made by Klement et al. [10, 11, 12]. In the present paper, the fuzzy B-subalgebra of the B-algebras with respect to a t-norm T is redefined and hence generalize the notion in [1, 7] and obtained some of their properties. Also the direct product and T-product of T-fuzzy subalgebra of B-algebra are introduced and discussed their properties in detail. Senapati et al. [29] introduced the notion of T-fuzzy subalgebras of B-algebra with respect to t-norm.
In this paper, we apply the notion of fuzzy sets with respect to a triangular norm to UP-algebras. We introduce the notions of T-fuzzy UP-subalgebras, T-fuzzy near UP-filters, T-fuzzy UP-filters, T-fuzzy UP-ideals, and T-fuzzy strongly UP-ideals, their properties are investigated and some useful examples are discussed. We discuss the relations between T-fuzzy UP-subalgebras (resp., T-fuzzy near UP-filters, T-fuzzy UP-filters, T-fuzzy UP-ideals, and T-fuzzy strongly UP-ideals) and a notion of UP-subalgebras (resp., near UP-filters, UP-filters, UP-ideals, strongly UP-ideals), and their level subsets and UP-homomorphisms are studied. Moreover, we introduce the notion of fuzzy sets with respect to a triangular norm of anti-type in UP-algebras, and study the properties as well as previous notions.

2. Basic results on UP-algebras

Before we begin our study, we will give the definition and useful properties of UP-algebras.

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Definition 2.1. *[5] An algebra $X = (X, \cdot, 0)$ of type $(2,0)$ is called a UP-algebra where X is a nonempty set, $\cdot$ is a binary operation on X, and 0 is a fixed element of X (i.e., a nullary operation) if it satisfies the following axioms:*

(UP-1) $(\forall x,y,z \in X)((y\cdot z)\cdot((x\cdot y)\cdot(x\cdot z)) = 0)$,
(UP-2) $(\forall x \in X)(0\cdot x = x)$,
(UP-3) $(\forall x \in X)(x\cdot 0 = 0)$, *and*
(UP-4) $(\forall x,y \in X)(x\cdot y = 0, y\cdot x = 0 \Rightarrow x = y)$.

From [5], we know that the notion of UP-algebras is a generalization of KU-algebras (see [17]).

Example 2.2. *[22] Let X be a universal set and let $\Omega \in \mathscr{P}(X)$ where $\mathscr{P}(X)$ means the power set of X. Let $\mathscr{P}_\Omega(X) = \{A \in \mathscr{P}(X) \mid \Omega \subseteq A\}$. Define a binary operation $\cdot$ on $\mathscr{P}_\Omega(X)$ by putting $A\cdot B = B\cap(A^C\cup\Omega)$ for all $A,B \in \mathscr{P}_\Omega(X)$ where A^C means the complement of a subset A. Then $(\mathscr{P}_\Omega(X),\cdot,\Omega)$ is a UP-algebra and we shall call it the generalized power UP-algebra of type 1 with respect to Ω. Let $\mathscr{P}^\Omega(X) = \{A \in \mathscr{P}(X) \mid A \subseteq \Omega\}$. Define a binary operation $*$ on $\mathscr{P}^\Omega(X)$ by putting $A*B = B\cup(A^C\cap\Omega)$ for all $A,B \in \mathscr{P}^\Omega(X)$. Then $(\mathscr{P}^\Omega(X),*,\Omega)$ is a UP-algebra and we shall call it the generalized power UP-algebra of type 2 with respect to Ω. In particular, $(\mathscr{P}(X),\cdot,\emptyset)$ is a UP-algebra and we shall call it the power UP-algebra of type 1, and $(\mathscr{P}(X),*,X)$ is a UP-algebra and we shall call it the power UP-algebra of type 2.*

Example 2.3. *[3] Let $\mathbb{N}$ be the set of all natural numbers with two binary operations $\circ$ and $\bullet$ defined by,*

$$(\forall x,y \in \mathbb{N})\left(x\circ y = \begin{cases} y & \text{if } x<y, \\ 0 & \text{otherwise} \end{cases}\right)$$

and

$$(\forall x,y \in \mathbb{N})\left(x\bullet y = \begin{cases} y & \text{if } x>y \text{ or } x=0, \\ 0 & \text{otherwise} \end{cases}\right).$$

Then $(\mathbb{N},\circ,0)$ and $(\mathbb{N},\bullet,0)$ are UP-algebras.

Example 2.4. *[32] Let $A = \{0,1,2,3,4,5,6\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	0	0	2	3	2	3	6
2	0	1	0	3	1	5	3
3	0	1	2	0	4	1	2
4	0	0	0	3	0	3	3
5	0	0	2	0	2	0	2
6	0	1	0	0	1	1	0

Then $(A,\cdot,0)$ is a UP-algebra.

For more examples of UP-algebras, see [2, 6, 21, 22].

The following proposition is very important for the study of UP-algebras.

Proposition 2.5. *[5, 6] In a UP-algebra X, the following properties hold:*

(1) $(\forall x \in X)(x\cdot x = 0)$,
(2) $(\forall x,y,z \in X)(x\cdot y = 0, y\cdot z = 0 \Rightarrow x\cdot z = 0)$,
(3) $(\forall x,y,z \in X)(x\cdot y = 0 \Rightarrow (z\cdot x)\cdot(z\cdot y) = 0)$,
(4) $(\forall x,y,z \in X)(x\cdot y = 0 \Rightarrow (y\cdot z)\cdot(x\cdot z) = 0)$,
(5) $(\forall x,y \in X)(x\cdot(y\cdot x) = 0)$,
(6) $(\forall x,y \in X)((y\cdot x)\cdot x = 0 \Leftrightarrow x = y\cdot x)$,
(7) $(\forall x,y \in X)(x\cdot(y\cdot y) = 0)$,
(8) $(\forall a,x,y,z \in X)((x\cdot(y\cdot z))\cdot(x\cdot((a\cdot y)\cdot(a\cdot z))) = 0)$,
(9) $(\forall a,x,y,z \in X)((((a\cdot x)\cdot(a\cdot y))\cdot z)\cdot((x\cdot y)\cdot z) = 0)$,
(10) $(\forall x,y,z \in X)(((x\cdot y)\cdot z)\cdot(y\cdot z) = 0)$,
(11) $(\forall x,y,z \in X)(x\cdot y = 0 \Rightarrow x\cdot(z\cdot y) = 0)$,
(12) $(\forall x,y,z \in X)(((x\cdot y)\cdot z)\cdot(x\cdot(y\cdot z)) = 0)$, *and*
(13) $(\forall a,x,y,z \in X)(((x\cdot y)\cdot z)\cdot(y\cdot(a\cdot z)) = 0)$.

On a UP-algebra $X = (X,\cdot,0)$, we define a binary relation $\leq$ on X [5] as follows:

$$(\forall x,y \in X)(x \leq y \Leftrightarrow x\cdot y = 0).$$

Definition 2.6. *[5, 31, 4] A nonempty subset S of a UP-algebra $(X,\cdot,0)$ is called*

(1) a UP-subalgebra of X if $(\forall x,y \in S)(x\cdot y \in S)$.
(2) a near UP-filter of X if

  (i) the constant 0 of X is in S, and
  (ii) $(\forall x,y \in X)(y \in S \Rightarrow x\cdot y \in S)$.

(3) a UP-filter of X if

(i) *the constant* 0 *of* X *is in* S*, and*
(ii) $(\forall x,y \in X)(x \cdot y \in S, x \in S \Rightarrow y \in S)$.

(4) *a UP-ideal of* X *if*

(i) *the constant* 0 *of* X *is in* S*, and*
(ii) $(\forall x,y,z \in X)(x \cdot (y \cdot z) \in S, y \in S \Rightarrow x \cdot z \in S)$.

(5) *a strongly UP-ideal of* X *if*

(i) *the constant* 0 *of* X *is in* S*, and*
(ii) $(\forall x,y,z \in X)((z \cdot y) \cdot (z \cdot x) \in S, y \in S \Rightarrow x \in S)$.

Guntasow et al. [4] proved the generalization that the notion of UP-subalgebras is a generalization of UP-filters, the notion of UP-filters is a generalization of UP-ideals, and the notion of UP-ideals is a generalization of strongly UP-ideals. Moreover, they also proved that a UP-algebra A is the only one strongly UP-ideal of itself.

3. Fuzzy sets with respect to a t-norm in UP-algebras

Definition 3.1. *[35] A fuzzy set* A *in a nonempty set* X *(or a fuzzy subset of* X*) is described by its membership function* α_A*. To every point* $x \in X$*, this function associates a real number* $\alpha_A(x)$ *in the unit interval* $[0,1]$*. The number* $\alpha_A(x)$ *is interpreted for the point as a degree of belonging* x *to the fuzzy set* A*, that is,* $A := \{(x, \alpha_A(x)) \mid x \in X\}$*. We say that a fuzzy set* A *in* X *is constant if its membership function* α_A *is constant.*

Definition 3.2. *[11] A triangular norm (briefly, t-norm) is a binary operation* T *on the unit interval [0,1], i.e., a function* $T : [0,1] \times [0,1] \rightarrow [0,1]$ *that satisfies the following axioms:*

(T1) *Boundary condition:* $(\forall x \in [0,1])(T(x,1) = x)$,
(T2) *Commutativity:* $(\forall x,y \in [0,1])(T(x,y) = T(y,x))$,
(T3) *Associativity:* $(\forall x,y,z \in [0,1])(T(x,T(y,z)) = T(T(x,y),z))$*, and*
(T4) *Monotonicity:* $(\forall x,y,z \in [0,1])(y \leq z \Rightarrow T(x,y) \leq T(x,z))$.

Definition 3.3. *Let* T *be a t-norm. Define the subset* $\triangle_T$ *of* $[0,1]$ *by*

$$\triangle_T := \{x \in [0,1] \mid T(x,x) = x\}.$$

A fuzzy set A *in a nonempty set* X *is said to satisfy the imaginable property with respect to* T *if* $\mathrm{Im}(\alpha_A) \subseteq \triangle_T$*, i.e.,*

$(\forall x \in X)(T(\alpha_A(x), \alpha_A(x)) = \alpha_A(x))$.

Lemma 3.4. *Let* T *be a t-norm. Then the following properties hold:*

(1) $(\forall x,y \in [0,1])(T(x,y) \leq x$ *and* $T(x,y) \leq y)$,
(2) $(\forall x \in [0,1])(T(x,0) = 0)$,
(3) $(\forall a,b,x,y \in [0,1])(x \leq a, y \leq b \Rightarrow T(x,y) \leq T(a,b))$*, and*
(4) $(\forall a,b,x,y \in [0,1])(x \leq a, y \leq a \Rightarrow T(x,y) \leq a)$.

Proof. (1) Let $x,y \in [0,1]$. Since $y \leq 1$, it follows from (T1), (T2), and (T4) that $T(x,y) = T(y,x) \leq T(1,x) = x$. Similarly, $T(x,y) = T(y,x) \leq y$.
(2) By (1), we have $0 \geq T(x,0) \geq 0$ and so $T(x,0) = 0$.
(3) Let $a,b,x,y \in [0,1]$ be such that $x \leq a$ and $y \leq b$. By (T4) and (T2), we have $T(x,y) \leq T(a,y)$ and $T(a,y) \leq T(a,b)$. Thus $T(x,y) \leq T(a,b)$.
(4) It is straightforward by (1) and (3). □

In what follows, let X denote a UP-algebra $(X, \cdot, 0)$ and T a t-norm unless otherwise specified.
Now, we introduce the notions of T-fuzzy UP-subalgebras, T-fuzzy near UP-filters, T-fuzzy UP-filters, T-fuzzy UP-ideals, and T-fuzzy strongly UP-ideals, their properties are investigated and some useful examples are discussed.

Definition 3.5. *A fuzzy set* A *in* X *is called*

(1) *a* T*-fuzzy UP-subalgebra of* X *if* $(\forall x,y \in X)(\alpha_A(x \cdot y) \geq T(\alpha_A(x), \alpha_A(y)))$.
(2) *a* T*-fuzzy near UP-filter of* X *if*

(i) $(\forall x \in X)(\alpha_A(0) \geq \alpha_A(x))$*, and*
(ii) $(\forall x,y \in X)(\alpha_A(x \cdot y) \geq T(\alpha_A(y), \alpha_A(y)))$.

(3) *a* T*-fuzzy UP-filter of* X *if*

(i) $(\forall x \in X)(\alpha_A(0) \geq \alpha_A(x))$*, and*
(ii) $(\forall x,y \in X)(\alpha_A(y) \geq T(\alpha_A(x \cdot y), \alpha_A(x)))$.

(4) *a* T*-fuzzy UP-ideal of* X *if*

(i) $(\forall x \in X)(\alpha_A(0) \geq \alpha_A(x))$*, and*
(ii) $(\forall x,y,z \in X)(\alpha_A(x \cdot z) \geq T(\alpha_A(x \cdot (y \cdot z)), \alpha_A(y)))$.

(5) *a* T*-fuzzy strongly UP-ideal of* X *if*

(i) $(\forall x \in X)(\alpha_A(0) \geq \alpha_A(x))$*, and*

(ii) $(\forall x,y,z \in X)(\alpha_A(x) \geq T(\alpha_A((z \cdot y) \cdot (z \cdot x)), \alpha_A(y)))$.

Theorem 3.6. *Every constant fuzzy set in X is a T-fuzzy strongly UP-ideal.*

Proof. Let A be a constant fuzzy set in X. Then $\alpha_A(0) \geq \alpha_A(x)$ for all $x \in X$. Let $x,y,z \in X$. Then

$$\begin{aligned} \alpha_A(x) &= \alpha_A(0) \\ &\geq T(\alpha_A(0), \alpha_A(0)) && \text{(Lemma 3.2 (1))} \\ &= T(\alpha_A((z \cdot y) \cdot (z \cdot x)), \alpha_A(y)). \end{aligned}$$

Hence, A is a T-fuzzy strongly UP-ideal of X. □

The following example show that the converse of Theorem 3.6 is not true.

Example 3.7. *Let $X = \{0,1,2,3,4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	4
3	0	0	0	0	4
4	0	0	0	0	0

Let T_{prod} be the product t-norm defined by

$$(\forall x,y \in [0,1])(T_{prod}(x,y) = xy). \tag{3.1}$$

Define a fuzzy set A in X by

$$\alpha_A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.8 & 0.7 & 0.7 & 0.7 & 0.7 \end{pmatrix}.$$

Then A is a T_{prod}-fuzzy strongly UP-ideal of X.

Theorem 3.8. *Every T-fuzzy strongly UP-ideal of X is a T-fuzzy UP-ideal.*

Proof. Let A be a T-fuzzy strongly UP-ideal of X. Then $\alpha_A(0) \geq \alpha_A(x)$ for all $x \in X$. Let $x,y,z \in X$. Then

$$\begin{aligned} \alpha_A(x \cdot z) &\geq T(\alpha_A((z \cdot y) \cdot (z \cdot (x \cdot z)), \alpha_A(y)) && \text{(Definition 3.5 (5) (ii))} \\ &= T(\alpha_A((z \cdot y) \cdot 0), \alpha_A(y)) && \text{(Proposition 2.5 (5))} \\ &= T(\alpha_A(0), \alpha_A(y)) && \text{((UP-3))} \\ &\geq T(\alpha_A(x \cdot (y \cdot z)), \alpha_A(y)). && \text{((T2) and (T4))} \end{aligned}$$

Hence, A is a T-fuzzy UP-ideal of X. □

The following example show that the converse of Theorem 3.8 is not true.

Example 3.9. *Let $X = \{0,1,2,3,4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	2
2	0	1	0	3	1
3	0	1	2	0	4
4	0	0	0	3	0

Let T_{Luk} be the Łukasiewicz t-norm defined by

$$(\forall x,y \in [0,1])(T_{Luk}(x,y) = \max\{x+y-1, 0\}). \tag{3.2}$$

Define a fuzzy set A in X by

$$\alpha_A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.7 & 0.4 & 0.2 & 0.5 & 0.3 \end{pmatrix}.$$

Then A is a T_{Luk}-fuzzy UP-ideal of X. Since

$$\alpha_A(2) = 0.2 < 0.4 = T_{Luk}(\alpha_A((2 \cdot 0) \cdot (2 \cdot 2)), \alpha_A(0)),$$

we have A is not a T_{Luk}-fuzzy strongly UP-ideal of X.

Theorem 3.10. *Every T-fuzzy UP-ideal of X is a T-fuzzy UP-filter.*

Proof. Let A be a T-fuzzy UP-ideal of X. Then $\alpha_A(0) \geq \alpha_A(x)$ for all $x \in X$. Let $x, y \in X$. Then

$$\begin{aligned} \alpha_A(y) &= \alpha_A(0 \cdot y) && \text{((UP-2))} \\ &\geq T(\alpha_A(0 \cdot (x \cdot y)), \alpha_A(x)) && \text{(Definition 3.5 (4) (ii))} \\ &= T(\alpha_A(x \cdot y), \alpha_A(x)). && \text{((UP-2))} \end{aligned}$$

Hence, A is a T-fuzzy UP-filter of X. □

The following example show that the converse of Theorem 3.10 is not true.

Example 3.11. *Let $X = \{0, 1, 2, 3, 4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	4
3	0	1	1	0	4
4	0	1	2	3	0

Let T_D be the Drastic t-norm defined by

$$(\forall x, y \in [0,1]) \left(T_D(x,y) = \begin{cases} y & \text{if } x = 1, \\ x & \text{if } y = 1, \\ 0 & \text{otherwise} \end{cases} \right). \tag{3.3}$$

Define a fuzzy set A in X by

$$\alpha_A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0.9 & 0.1 & 0.2 & 0 \end{pmatrix}.$$

Then A is a T_D-fuzzy UP-filter of X. Since

$$\alpha_A(1 \cdot 2) = 0.1 < 0.2 = T(\alpha_A(1 \cdot (3 \cdot 2)), \alpha_A(3)),$$

we have A is not a T_D-fuzzy UP-ideal of X.

Theorem 3.12. *Every T-fuzzy UP-filter of X is a T-fuzzy near UP-filter.*

Proof. Let A be a T-fuzzy UP-filter of X. Then $\alpha_A(0) \geq \alpha_A(x)$ for all $x \in X$. Let $x, y \in X$. Then

$$\begin{aligned} \alpha_A(x \cdot y) &\geq T(\alpha_A(y \cdot (x \cdot y)), \alpha_A(y)) && \text{(Definition 3.5 (3) (ii))} \\ &= T(\alpha_A(0), \alpha_A(y)) && \text{(Proposition 2.5 (5))} \\ &\geq T(\alpha_A(y), \alpha_A(y)). && \text{((T2) and (T4))} \end{aligned}$$

Hence, A is a T-fuzzy near UP-filter of X. □

The following example show that the converse of Theorem 3.12 is not true.

Example 3.13. *Let $X = \{0, 1, 2, 3, 4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	0
3	0	0	2	0	4
4	0	0	1	3	0

Define a fuzzy set A in X by

$$\alpha_A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.72 & 0.5 & 0.5 & 0.2 & 0.2 \end{pmatrix}.$$

Then A is a T_{Luk}-fuzzy near UP-filter of X (see T_{Luk} in Example 3.9). Since

$$\alpha_A(4) = 0.2 < 0.22 = T_{Luk}(\alpha_A(2 \cdot 4), \alpha_A(2)),$$

we have A is not a T_{Luk}-fuzzy UP-filter of X.

Theorem 3.14. *Every T-fuzzy UP-filter of X is a T-fuzzy UP-subalgebra.*

Proof. Let A be a T-fuzzy UP-filter of X. Then $\alpha_A(0) \geq \alpha_A(x)$ for all $x \in X$. Let $x, y \in X$. Then

$$\begin{aligned} \alpha_A(x \cdot y) &\geq T(\alpha_A(y \cdot (x \cdot y)), \alpha_A(y)) \\ &= T(\alpha_A(0), \alpha_A(y)) && \text{(Proposition 2.5 (5))} \\ &\geq T(\alpha_A(x), \alpha_A(y)). && \text{((T2) and (T4))} \end{aligned}$$

Hence, A is a T-fuzzy UP-subalgebra of X. □

The following example show that the converse of Theorem 3.14 is not true.

Example 3.15. *Let $X = \{0,1,2,3,4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	0	0	0
2	0	2	0	0	0
3	0	2	2	0	0
4	0	2	2	4	0

Define a fuzzy set A in X by

$$\alpha_A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.8 & 0.7 & 0.9 & 0.9 & 0.9 \end{pmatrix}.$$

Then A is a T_{Luk}-fuzzy UP-subalgebra of X (see T_{Luk} in Example 3.9). Since

$$\alpha_A(1) = 0.7 < 0.8 = T_{Luk}(\alpha_A(4\cdot 1), \alpha_A(4)),$$

we have A is not a T_{Luk}-fuzzy UP-filter of X. Since $\alpha_A(0) < \alpha_A(2)$, we have A does not satisfy the condition: $(\forall x \in X)(\alpha_A(0) \geq \alpha_A(x))$.

Examples 3.16 and 3.17 show that the notion of T-fuzzy near UP-filters does not coincide with the notion of T-fuzzy UP-subalgebras.

Example 3.16. *Let $X = \{0,1,2,3\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	1	2
2	0	0	0	2
3	0	0	0	0

Define a fuzzy set A in X by

$$\alpha_A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0.8 & 0.5 & 0.05 & 0.2 \end{pmatrix}.$$

Then A is a T_{prod}-fuzzy near UP-filter of X (see T_{prod} in Example 3.7). Since

$$\alpha_A(1\cdot 3) = 0.05 < 0.1 = T_{prod}(\alpha_A(1), \alpha_A(3)),$$

we have A is not a T_{prod}-fuzzy UP-subalgebra of X.

Example 3.17. *Let $X = \{0,1,2,3,4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	0	0	0
2	0	2	0	0	0
3	0	2	2	0	0
4	0	2	2	4	0

Let T_{min} be the Gödel t-norm defined by

$$(\forall x, y \in [0,1])(T_{min}(x,y) = \min\{x,y\}). \tag{3.4}$$

Define a fuzzy set A in X by

$$\alpha_A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.9 & 0.1 & 0.3 & 0.2 & 0 \end{pmatrix}.$$

Then A is a T_{min}-fuzzy UP-subalgebra of X (see T_{min} in Example 3.15). Since

$$\alpha_A(4\cdot 3) = 0 < 0.2 = T_{min}(\alpha_A(3), \alpha_A(3)),$$

we have A is not a T_{min}-fuzzy near UP-filter of X.

By Theorems 3.8, 3.10, 3.12, and 3.14 and Examples 3.9, 3.11, 3.13, and 3.15, we have that the notion of T-fuzzy UP-ideals is a generalization of T-fuzzy strongly UP-ideals, the notion of T-fuzzy UP-filters is a generalization of T-fuzzy UP-ideals, the notion of T-fuzzy near UP-filters is a generalization of T-fuzzy UP-filters, and the notion of T-fuzzy UP-subalgebras is a generalization of T-fuzzy UP-filters. Examples 3.16 and 3.17, we have that the notion of T-fuzzy near UP-filters does not coincide with the notion of T-fuzzy UP-subalgebras.
Next, we introduce the notions of anti-T-fuzzy UP-subalgebras, anti-T-fuzzy near UP-filters, anti-T-fuzzy UP-filters, anti-T-fuzzy UP-ideals, and anti-T-fuzzy strongly UP-ideals, their properties are investigated and some useful examples are discussed.

Definition 3.18. *A fuzzy set A in X is called*

(1) an anti-T-fuzzy UP-subalgebra of X if $(\forall x, y \in X)(\alpha_A(x\cdot y) \leq T(\alpha_A(x), \alpha_A(y)))$.
(2) an anti-T-fuzzy near UP-filter of X if

(i) $(\forall x \in X)(\alpha_A(0) \leq \alpha_A(x))$, *and*
(ii) $(\forall x, y \in X)(\alpha_A(x \cdot y) \leq T(\alpha_A(y), \alpha_A(y)))$.

(3) an anti-T-fuzzy UP-filter of X if

(i) $(\forall x \in X)(\alpha_A(0) \leq \alpha_A(x))$, *and*
(ii) $(\forall x, y \in X)(\alpha_A(y) \leq T(\alpha_A(x \cdot y), \alpha_A(x)))$.

(4) an anti-T-fuzzy UP-ideal of X if

(i) $(\forall x \in X)(\alpha_A(0) \leq \alpha_A(x))$, *and*
(ii) $(\forall x, y, z \in X)(\alpha_A(x \cdot z) \leq T(\alpha_A(x \cdot (y \cdot z)), \alpha_A(y)))$.

(5) an anti-T-fuzzy strongly UP-ideal of X if

(i) $(\forall x \in X)(\alpha_A(0) \leq \alpha_A(x))$, *and*
(ii) $(\forall x, y, z \in X)(\alpha_A(x) \leq T(\alpha_A((z \cdot y) \cdot (z \cdot x)), \alpha_A(y)))$.

Lemma 3.19. *If A is an anti-T-fuzzy UP-subalgebra of X, then the following properties hold:*

(1) $(\forall x \in X)(\alpha_A(0) = T(\alpha_A(0), \alpha_A(x)))$, *and*
(2) $(\forall x \in X)(\alpha_A(0) \leq \alpha_A(x))$.

Proof. (1) Let $x \in X$. Then

$$\begin{aligned} T(\alpha_A(0), \alpha_A(x)) &\leq \alpha_A(0) && \text{(Lemma 3.4 (1))} \\ &= \alpha_A(x \cdot 0) && \text{((UP-3))} \\ &\leq T(\alpha_A(x), \alpha_A(0)) && \text{(Definition 3.18 (1))} \\ &= T(\alpha_A(0), \alpha_A(x)). && \text{((T2))} \end{aligned}$$

Thus $\alpha_A(0) = T(\alpha_A(0), \alpha_A(x))$.
(2) By (1) and Lemma 3.4 (1), we have $\alpha_A(0) = T(\alpha_A(0), \alpha_A(x)) \leq \alpha_A(x)$. □

Theorem 3.20. *A fuzzy set A in X is an anti-T-fuzzy UP-filter of X if and only if it is an anti-T-fuzzy UP-subalgebra of X.*

Proof. Assume that A is an anti-T-fuzzy UP-filter of X. Then $\alpha_A(0) \leq \alpha_A(x)$ for all $x \in X$. Let $x, y \in X$. Then

$$\begin{aligned} \alpha_A(x \cdot y) &\leq T(\alpha_A(y \cdot (x \cdot y)), \alpha_A(y)) && \text{(Definition 3.18 (3) (ii))} \\ &= T(\alpha_A(0), \alpha_A(y)) && \text{(Proposition 2.5 (5))} \\ &\leq T(\alpha_A(x), \alpha_A(y)). && \text{((T2) and (T4))} \end{aligned}$$

Hence, A is an anti-T-fuzzy UP-subalgebra of X.
Conversely, assume that A is an anti-T-fuzzy UP-subalgebra of X. By Lemma 3.19 (2), we have $\alpha_A(0) \leq \alpha_A(x)$ for all $x \in X$. Let $x, y \in X$. Then

$$\begin{aligned} \alpha_A(y) &= \alpha_A(0 \cdot y) && \text{((UP-2))} \\ &\leq T(\alpha_A(0), \alpha_A(y)) && \text{(Definition 3.18 (1))} \\ &= \alpha_A(0) && \text{(Lemma 3.19 (1))} \\ &= T(\alpha_A(0), \alpha_A(x)) && \text{(Lemma 3.19 (1))} \\ &\leq T(\alpha_A(x \cdot y), \alpha_A(x)). && \text{((T2) and (T4))} \end{aligned}$$

Hence, A is an anti-T-fuzzy UP-filter of X. □

Theorem 3.21. *A fuzzy set A in X is an anti-T-fuzzy UP-ideal of X if and only if it is an anti-T-fuzzy UP-filter of X.*

Proof. Assume that A is an anti-T-fuzzy UP-ideal of X. Then $\alpha_A(0) \leq \alpha_A(x)$ for all $x \in X$. Let $x, y \in X$. Then

$$\begin{aligned} \alpha_A(y) &= \alpha_A(0 \cdot y) && \text{((UP-2))} \\ &\leq T(\alpha_A(0 \cdot (x \cdot y)), \alpha_A(x)) && \text{(Definition 3.18 (4) (ii))} \\ &= T(\alpha_A(x \cdot y), \alpha_A(x)). && \text{((UP-2))} \end{aligned}$$

Hence, A is an anti-T-fuzzy UP-filter of X.
Conversely, assume that A is an anti-T-fuzzy UP-filter of X. Then $\alpha_A(0) \leq \alpha_A(x)$ for all $x \in X$. Let $x, y, z \in X$. Then

$$\begin{aligned} \alpha_A(x \cdot z) &\leq T(\alpha_A(z \cdot (x \cdot z)), \alpha_A(z)) && \text{(Definition 3.18 (3) (ii))} \\ &= T(\alpha_A(0), \alpha_A(z)) && \text{(Proposition 2.5 (5))} \\ &= \alpha_A(0) && \text{(Lemma 3.19 (1) and Theorem 3.20)} \\ &= T(\alpha_A(0), \alpha_A(y)) && \text{(Lemma 3.19 (1) and Theorem 3.20)} \\ &\leq T(\alpha_A(x \cdot (y \cdot z)), \alpha_A(y)). && \text{((T2) and (T4))} \end{aligned}$$

Hence, A is an anti-T-fuzzy UP-ideal of X. □

Theorem 3.22. *A fuzzy set A in X is an anti-T-fuzzy strongly UP-ideal of X if and only if it is an anti-T-fuzzy UP-ideal of X.*

Proof. Assume that A is an anti-T-fuzzy strongly UP-ideal of X. Then $\alpha_A(0) \leq \alpha_A(x)$ for all $x \in X$. Let $x, y, z \in X$. Then

$$\begin{aligned}
\alpha_A(x \cdot z) &\leq T(\alpha_A((z \cdot y) \cdot (z \cdot (x \cdot z))), \alpha_A(y)) && \text{(Definition 3.18 (5) (ii))}\\
&= T(\alpha_A((z \cdot y) \cdot 0), \alpha_A(y)) && \text{(Proposition 2.5 (5))}\\
&= T(\alpha_A(0), \alpha_A(y)) && \text{((UP-3))}\\
&\leq T(\alpha_A(x \cdot (y \cdot z)), \alpha_A(y)). && \text{((T2) and (T4))}
\end{aligned}$$

Hence, A is an anti-T-fuzzy UP-ideal of X.
Conversely, assume that A is an anti-T-fuzzy UP-ideal of X. Then $\alpha_A(0) \leq \alpha_A(x)$ for all $x \in X$. Let $x, y, z \in X$. Then

$$\begin{aligned}
\alpha_A(x) &= \alpha_A(0 \cdot x) && \text{((UP-2))}\\
&\leq T(\alpha_A(0 \cdot (0 \cdot x)), \alpha_A(0)) && \text{(Definition 3.18 (4) (ii))}\\
&= T(\alpha_A(x), \alpha_A(0)) && \text{((UP-2))}\\
&= \alpha_A(0) && \text{(Lemma 3.19 (1) and Theorems 3.20 and 3.21)}\\
&= T(\alpha_A(0), \alpha_A(y)) && \text{(Lemma 3.19 (1) and Theorems 3.20 and 3.21)}\\
&\leq T(\alpha_A((z \cdot y) \cdot (z \cdot x)), \alpha_A(y)). && \text{((T2) and (T4))}
\end{aligned}$$

Hence, A is an anti-T-fuzzy strongly UP-ideal of X. □

Theorem 3.23. *Every anti-T-fuzzy UP-subalgebra of X is an anti-T-fuzzy near UP-filter.*

Proof. Let A be an anti-T-fuzzy UP-subalgebra of X. By Lemma 3.19 (2), we have $\alpha_A(0) \leq \alpha_A(x)$ for all $x \in X$. Let $x, y \in X$. Then

$$\begin{aligned}
\alpha_A(x \cdot y) &= \alpha_A(0 \cdot (x \cdot y)) && \text{((UP-2))}\\
&\leq T(\alpha_A(0), \alpha_A(x \cdot y)) && \text{(Definition 3.18 (1))}\\
&= \alpha_A(0) && \text{(Lemma 3.19 (1))}\\
&= T(\alpha_A(0), \alpha_A(y)) && \text{(Lemma 3.19 (1))}\\
&\leq T(\alpha_A(y), \alpha_A(y)). && \text{((T2) and (T4))}
\end{aligned}$$

Hence, A is an anti-T-fuzzy near UP-filter of X. □

The following example show that the converse of Theorem 3.23 is not true.

Example 3.24. *Let $X = \{0, 1, 2, 3, 4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	0
3	0	0	2	0	4
4	0	0	1	3	0

Let T_{nM} be the nilpotent minimum defined by

$$(\forall x, y \in [0,1]) \left(T_{nM}(x,y) = \begin{cases} \min\{x,y\} & \text{if } x + y > 1, \\ 0 & \text{otherwise} \end{cases} \right). \tag{3.5}$$

Define a fuzzy set A in X by

$$\alpha_A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.7 & 0.9 & 0.9 & 0.7 & 0.7 \end{pmatrix}.$$

Then A is an anti-T_{nM}-fuzzy near UP-filter of X. Since

$$\alpha_A(0 \cdot 1) = 0.9 > 0.7 = T_{nM}(\alpha_A(0), \alpha_A(1)),$$

we have A is not an anti-T_{nM}-fuzzy UP-subalgebra of X.

Theorem 3.25. *If A is an anti-T-fuzzy UP-subalgebra of X, then A is constant.*

Proof. Assume that A is an anti-T-fuzzy UP-subalgebra of X. Let $x \in X$. Then

$$\begin{aligned}
\alpha_A(x) &= \alpha_A(0 \cdot x) && \text{((UP-2))}\\
&\leq T(\alpha_A(0), \alpha_A(x)) && \text{(Definition 3.18 (1))}\\
&= \alpha_A(0). && \text{(Lemma 3.19 (1))}\\
&\leq \alpha_A(x). && \text{(Lemma 3.19 (2))}
\end{aligned}$$

Hence, α_A is constant, so A is constant. □

The following example show that the converse of Theorem 3.25 is not true.

Example 3.26. *Let $X=\{0,1,2,3,4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	4
3	0	0	0	0	4
4	0	0	0	0	0

Define a fuzzy set A in X by $\alpha_A(x)=0.2$ for all $x\in X$. Since

$$\alpha_A(3\cdot 2)=0.2>0.04=T_{prod}(\alpha_A(3),\alpha_A(2)),$$

we have A is not an anti-T_{prod}-fuzzy UP-subalgebra of X (see T_{prod} in Example 3.7).

By Theorems 3.20, 3.21, and 3.22, we have that the notions of anti-T-fuzzy strongly UP-ideals, anti-T-fuzzy UP-ideals, anti-T-fuzzy UP-filters, and anti-T-fuzzy UP-subalgebras coincide. By Theorem 3.23 and Example 3.24, we have that the notion of anti-T-fuzzy near UP-filters is a generalization of anti-T-fuzzy UP-subalgebras.

Definition 3.27. *Let A be a fuzzy set in X. Define the subset I_{α_A} of X by*

$$I_{\alpha_A}:=\{x\in X\mid \alpha_A(x)=\alpha_A(0)\}.$$

Since $\alpha_A(0)=\alpha_A(0)$, we have $0\in I_{\alpha_A}\neq\emptyset$.

Definition 3.28. *A T-fuzzy UP-subalgebra (resp., T-fuzzy near UP-filter, T-fuzzy UP-filter, T-fuzzy UP-ideal, T-fuzzy strongly UP-ideal) A of X is called an imaginable T-fuzzy UP-subalgebra (resp., imaginable T-fuzzy near UP-filter, imaginable T-fuzzy UP-filter, imaginable T-fuzzy UP-ideal, imaginable T-fuzzy strongly UP-ideal) of X if A satisfies the imaginable property with respect to T.*

Proposition 3.29. *If A is an imaginable T-fuzzy UP-subalgebra of X, then*

$$(\forall x\in X)(\alpha_A(0)\geq\alpha_A(x)).$$

Proof. Assume that A is an imaginable T-fuzzy UP-subalgebra of X. Let $x\in X$. By Proposition 2.5 (1), we have

$$\alpha_A(0)=\alpha_A(x\cdot x)\geq T(\alpha_A(x),\alpha_A(x))=\alpha_A(x).$$

□

Theorem 3.30. *If A is an imaginable T-fuzzy UP-subalgebra of X, then I_{α_A} is a UP-subalgebra of X.*

Proof. Assume that A is an imaginable T-fuzzy UP-subalgebra of X. Let $x,y\in I_{\alpha_A}$. Then $\alpha_A(x)=\alpha_A(0)$ and $\alpha_A(y)=\alpha_A(0)$. Thus

$$\begin{aligned}\alpha_A(x\cdot y)&\geq T(\alpha_A(x),\alpha_A(y)) && \text{(Definition 3.5 (1))}\\&=T(\alpha_A(0),\alpha_A(0))\\&=\alpha_A(0)\\&\geq\alpha_A(x\cdot y). && \text{(Proposition 3.29)}\end{aligned}$$

Thus $\alpha_A(x\cdot y)=\alpha_A(0)$, that is, $x\cdot y\in I_{\alpha_A}$. Therefore, I_{α_A} is a UP-subalgebra of X. □

Theorem 3.31. *If A is a T-fuzzy near UP-filter of X with $T(\alpha_A(0),\alpha_A(0))=\alpha_A(0)$, then I_{α_A} is a near UP-filter of X.*

Proof. Assume that A is a T-fuzzy near UP-filter of X with $T(\alpha_A(0),\alpha_A(0))=\alpha_A(0)$. By Definition 3.27, we have $0\in I_{\alpha_A}$. Let $x\in X$ and $y\in I_{\alpha_A}$. Then $\alpha_A(y)=\alpha_A(0)$. Thus

$$\begin{aligned}\alpha_A(x\cdot y)&\geq T(\alpha_A(y),\alpha_A(y)) && \text{(Definition 3.5 (2) (ii))}\\&=T(\alpha_A(0),\alpha_A(0))\\&=\alpha_A(0)\\&\geq\alpha_A(x\cdot y). && \text{(Definition 3.5 (2) (i))}\end{aligned}$$

Thus $\alpha_A(x\cdot y)=\alpha_A(0)$, that is, $x\cdot y\in I_{\alpha_A}$. Therefore, I_{α_A} is a near UP-filter of X. □

Theorem 3.32. *If A is a T-fuzzy UP-filter of X with $T(\alpha_A(0),\alpha_A(0))=\alpha_A(0)$, then I_{α_A} is a UP-filter of X.*

Proof. Assume that A is a T-fuzzy UP-filter of X with $T(\alpha_A(0),\alpha_A(0))=\alpha_A(0)$. By Definition 3.27, we have $0\in I_{\alpha_A}$. Let $x,y\in X$ be such that $x\cdot y\in I_{\alpha_A}$ and $x\in I_{\alpha_A}$. Then $\alpha_A(x\cdot y)=\alpha_A(0)$ and $\alpha_A(x)=\alpha_A(0)$. Thus

$$\begin{aligned}\alpha_A(y)&\geq T(\alpha_A(x\cdot y),\alpha_A(x)) && \text{(Definition 3.5 (3) (ii))}\\&=T(\alpha_A(0),\alpha_A(0))\\&=\alpha_A(0)\\&\geq\alpha_A(y). && \text{(Definition 3.5 (3) (i))}\end{aligned}$$

Thus $\alpha_A(y)=\alpha_A(0)$, that is, $y\in I_{\alpha_A}$. Therefore, I_{α_A} is a UP-filter of X. □

Theorem 3.33. *If A is a T-fuzzy UP-ideal of X with $T(\alpha_A(0),\alpha_A(0)) = \alpha_A(0)$, then I_{α_A} is a UP-ideal of X.*

Proof. Assume that A is a T-fuzzy UP-ideal of X with $T(\alpha_A(0),\alpha_A(0)) = \alpha_A(0)$. By Definition 3.27, we have $0 \in I_{\alpha_A}$. Let $x,y,z \in X$ be such that $x\cdot(y\cdot z) \in I_{\alpha_A}$ and $y \in I_{\alpha_A}$. Then $\alpha_A(x\cdot(y\cdot z)) = \alpha_A(0)$ and $\alpha_A(y) = \alpha_A(0)$. Thus

$$\begin{aligned}\alpha_A(x\cdot z) &\geq T(\alpha_A(x\cdot(y\cdot z)),\alpha_A(y)) && \text{(Definition 3.5 (4) (ii))}\\ &= T(\alpha_A(0),\alpha_A(0))\\ &= \alpha_A(0)\\ &\geq \alpha_A(x\cdot z). && \text{(Definition 3.5 (4) (i))}\end{aligned}$$

Thus $\alpha_A(x\cdot z) = \alpha_A(0)$, that is, $x\cdot z \in I_{\alpha_A}$. Therefore, I_{α_A} is a UP-ideal of X. □

Theorem 3.34. *If A is a T-fuzzy strongly UP-ideal of X with $T(\alpha_A(0),\alpha_A(0)) = \alpha_A(0)$, then I_{α_A} is a strongly UP-ideal of X. That is, $I_{\alpha_A} = X$.*

Proof. Assume that A is a T-fuzzy strongly UP-ideal of X with $T(\alpha_A(0),\alpha_A(0)) = \alpha_A(0)$. Let $x \in X$. Then

$$\begin{aligned}\alpha_A(x) &\geq T(\alpha_A((x\cdot 0)\cdot(x\cdot x)),\alpha_A(0)) && \text{(Definition 3.5 (5) (ii))}\\ &= T(\alpha_A(0),\alpha_A(0)) && \text{(Proposition 2.5 (1) and (UP-3))}\\ &= \alpha_A(0)\\ &\geq \alpha_A(x). && \text{(Definition 3.5 (5) (i))}\end{aligned}$$

Thus $\alpha_A(x) = \alpha_A(0)$ for all $x \in X$, so $I_{\alpha_A} = X$. Therefore, I_{α_A} is a strongly UP-ideal of X. □

Definition 3.35. *A anti-T-fuzzy UP-subalgebra (resp., anti-T-fuzzy near UP-filter, anti-T-fuzzy UP-filter, anti-T-fuzzy UP-ideal, anti-T-fuzzy strongly UP-ideal) A of X is called an imaginable anti-T-fuzzy UP-subalgebra (resp., imaginable anti-T-fuzzy near UP-filter, imaginable anti-T-fuzzy UP-filter, imaginable anti-T-fuzzy UP-ideal, imaginable anti-T-fuzzy strongly UP-ideal) of X if A satisfies the imaginable property with respect to T.*

Theorem 3.36. *If A is an anti-T-fuzzy near UP-filter of X, then I_{α_A} is a near UP-filter of X.*

Proof. Assume that A an anti-T-fuzzy near UP-filter of X. By Definition 3.27, we have $0 \in I_{\alpha_A}$. Let $x \in X$ and $y \in I_{\alpha_A}$. Then $\alpha_A(y) = \alpha_A(0)$. Thus

$$\begin{aligned}\alpha_A(x\cdot y) &\leq T(\alpha_A(y),\alpha_A(y)) && \text{(Definition 3.18 (2) (ii))}\\ &= T(\alpha_A(0),\alpha_A(0))\\ &\leq \alpha_A(0) && \text{(Lemma 3.4 (1))}\\ &\leq \alpha_A(x\cdot y). && \text{(Definition 3.18 (2) (i))}\end{aligned}$$

Thus $\alpha_A(x\cdot y) = \alpha_A(0)$, that is, $x\cdot y \in I_{\alpha_A}$. Therefore, I_{α_A} is a near UP-filter of X. □

Theorem 3.37. *If A is an anti-T-fuzzy UP-subalgebra (resp., anti-T-fuzzy UP-filter, anti-T-fuzzy UP-ideal, anti-T-fuzzy strongly UP-ideal) of X, then I_{α_A} is a strongly UP-ideal of X. That is, $I_{\alpha_A} = X$.*

Proof. Assume that A is an anti-T-fuzzy UP-subalgebra of X. By Theorem 3.25, we have A is constant. Thus $\alpha_A(x) = \alpha_A(0)$ for all $x \in X$, so $I_{\alpha_A} = X$. Therefore, I_{α_A} is a strongly UP-ideal of X. □

4. (anti-)(λ,τ)-characteristic fuzzy sets

If S is a nonempty subset of X and $\lambda,\tau \in [0,1]$ with $\lambda > \tau$, the (λ,τ)-*characteristic function* $\chi_S^{\lambda,\tau}$ of X is a function of X into $\{\lambda,\tau\}$ defined as follows:

$$(\forall x \in X)\left(\chi_S^{\lambda,\tau}(x) = \begin{cases}\lambda & \text{if } x \in S,\\ \tau & \text{otherwise}\end{cases}\right). \tag{4.1}$$

By the definition of (λ,τ)-characteristic function, $\chi_S^{\lambda,\tau}$ is a function of X into $\{\lambda,\tau\} \subset [0,1]$. We denote the fuzzy set $A_S^{\lambda,\tau}$ in X is described by its membership function $\chi_S^{\lambda,\tau}$, is called the (λ,τ)-*characteristic fuzzy set* of S in X.

Lemma 4.1. *Let S be a nonempty subset of X. Then the following statements hold:*

(1) *if the constant 0 of X is in S, then*

$(\forall x \in X)(\chi_S^{\lambda,\tau}(0) \geq \chi_S^{\lambda,\tau}(x))$, *and*

(2) *if there exists an element $x \in S$ such that $\chi_S^{\lambda,\tau}(0) \geq \chi_S^{\lambda,\tau}(x)$, then the constant 0 of X is in S.*

Proof. (1) If $0 \in S$, then $\chi_S^{\lambda,\tau}(0) = \lambda \geq \chi_S^{\lambda,\tau}(x)$ for all $x \in X$.
(2) Assume that there exists an element $x \in S$ such that $\chi_S^{\lambda,\tau}(0) \geq \chi_S^{\lambda,\tau}(x)$. Thus $\chi_S^{\lambda,\tau}(0) \geq \lambda$, so $\chi_S^{\lambda,\tau}(0) = \lambda$. Hence, $0 \in S$. □

Theorem 4.2. *If S is a UP-subalgebra of X, then $A_S^{\lambda,\tau}$ is a T-fuzzy UP-subalgebra of X.*

Proof. Assume that S is a UP-subalgebra of X. Let $x,y \in X$.
Case 1: $x \in S$ and $y \in S$. Then $\chi_S^{\lambda,\tau}(x) = \lambda = \chi_S^{\lambda,\tau}(y)$. Since S is a UP-subalgebra of X, we have $x \cdot y \in S$ and so $\chi_S^{\lambda,\tau}(x \cdot y) = \lambda$. By Lemma 3.4 (1), we have

$$T(\chi_S^{\lambda,\tau}(x), \chi_S^{\lambda,\tau}(y)) = T(\lambda,\lambda) \leq \lambda = \chi_S^{\lambda,\tau}(x \cdot y).$$

Case 2: $x \notin S$ or $y \notin S$. Then $\chi_S^{\lambda,\tau}(x) = \tau$ or $\chi_S^{\lambda,\tau}(y) = \tau$. By Lemma 3.4 (1), we have

$$T(\chi_S^{\lambda,\tau}(x), \chi_S^{\lambda,\tau}(y)) \leq \tau \leq \chi_S^{\lambda,\tau}(x \cdot y).$$

Therefore, $A_S^{\lambda,\tau}$ is a T-fuzzy UP-subalgebra of X. □

Theorem 4.3. *If S is a nonempty subset of X such that $A_S^{\lambda,\tau}$ is a T-fuzzy UP-subalgebra of X with $T(\lambda,\lambda) = \lambda$, then S is a UP-subalgebra of X.*

Proof. Assume that S is a nonempty subset of X such that $A_S^{\lambda,\tau}$ is a T-fuzzy UP-subalgebra of X with $T(\lambda,\lambda) = \lambda$. Let $x,y \in S$. Then $\chi_S^{\lambda,\tau}(x) = \lambda = \chi_S^{\lambda,\tau}(y)$. By Definition 3.5 (1), we have

$$\chi_S^{\lambda,\tau}(x \cdot y) \geq T(\chi_S^{\lambda,\tau}(x), \chi_S^{\lambda,\tau}(y)) = T(\lambda,\lambda) = \lambda \geq \chi_S^{\lambda,\tau}(x \cdot y).$$

Thus $\chi_S^{\lambda,\tau}(x \cdot y) = \lambda$, that is, $x \cdot y \in S$. Hence, S is a UP-subalgebra of X. □

Examples 4.4, 4.8, 4.12, 4.16 and 4.20 show that the condition $T(\lambda,\lambda) = \lambda$ is necessary.

Example 4.4. *Let $X = \{0,1,2,3,4,5,6\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	0	0	2	3	2	3	6
2	0	1	0	3	1	5	3
3	0	1	2	0	4	1	2
4	0	0	0	3	0	3	3
5	0	0	2	0	2	0	2
6	0	1	0	0	1	1	0

Let $S = \{0,3,5,6\}$. Define a fuzzy set $A_S^{0.7,0.5}$ in X by

$$\chi_S^{0.7,0.5} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0.7 & 0.5 & 0.5 & 0.7 & 0.5 & 0.7 & 0.7 \end{pmatrix}.$$

Then $A_S^{0.7,0.5}$ is a T_{Luk}-fuzzy UP-subalgebra of X but $T_{Luk}(0.7,0.7) = 0.4 \neq 0.7$ (see T_{Luk} in Example 3.9). Since $3 \in S$ and $6 \in S$ but $3 \cdot 6 = 2 \notin S$, we have S is not a UP-subalgebra of X.

Corollary 4.5. *A nonempty subset S of X is a UP-subalgebra of X if and only if $A_S^{1,\tau}$ is a T-fuzzy UP-subalgebra of X.*

Proof. It is straightforward by Theorems 4.2 and 4.3. □

Theorem 4.6. *If S is a near UP-filter of X, then $A_S^{\lambda,\tau}$ is a T-fuzzy near UP-filter of X.*

Proof. Assume that S is a near UP-filter of X. Since $0 \in S$, it follows from Lemma 4.1 (1) that $\chi_S^{\lambda,\tau}(0) \geq \chi_S^{\lambda,\tau}(x)$ for all $x \in X$. Let $x,y \in X$.
Case 1: $y \in S$. Then $\chi_S^{\lambda,\tau}(y) = \lambda$. Since S is a near UP-filter of X, we have $x \cdot y \in S$ and so $\chi_S^{\lambda,\tau}(x \cdot y) = \lambda$. By Lemma 3.4 (1), we have

$$T(\chi_S^{\lambda,\tau}(y), \chi_S^{\lambda,\tau}(y)) = T(\lambda,\lambda) \leq \lambda = \chi_S^{\lambda,\tau}(x \cdot y).$$

Case 2: $y \notin S$. Then $\chi_S^{\lambda,\tau}(y) = \tau$. By Lemma 3.4 (1), we have

$$T(\chi_S^{\lambda,\tau}(y), \chi_S^{\lambda,\tau}(y)) = T(\tau,\tau) \leq \tau \leq \chi_S^{\lambda,\tau}(x \cdot y).$$

Therefore, $A_S^{\lambda,\tau}$ is a T-fuzzy near UP-filter of X. □

Theorem 4.7. *If S is a nonempty subset of X such that $A_S^{\lambda,\tau}$ is a T-fuzzy near UP-filter of X with $T(\lambda,\lambda) = \lambda$, then S is a near UP-filter of X.*

Proof. Assume that S is a nonempty subset of X such that $A_S^{\lambda,\tau}$ is a T-fuzzy near UP-filter of X with $T(\lambda,\lambda)=\lambda$. Then $\chi_S^{\lambda,\tau}(0)\geq\chi_S^{\lambda,\tau}(x)$ for all $x\in X$, it follows from Lemma 4.1 (2) that $0\in S$. Let $x\in X$ and $y\in S$. Then $\chi_S^{\lambda,\tau}(y)=\lambda$. By Definition 3.5 (2) (ii), we have

$$\chi_S^{\lambda,\tau}(x\cdot y)\geq T(\chi_S^{\lambda,\tau}(y),\chi_S^{\lambda,\tau}(y))=T(\lambda,\lambda)=\lambda\geq\chi_S^{\lambda,\tau}(x\cdot y).$$

Thus $\chi_S^{\lambda,\tau}(x\cdot y)=\lambda$, that is, $x\cdot y\in S$. Hence, S is a near UP-filter of X. □

Example 4.8. *From Example 4.4, we let $S=\{0,3,5,6\}$. Then $A_S^{0.7,0.5}$ is a T_{Luk}-fuzzy near UP-filter of X but $T_{Luk}(0.7,0.7)=0.4\neq 0.7$ (see T_{Luk} in Example 3.9). Since $6\in S$, but $3\cdot 6=2\notin S$, we have S is not a near UP-filter of X.*

Corollary 4.9. *A nonempty subset S of X is a near UP-filter of X if and only if $A_S^{1,\tau}$ is a T-fuzzy near UP-filter of X.*

Proof. It is straightforward by Theorems 4.6 and 4.7. □

Theorem 4.10. *If S is a UP-filter of X, then $A_S^{\lambda,\tau}$ is a T-fuzzy UP-filter of X.*

Proof. Assume that S is a UP-filter of X. Since $0\in S$, it follows from Lemma 4.1 (1) that $\chi_S^{\lambda,\tau}(0)\geq\chi_S^{\lambda,\tau}(x)$ for all $x\in X$. Let $x,y\in X$.
Case 1: $x\cdot y\in S$ and $x\in S$. Then $\chi_S^{\lambda,\tau}(x\cdot y)=\lambda=\chi_S^{\lambda,\tau}(x)$. Since S is a UP-filter of X, we have $y\in S$ and so $\chi_S^{\lambda,\tau}(y)=\lambda$. By Lemma 3.4 (1), we have

$$T(\chi_S^{\lambda,\tau}(x\cdot y),\chi_S^{\lambda,\tau}(x))=T(\lambda,\lambda)\leq\lambda=\chi_S^{\lambda,\tau}(y).$$

Case 2: $x\cdot y\notin S$ or $x\notin S$. Then $\chi_S^{\lambda,\tau}(x\cdot y)=\tau$ or $\chi_S^{\lambda,\tau}(x)=\tau$. By Lemma 3.4 (1), we have

$$T(\chi_S^{\lambda,\tau}(x\cdot y),\chi_S^{\lambda,\tau}(x))\leq\tau\leq\chi_S^{\lambda,\tau}(y).$$

Therefore, $A_S^{\lambda,\tau}$ is a T-fuzzy UP-filter of X. □

Theorem 4.11. *If S is a nonempty subset of X such that $A_S^{\lambda,\tau}$ is a T-fuzzy UP-filter of X with $T(\lambda,\lambda)=\lambda$, then S is a UP-filter of X.*

Proof. Assume that S is a nonempty subset of X such that $A_S^{\lambda,\tau}$ is a T-fuzzy UP-filter of X with $T(\lambda,\lambda)=\lambda$. Then $\chi_S^{\lambda,\tau}(0)\geq\chi_S^{\lambda,\tau}(x)$ for all $x\in X$, it follows from Lemma 4.1 (2) that $0\in S$. Let $x,y\in X$ be such that $x\cdot y\in S$ and $x\in S$. Then $\chi_S^{\lambda,\tau}(x\cdot y)=\lambda=\chi_S^{\lambda,\tau}(x)$. By Definition 3.5 (3) (ii), we have

$$\chi_S^{\lambda,\tau}(y)\geq T(\chi_S^{\lambda,\tau}(x\cdot y)),\chi_S^{\lambda,\tau}(x))=T(\lambda,\lambda)=\lambda\geq\chi_S^{\lambda,\tau}(y).$$

Thus $\chi_S^{\lambda,\tau}(y)=\lambda$, that is, $y\in S$. Hence, S is a UP-filter of X. □

Example 4.12. *From Example 4.4, we let $S=\{0,3,5,6\}$. Then $A_S^{0.7,0.5}$ is a T_{Luk}-fuzzy UP-filter of X but $T_{Luk}(0.7,0.7)=0.4\neq 0.7$ (see T_{Luk} in Example 3.9). Since $5\cdot 1=0\in S$ and $5\in S$, but $1\notin S$, we have S is not a UP-filter of X.*

Corollary 4.13. *A nonempty subset S of X is a UP-filter of X if and only if $A_S^{1,\tau}$ is a T-fuzzy UP-filter of X.*

Proof. It is straightforward by Theorems 4.10 and 4.11. □

Theorem 4.14. *If S is a UP-ideal of X, then $A_S^{\lambda,\tau}$ is a T-fuzzy UP-ideal of X.*

Proof. Assume that S is a UP-ideal of X. Since $0\in S$, it follows from Lemma 4.1 (1) that $\chi_S^{\lambda,\tau}(0)\geq\chi_S^{\lambda,\tau}(x)$ for all $x\in X$. Next, let $x,y,z\in X$.
Case 1: $x\cdot(y\cdot z)\in S$ and $y\in S$. Then $\chi_S^{\lambda,\tau}(x\cdot(y\cdot z))=\lambda$ and $\chi_S^{\lambda,\tau}(y)=\lambda$. Since S is a UP-ideal of X, we have $x\cdot z\in S$ and so $\chi_S^{\lambda,\tau}(x\cdot z)=\lambda$. By Lemma 3.4 (1), we have

$$T(\chi_S^{\lambda,\tau}(x\cdot(y\cdot z)),\chi_S^{\lambda,\tau}(y))=T(\lambda,\lambda)\leq\lambda=\chi_S^{\lambda,\tau}(x\cdot z).$$

Case 2: $x\cdot(y\cdot z)\notin S$ or $y\notin S$. Then $\chi_S^{\lambda,\tau}(x\cdot(y\cdot z))=\tau$ or $\chi_S^{\lambda,\tau}(y)=\tau$. By Lemma 3.4 (1), we have

$$T(\chi_S^{\lambda,\tau}(x\cdot(y\cdot z)),\chi_S^{\lambda,\tau}(y))\leq\tau\leq\chi_S^{\lambda,\tau}(x\cdot z).$$

Therefore, $A_S^{\lambda,\tau}$ is a T-fuzzy UP-ideal of X. □

Theorem 4.15. *If S is a nonempty subset of X such that $A_S^{\lambda,\tau}$ is a T-fuzzy UP-ideal of X with $T(\lambda,\lambda)=\lambda$, then S is a UP-ideal of X.*

Proof. Assume that S is a nonempty subset of X such that $A_S^{\lambda,\tau}$ is a T-fuzzy UP-ideal of X with $T(\lambda,\lambda)=\lambda$. Then $\chi_S^{\lambda,\tau}(0)\geq\chi_S^{\lambda,\tau}(x)$ for all $x\in X$, it follows from Lemma 4.1 (2) that $0\in S$. Let $x,y,z\in X$ be such that $x\cdot(y\cdot z)\in S$ and $y\in S$. Then $\chi_S^{\lambda,\tau}(x\cdot(y\cdot z))=\lambda=\chi_S^{\lambda,\tau}(y)$. By Definition 3.5 (4) (ii), we have

$$\chi_S^{\lambda,\tau}(x\cdot z)\geq T(\chi_S^{\lambda,\tau}(x\cdot(y\cdot z)),\chi_S^{\lambda,\tau}(y))=T(\lambda,\lambda)=\lambda\geq\chi_S^{\lambda,\tau}(x\cdot z).$$

Thus $\chi_S^{\lambda,\tau}(x\cdot z)=\lambda$, that is, $x\cdot z\in S$. Hence, S is a UP-ideal of X. □

Example 4.16. *From Example 4.4, we let $S = \{0,3,5,6\}$. Then $A_S^{0.7,0.5}$ is a T_{Luk}-fuzzy UP-ideal of X but $T_{Luk}(0.7,0.7) = 0.4 \neq 0.7$ (see T_{Luk} in Example 3.9). Since $6\cdot(5\cdot 1) = 0 \in S$ and $5 \in S$, but $(6\cdot 1) = 1 \notin S$, we have S is not a UP-ideal of X.*

Corollary 4.17. *A nonempty subset S of X is a UP-ideal of X if and only if $A_S^{1,\tau}$ is a T-fuzzy UP-ideal of X.*

Proof. It is straightforward by Theorems 4.14 and 4.15. □

Theorem 4.18. *If S is a strongly UP-ideal of X, then $A_S^{\lambda,\tau}$ is a T-fuzzy strongly UP-ideal of X.*

Proof. Assume that S is a strongly UP-ideal of X. Then $S = X$, so $A_S^{\lambda,\tau}$ is constant. By Theorem 3.6, we have $A_S^{\lambda,\tau}$ is a T-fuzzy strongly UP-ideal of X. □

Theorem 4.19. *If S is a nonempty subset of X such that $A_S^{\lambda,\tau}$ is a T-fuzzy strongly UP-ideal of X with $T(\lambda,\lambda) = \lambda$, then S is a strongly UP-ideal of X.*

Proof. Assume that S is a nonempty subset of X such that $A_S^{\lambda,\tau}$ is a T-fuzzy strongly UP-ideal of X with $T(\lambda,\lambda) = \lambda$. Then $\chi_S^{\lambda,\tau}(0) \geq \chi_S^{\lambda,\tau}(x)$ for all $x \in X$, it follows from Lemma 4.1 (2) that $0 \in S$. Thus $\chi_S^{\lambda,\tau}(0) = \lambda$. Let $x \in X$. By (UP-3) and Proposition 2.5 (1), we have

$$\chi_S^{\lambda,\tau}(x) \geq T(\chi_S^{\lambda,\tau}((x\cdot 0)\cdot(x\cdot x)), \chi_S^{\lambda,\tau}(0)) = T(\chi_S^{\lambda,\tau}(0), \chi_S^{\lambda,\tau}(0)) = T(\lambda,\lambda) = \lambda \geq \chi_S^{\lambda,\tau}(x).$$

Thus $X = S$, that is, S is a strongly UP-ideal of X. □

Example 4.20. *From Example 4.4, we let $S = \{0,3,5,6\}$. Then $A_S^{0.7,0.5}$ is a T_{Luk}-fuzzy strongly UP-ideal of X but $T_{Luk}(0.7,0.7) = 0.4 \neq 0.7$ (see T_{Luk} in Example 3.9). Since $(2\cdot 5)\cdot(2\cdot 4) = 0 \in S$ and $5 \in S$, but $4 \notin S$, we have S is not a strongly UP-ideal of X.*

Corollary 4.21. *A nonempty subset S of X is a strongly UP-ideal of X if and only if $A_S^{1,\tau}$ is a T-fuzzy strongly UP-ideal of X.*

Proof. It is straightforward by Theorems 4.18 and 4.19. □

If S is a nonempty subset of X and $\lambda,\tau \in [0,1]$ with $\lambda > \tau$, the *anti-(λ,τ)-characteristic function* $\overline{\chi}_S^{\lambda,\tau}$ of X is a function of X into $\{\lambda,\tau\}$ defined as follows:

$$(\forall x \in X)\left(\overline{\chi}_S^{\lambda,\tau}(x) = \begin{cases} \tau & \text{if } x \in S, \\ \lambda & \text{otherwise} \end{cases}\right). \tag{4.2}$$

By the definition of anti-(λ,τ)-characteristic function, $\overline{\chi}_S^{\lambda,\tau}$ is a function of X into $\{\lambda,\tau\} \subset [0,1]$. We denote the fuzzy set $\overline{A}_S^{\lambda,\tau}$ in X is described by its membership function $\overline{\chi}_S^{\lambda,\tau}$, is called the *anti-$(\lambda,\tau)$-characteristic fuzzy set* of S in X.

Lemma 4.22. *Let S be a nonempty subset of X. Then the following statements hold:*

(1) *if the constant 0 of X is in S, then*

$$(\forall x \in X)(\overline{\chi}_S^{\lambda,\tau}(0) \leq \overline{\chi}_S^{\lambda,\tau}(x)), \text{ and}$$

(2) *if there exists an element $x \in S$ such that $\overline{\chi}_S^{\lambda,\tau}(0) \leq \overline{\chi}_S^{\lambda,\tau}(x)$, then the constant 0 of X is in S.*

Proof. (1) If $0 \in S$, then $\overline{\chi}_S^{\lambda,\tau}(0) = \tau \leq \overline{\chi}_S^{\lambda,\tau}(x)$ for all $x \in X$.
(2) Assume that there exists an element $x \in S$ such that $\overline{\chi}_S^{\lambda,\tau}(0) \leq \overline{\chi}_S^{\lambda,\tau}(x)$. Then $\overline{\chi}_S^{\lambda,\tau}(0) \leq \overline{\chi}_S^{\lambda,\tau}(x) = \tau$, so $\overline{\chi}_S^{\lambda,\tau}(0) = \tau$. Hence, $0 \in S$. □

Theorem 4.23. *If S is a strongly UP-ideal of X and $T(\tau,\tau) = \tau$, then $\overline{A}_S^{\lambda,\tau}$ is an anti-T-fuzzy UP-subalgebra (resp., anti-T-fuzzy UP-filter, anti-T-fuzzy UP-ideal, anti-T-fuzzy strongly UP-ideal) of X.*

Proof. Assume that S is a strongly UP-ideal of X. Then $S = X$, so $\overline{\chi}_S^{\lambda,\tau}(x) = \tau$ for all $x \in X$. Let $x,y \in X$. Then $\overline{\chi}_S^{\lambda,\tau}(x) = \tau = \overline{\chi}_S^{\lambda,\tau}(y)$. Since $S = X$, we have $x\cdot y \in S$ and so $\overline{\chi}_S^{\lambda,\tau}(x\cdot y) = \tau$. Thus

$$T(\overline{\chi}_S^{\lambda,\tau}(x), \overline{\chi}_S^{\lambda,\tau}(y)) = T(\tau,\tau) = \tau = \overline{\chi}_S^{\lambda,\tau}(x\cdot y).$$

Therefore, $\overline{A}_S^{\lambda,\tau}$ is an anti-T-fuzzy UP-subalgebra of X. □

Example 4.24 shows that the condition $T(\tau,\tau) = \tau$ is necessary.

Example 4.24. *Let $X = \{0,1,2,3,4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	4
3	0	0	0	0	4
4	0	0	0	0	0

Let S={0,1,2,3,4}. Then S is a strongly UP-ideal of X but $T_{H_0}(0.6,0.6)$. Let T_{H_0} be the Hamacher product defined by

$$(\forall x,y \in [0,1]) \left(T_{H_0}(x,y) = \begin{cases} 0 & \text{if } x=y=0, \\ \dfrac{xy}{x+y-xy} & \text{otherwise} \end{cases} \right). \tag{4.3}$$

Define a fuzzy set A in X by

$$\overline{\chi}_S^{0.9,0.6} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.6 & 0.6 & 0.6 & 0.6 & 0.6 \end{pmatrix}.$$

Since

$$\overline{\chi}_S^{0.9,0.6}(2 \cdot 4) = 0.6 > 0.43 = T_{H_0}(\overline{\chi}_S^{0.9,0.6}(2), \overline{\chi}_S^{0.9,0.6}(4)),$$

we have $\overline{A}_S^{0.9,0.6}$ is not an anti-T_{H_0}-fuzzy UP-subalgebra of X.

Theorem 4.25. *If S is a nonempty subset of X such that $\overline{A}_S^{\lambda,\tau}$ is an anti-T-fuzzy UP-subalgebra (resp., anti-T-fuzzy UP-filter, anti-T-fuzzy UP-ideal, anti-T-fuzzy strongly UP-ideal) of X, then S is a strongly UP-ideal of X.*

Proof. Assume that S is a nonempty subset of X such that $\overline{A}_S^{\lambda,\tau}$ is an anti-T-fuzzy UP-subalgebra of X. By Theorem 3.25, we have $\overline{A}_S^{\lambda,\tau}$ is constant. Thus $S = X$, that is, S is a strongly UP-ideal of X. □

Corollary 4.26. *A nonempty subset S of X is a strongly UP-ideal of X if and only if $\overline{A}_S^{\lambda,0}$ is an anti-T-fuzzy UP-subalgebra (resp., anti-T-fuzzy UP-filter, anti-T-fuzzy UP-ideal, anti-T-fuzzy strongly UP-ideal) of X.*

Proof. It is straightforward by Theorems 4.23 and 4.25. □

Theorem 4.27. *If S is a near UP-filter of X, $T(\tau,\tau) = \tau$, and $T(\lambda,\lambda) = \lambda$, then $\overline{A}_S^{\lambda,\tau}$ is an anti- T-fuzzy near UP-filter of X.*

Proof. Assume that S is a near UP-filter of X. Since $0 \in S$, it follows from Lemma 4.22 (1) that $\overline{\chi}_S^{\lambda,\tau}(0) \leq \overline{\chi}_S^{\lambda,\tau}(x)$ for all $x \in X$. Let $x,y \in X$.
Case 1: $y \in S$. Then $\overline{\chi}_S^{\lambda,\tau}(y) = \tau$. Since S is a near UP-filter of X, we have $x \cdot y \in S$ and so $\overline{\chi}_S^{\lambda,\tau}(x \cdot y) = \tau$. Thus

$$T(\overline{\chi}_S^{\lambda,\tau}(y), \overline{\chi}_S^{\lambda,\tau}(y)) = T(\tau,\tau) = \tau = \overline{\chi}_S^{\lambda,\tau}(x \cdot y).$$

Case 2: $y \notin S$. Then $\overline{\chi}_S^{\lambda,\tau}(y) = \lambda$. Thus

$$T(\overline{\chi}_S^{\lambda,\tau}(y), \overline{\chi}_S^{\lambda,\tau}(y)) = T(\lambda,\lambda) = \lambda \geq \overline{\chi}_S^{\lambda,\tau}(x \cdot y).$$

Therefore, $\overline{A}_S^{\lambda,\tau}$ is an anti-T-fuzzy near UP-filter of X. □

Example 4.28 shows that the conditions $T(\tau,\tau) = \tau$ and $T(\lambda,\lambda) = \lambda$ are necessary.

Example 4.28. *Let $X = \{0,1,2\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2
0	0	1	2
1	0	0	2
2	0	0	0

Let $S = \{0,1\}$. Then S is a near UP-filter of X but $T_{Luk}(0.7,0.7) = 0.4 \neq 0.7$ and $T_{Luk}(0.5,0.5) = 0 \neq 0.5$ (see T_{Luk} in Example 3.9). Define a fuzzy set $\overline{A}_S^{0.7,0.5}$ in X by

$$\overline{\chi}_S^{0.7,0.5} = \begin{pmatrix} 0 & 1 & 2 \\ 0.5 & 0.5 & 0.7 \end{pmatrix}.$$

Since

$$\overline{\chi}_S^{0.7,0.5}(0 \cdot 1) = 0.5 > 0 = T_{Luk}(\overline{\chi}_S^{0.7,0.5}(1), \overline{\chi}_S^{0.7,0.5}(1)),$$

we have $\overline{A}_S^{0.7,0.5}$ is not an anti-T_{Luk}-fuzzy near UP-filter of X.

Theorem 4.29. *If S is a nonempty subset of X such that $\overline{A}_S^{\lambda,\tau}$ is an anti-T-fuzzy near UP-filter of X, then S is a near UP-filter of X.*

Proof. Assume that S is a nonempty subset of X such that $\overline{A}_S^{\lambda,\tau}$ is an anti-T-fuzzy near UP-filter of X. Then $\overline{\chi}_S^{\lambda,\tau}(0) \leq \overline{\chi}_S^{\lambda,\tau}(x)$ for all $x \in X$, it follows from Lemma 4.22 (2) that $0 \in S$. Let $x \in X$ and $y \in S$. Then $\overline{\chi}_S^{\lambda,\tau}(y) = \tau$. By Definition 3.18 (2) (ii) and Lemma 3.4 (1), we have

$$\overline{\chi}_S^{\lambda,\tau}(x \cdot y) \leq T(\overline{\chi}_S^{\lambda,\tau}(y), \overline{\chi}_S^{\lambda,\tau}(y)) = T(\tau,\tau) \leq \tau \leq \overline{\chi}_S^{\lambda,\tau}(x \cdot y).$$

Thus $\overline{\chi}_S^{\lambda,\tau}(x \cdot y) = \tau$, that is, $x \cdot y \in S$. Hence, S is a near UP-filter of X. □

Corollary 4.30. *A nonempty subset S of X is a near UP-filter of X if and only if $\overline{A}_S^{1,0}$ is an anti-T-fuzzy near UP-filter of X.*

Proof. It is straightforward by Theorems 4.27 and 4.29. □

5. Level subsets of a fuzzy set with respect to a t-norm

Definition 5.1. *[31] Let A be a fuzzy set in a nonempty set X. For any $s \in [0,1]$, the sets*

$$U(A;s) = \{x \in X \mid \alpha_A(x) \geq s\},$$
$$L(A;s) = \{x \in X \mid \alpha_A(x) \leq s\},$$
$$E(A;s) = \{x \in X \mid \alpha_A(x) = s\}$$

are called an upper s-level subset, a lower s-level subset, and an equal s-level subset of A, respectively.

For a fuzzy set A in a nonempty set X, we see that $U(A;1) = E(A;1), L(A;0) = E(A;0)$, and $U(A;0) = X = L(A;1)$.

Lemma 5.2. *Let A be a fuzzy set in X. Then the following statements hold:*

(1) *if the constant 0 of X is in $E(A;1)$, then $\alpha_A(0) \geq \alpha_A(x)$ for all $x \in X$, and*
(2) *for any $s \in [0,1]$, if there exists an element $x \in U(A;s)$ such that $\alpha_A(0) \geq \alpha_A(x)$, then the constant 0 of X is in $U(A;s)$.*

Proof. (1) If $0 \in E(A;1)$, then $\alpha_A(0) = 1 \geq \alpha_A(x)$ for all $x \in X$.
(2) Assume that $\alpha_A(0) \geq \alpha_A(x)$ for some $x \in U(A;s)$. Then $\alpha_A(0) \geq \alpha_A(x) \geq s$, that is, $0 \in U(A;s)$. □

Theorem 5.3. *If A is a T-fuzzy UP-subalgebra of X, then for all $s \in [0,1]$ with $T(s,s) = s$, $U(A;s)$ is a UP-subalgebra of X if $U(A;s)$ is nonempty. In particular, $E(A;1)$ is a UP-subalgebra of X if $E(A;1)$ is nonempty.*

Proof. Assume that A is a T-fuzzy UP-subalgebra of X. Let $s \in [0,1]$ be such that $T(s,s) = s$ and $U(A;s) \neq \emptyset$ and let $x, y \in U(A;s)$. Then $\alpha_A(x) \geq s$ and $\alpha_A(y) \geq s$. Thus

$$\begin{aligned} \alpha_A(x \cdot y) &\geq T(\alpha_A(x), \alpha_A(y)) && \text{(Definition 3.5 (1))} \\ &\geq T(s,s) && \text{(Lemma 3.4 (3))} \\ &= s. \end{aligned}$$

Hence, $x \cdot y \in U(A;s)$, that is, $U(A;s)$ is a UP-subalgebra of X. □

Theorem 5.4. *If a fuzzy set A in X is such that a nonempty subset $U(A;s)$ is a UP-subalgebra of X for all $s \in [0,1]$, then A is a T-fuzzy UP-subalgebra of X.*

Proof. Assume that a fuzzy set A in X is such that a nonempty subset $U(A;s)$ is a UP-subalgebra of X for all $s \in [0,1]$. Suppose that there exist $x, y \in X$ such that $\alpha_A(x \cdot y) < T(\alpha_A(x), \alpha_A(y))$. Put $s_0 = \frac{1}{2}[\alpha_A(x \cdot y) + T(\alpha_A(x), \alpha_A(y))]$. Thus $s_0 \in [0,1]$ and $\alpha_A(x \cdot y) < s_0 < T(\alpha_A(x), \alpha_A(y))$. This implies that $x \cdot y \notin U(A;s_0)$. By Lemma 3.4 (1), we have $x, y \in U(A;s_0) \neq \emptyset$. Thus $U(A;s_0)$ is not a UP-subalgebra of X. This is a contradiction to the fact that $\emptyset \neq U(A;s_0)$ is a UP-subalgebra of X. Hence, $\alpha_A(x \cdot y) \geq T(\alpha_A(x), \alpha_A(y))$ for all $x, y \in X$. Therefore, A is a T-fuzzy UP-subalgebra of X. □

The following example show that the converse of Theorem 5.4 is not true.

Example 5.5. *Let $X = \{0,1,2,3,4,5,6\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	0	0	2	3	2	3	6
2	0	1	0	3	1	5	3
3	0	1	2	0	4	1	2
4	0	0	0	3	0	3	3
5	0	0	2	0	2	0	2
6	0	1	0	0	1	1	0

Define a fuzzy set A in X by

$$\alpha_A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0.75 & 0.55 & 0.5 & 0.7 & 0.5 & 0.7 & 0.7 \end{pmatrix}.$$

Then A is a T_{Luk}-fuzzy UP-subalgebra of X (see T_{Luk} in Example 3.9). If $s = 0.7$, then $U(A;0.7) = \{0,3,5,6\} \neq \emptyset$. Since $3,6 \in U(A;0.7)$ but $3 \cdot 6 = 2 \notin U(A;0.7)$, we have $U(A;0.7)$ is not a UP-subalgebra of X.

Theorem 5.6. *If A is a T-fuzzy near UP-filter of X, then for all $s \in [0,1]$ with $T(s,s) = s$, $U(A;s)$ is a near UP-filter of X if $U(A;s)$ is nonempty. In particular, $E(A;1)$ is a near UP-filter of X if $E(A;1)$ is nonempty.*

Proof. Assume that A is a T-fuzzy near UP-filter of X. Let $s \in [0,1]$ be such that $T(s,s) = s$ and $U(A;s) \neq \emptyset$. Since A is a T-fuzzy near UP-filter of X, we have $\alpha_A(0) \geq \alpha_A(x)$ for all $x \in X$, it follows from Lemma 5.2 (2) that $0 \in U(A;s)$. Let $x \in X$ and $y \in U(A;s)$. Then $\alpha_A(y) \geq s$. Thus

$$\begin{aligned} \alpha_A(x \cdot y) &\geq T(\alpha_A(y), \alpha_A(y)) && \text{(Definition 3.5 (2) (ii))} \\ &\geq T(s,s) && \text{(Lemma 3.4 (3))} \\ &= s. \end{aligned}$$

Hence, $x \cdot y \in U(A;s)$, that is, $U(A;s)$ is a near UP-filter of X. □

Theorem 5.7. *If a fuzzy set A in X is such that a nonempty subset $U(A;s)$ is a near UP-filter of X for all $s \in [0,1]$, then A is a T-fuzzy near UP-filter of X.*

Proof. Assume that a fuzzy set A in X is such that a nonempty subset $U(A;s)$ is a near UP-filter of X for all $s \in [0,1]$. For any $x \in X$, let $\alpha_A(x) = s$. Then $x \in U(A;s) \neq \emptyset$. By assumption, we have $U(A;s)$ is a near UP-filter of X. Thus $0 \in U(A;s)$, that is, $\alpha_A(0) \geq s = \alpha_A(x)$. Suppose that there exist $x, y \in X$ such that $\alpha_A(x \cdot y) < T(\alpha_A(y), \alpha_A(y))$. Put $s_0 = \frac{1}{2}[\alpha_A(x \cdot y) + T(\alpha_A(y), \alpha_A(y))]$. Thus $s_0 \in [0,1]$ and $\alpha_A(x \cdot y) < s_0 < T(\alpha_A(y), \alpha_A(y))$. This implies that $x \cdot y \notin U(A;s_0)$. By Lemma 3.4 (1), we have $y \in U(A;s_0) \neq \emptyset$. Thus $U(A;s_0)$ is not a near UP-filter of X. This is a contradiction to the fact that $\emptyset \neq U(A;s_0)$ is a near UP-filter of X. Hence, $\alpha_A(x \cdot y) \geq T(\alpha_A(y), \alpha_A(y))$ for all $x, y \in X$. Therefore, A is a T-fuzzy near UP-filter of X. □

The following example show that the converse of Theorem 5.7 is not true.

Example 5.8. *From Example 5.5, we have A is a T_{Luk}-fuzzy near UP-filter of X and $U(A;0.7) = \{0,3,5,6\}$ (see T_{Luk} in Example 3.9). Since $5 \in U(A;0.7)$ but $3 \cdot 5 = 1 \notin U(A;0.7)$, we have $U(A;0.7)$ is not a near UP-filter of X.*

Theorem 5.9. *If A is a T-fuzzy UP-filter of X, then for all $s \in [0,1]$ with $T(s,s) = s$, $U(A;s)$ is a UP-filter of X if $U(A;s)$ is nonempty. In particular, $E(A;1)$ is a UP-filter of X if $E(A;1)$ is nonempty.*

Proof. Assume that A is a T-fuzzy UP-filter of X. Let $s \in [0,1]$ be such that $T(s,s) = s$ and $U(A;s) \neq \emptyset$. Since A is a T-fuzzy UP-filter of X, we have $\alpha_A(0) \geq \alpha_A(x)$ for all $x \in X$, it follows from Lemma 5.2 (2) that $0 \in U(A;s)$. Let $x, y \in X$ be such that $x \cdot y \in U(A;s)$ and $x \in U(A;s)$. Then $\alpha_A(x \cdot y) \geq s$ and $\alpha_A(x) \geq s$. Thus

$$\begin{aligned} \alpha_A(y) &\geq T(\alpha_A(x \cdot y), \alpha_A(x)) && \text{(Definition 3.5 (3) (ii))} \\ &\geq T(s,s) && \text{(Lemma 3.4 (3))} \\ &= s. \end{aligned}$$

Hence, $y \in U(A;s)$, that is, $U(A;s)$ is a UP-filter of X. □

Theorem 5.10. *If a fuzzy set A in X is such that a nonempty subset $U(A;s)$ is a UP-filter of X for all $s \in [0,1]$, then A is a T-fuzzy UP-filter of X.*

Proof. Assume that a fuzzy set A in X is such that a nonempty subset $U(A;s)$ is a UP-filter of X for all $s \in [0,1]$. For any $x \in X$, let $\alpha_A(x) = s$. Then $x \in U(A;s) \neq \emptyset$. By assumption, we have $U(A;s)$ is a UP-filter of X. Thus $0 \in U(A;s)$, that is, $\alpha_A(0) \geq s = \alpha_A(x)$. Suppose that there exist $x, y \in X$ such that $\alpha_A(y) < T(\alpha_A(x \cdot y), \alpha_A(x))$. Put $s_0 = \frac{1}{2}[\alpha_A(y) + T(\alpha_A(x \cdot y), \alpha_A(x))]$. Thus $s_0 \in [0,1]$ and $\alpha_A(y) < s_0 < T(\alpha_A(x \cdot y), \alpha_A(x))$. This implies that $y \notin U(A;s_0)$. By Lemma 3.4 (1), we have $x \cdot y, x \in U(A;s_0) \neq \emptyset$. Thus $U(A;s_0)$ is not a UP-filter of X. This is a contradiction to the fact that $\emptyset \neq U(A;s_0)$ is a UP-filter of X. Hence, $\alpha_A(y) \geq T(\alpha_A(x \cdot y), \alpha_A(x))$ for all $x, y \in X$. Therefore, A is a T-fuzzy UP-filter of X. □

The following example show that the converse of Theorem 5.10 is not true.

Example 5.11. *From Example 5.5, we have A is a T_{Luk}-fuzzy UP-filter of X and $U(A;0.7) = \{0,3,5,6\}$ (see T_{Luk} in Example 3.9). Since $5 \cdot 1 = 0 \in U(A;0.7)$ and $5 \in U(A;0.7)$ but $1 \notin U(A;0.7)$, we have $U(A;0.7)$ is not a UP-filter of X.*

Theorem 5.12. *If A is a T-fuzzy UP-ideal of X, then for all $s \in [0,1]$ with $T(s,s) = s$, $U(A;s)$ is a UP-ideal of X if $U(A;s)$ is nonempty. In particular, $E(A;1)$ is a UP-ideal of X if $U(A;1)$ is nonempty.*

Proof. Assume that A is a T-fuzzy UP-ideal of X. Let $s \in [0,1]$ be such that $T(s,s) = s$ and $U(A;s) \neq \emptyset$. Since A is a T-fuzzy UP-ideal of X, we have $\alpha_A(0) \geq \alpha_A(x)$ for all $x \in X$, it follows from Lemma 5.2 (2) that $0 \in U(A;s)$. Let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in U(A;s)$ and $y \in U(A;s)$. Then $\alpha_A(x \cdot (y \cdot z)) \geq s$ and $\alpha_A(y) \geq s$. Thus

$$\begin{aligned} \alpha_A(x \cdot z) &\geq T(\alpha_A(x \cdot (y \cdot z)), \alpha_A(y)) && \text{(Definition 3.5 (4) (ii))} \\ &\geq T(s,s) && \text{(Lemma 3.4 (3))} \\ &= s. \end{aligned}$$

Hence, $x \cdot z \in U(A;s)$, that is, $U(A;s)$ is a UP-ideal of X. □

Theorem 5.13. *If a fuzzy set A in X is such that a nonempty subset $U(A;s)$ is a UP-ideal of X for all $s \in [0,1]$, then A is a T-fuzzy UP-ideal of X.*

Proof. Assume that a fuzzy set A in X is such that a nonempty subset $U(A;s)$ is a UP-ideal of X for all $s \in [0,1]$. For any $x \in X$, let $\alpha_A(x) = s$. Then $x \in U(A;s) \neq \emptyset$. By assumption, we have $U(A;s)$ is a UP-ideal of X. Thus $0 \in U(A;s)$, that is, $\alpha_A(0) \geq s = \alpha_A(x)$. Suppose that there exist $x,y,z \in X$ such that $\alpha_A(x\cdot z) < T(\alpha_A(x\cdot(y\cdot z)),\alpha_A(y))$. Put $s_0 = \frac{1}{2}[\alpha_A(x\cdot z)+T(\alpha_A(x\cdot(y\cdot z)),\alpha_A(y))]$. Thus $s_0 \in [0,1]$ and $\alpha_A(x\cdot z) < s_0 < T(\alpha_A(x\cdot(y\cdot z)),\alpha_A(y))$. This implies that $x\cdot z \notin U(A;s_0)$. By Lemma 3.4 (1), we have $x\cdot(y\cdot z), y \in U(A;s_0) \neq \emptyset$. Thus $U(A;s_0)$ is not a UP-ideal of X. This is a contradiction to the fact that $\emptyset \neq U(A;s_0)$ is a UP-ideal of X. Hence, $\alpha_A(x\cdot z) \geq T(\alpha_A(x\cdot(y\cdot z)),\alpha_A(y))$ for all $x,y,z \in X$. Therefore, A is a T-fuzzy UP-ideal of X. □

The following example show that the converse of Theorem 5.13 is not true.

Example 5.14. *From Example 5.5, we have A is a T_{Luk}-fuzzy UP-ideal of X and $U(A;0.7) = \{0,3,5,6\}$ (see T_{Luk} in Example 3.9). Since $1\cdot(6\cdot 2) = 0 \in U(A;0.7)$ and $6 \in U(A;0.7)$ but $1\cdot 2 = 2 \notin U(A;0.7)$, we have $U(A;0.7)$ is not a UP-ideal of X.*

Theorem 5.15. *If A is a T-fuzzy strongly UP-ideal of X, then for all $s \in [0,1]$ with $T(s,s) = s$, $U(A;s)$ is a strongly UP-ideal of X if $U(A;s)$ is nonempty. In particular, $E(A;1)$ is a strongly UP-ideal of X if $U(A;1)$ is nonempty.*

Proof. Assume that A is a T-fuzzy strongly UP-ideal of X. Let $s \in [0,1]$ be such that $T(s,s) = s$ and $U(A;s) \neq \emptyset$. Since A is a T-fuzzy strongly UP-ideal of X, we have $\alpha_A(0) \geq \alpha_A(x)$ for all $x \in X$, it follows from Lemma 5.2 (2) that $0 \in U(A;s)$. Let $x \in X$. Then

$$\begin{aligned}
\alpha_A(x) &\geq T(\alpha_A((x\cdot 0)\cdot(x\cdot x)),\alpha_A(0)) && \text{(Definition 3.5 (5) (ii))}\\
&= T(\alpha_A(0),\alpha_A(0)) && \text{(Proposition 2.5 (1) and (UP-3))}\\
&\geq T(s,s) && \text{(Lemma 3.4 (3))}\\
&= s.
\end{aligned}$$

Thus $x \in U(A;s)$, that is, $U(A;s) = X$. Hence, $U(A;s)$ is a strongly UP-ideal of X. □

Theorem 5.16. *If a fuzzy set A in X is such that a nonempty subset $U(A;s)$ is a strongly UP-ideal of X for all $s \in [0,1]$, then A is a T-fuzzy strongly UP-ideal of X.*

Proof. It is straightforward by Theorem 3.6. □

The following example show that the converse of Theorem 5.16 is not true.

Example 5.17. *From Example 5.5, we have A is a T_{Luk}-fuzzy strongly UP-ideal of X and $U(A;0.7) = \{0,3,5,6\}$ (see T_{Luk} in Example 3.9). Since $U(A;0.7) \neq X$, we have $U(A;0.7)$ is not a strongly UP-ideal of X.*

Lemma 5.18. *Let A be a fuzzy set in X. Then the following statements hold:*

(1) *if the constant 0 of X is in $E(A;0)$, then $\alpha_A(0) \leq \alpha_A(x)$ for all $x \in X$, and*
(2) *for any $s \in [0,1]$, if there exists an element $x \in L(A;s)$ such that $\alpha_A(0) \leq \alpha_A(x)$, then the constant 0 of X is in $L(A;s)$.*

Proof. (1) If $0 \in E(A;0)$, then $\alpha_A(0) = 0 \leq \alpha_A(x)$ for all $x \in X$.
(2) Assume that $\alpha_A(0) \leq \alpha_A(x)$ for some $x \in L(A;s)$. Then $\alpha_A(0) \leq \alpha_A(x) \leq s$, that is, $0 \in L(A;s)$. □

Theorem 5.19. *If A is an anti-T-fuzzy near UP-filter of X, then for all $s \in [0,1]$, $L(A;s)$ is a near UP-filter of X if $L(A;s)$ is nonempty. In particular, $E(A;0)$ is a near UP-filter of X if $E(A;0)$ is nonempty.*

Proof. Assume that A is an anti-T-fuzzy near UP-filter of X. Let $s \in [0,1]$ be such that $L(A;s) \neq \emptyset$. Since A is an anti-T-fuzzy near UP-filter of X, we have $\alpha_A(0) \leq \alpha_A(x)$ for all $x \in X$, it follows from Lemma 5.18 (2) that $0 \in L(A;s)$. Let $x \in X$ and $y \in L(A;s)$. Then $\alpha_A(y) \leq s$. Thus

$$\begin{aligned}
\alpha_A(x\cdot y) &\leq T(\alpha_A(y),\alpha_A(y)) && \text{(Definition 3.18 (2) (ii))}\\
&\leq T(s,s) && \text{(Lemma 3.4 (3))}\\
&\leq s. && \text{(Lemma 3.2 (1))}
\end{aligned}$$

Hence, $x\cdot y \in L(A;s)$, that is, $L(A;s)$ is a near UP-filter of X. □

Theorem 5.20. *If a fuzzy set A in X is such that a nonempty subset $L(A;s)$ is a near UP-filter of X for all $s \in [0,1]$ and satisfying the imaginable property with respect to T, then A is an anti-T-fuzzy near UP-filter of X.*

Proof. Assume that a fuzzy set A in X is such that a nonempty subset $L(A;s)$ is a near UP-filter of X for all $s \in [0,1]$ and satisfying the imaginable property with respect to T. For any $x \in X$, let $\alpha_A(x) = s$. Then $x \in L(A;s) \neq \emptyset$. By assumption, we have $L(A;s)$ is a near UP-filter of X. Thus $0 \in L(A;s)$, that is, $\alpha_A(0) \leq s = \alpha_A(x)$. Suppose that there exist $x,y \in X$ such that $\alpha_A(x\cdot y) > T(\alpha_A(y),\alpha_A(y))$. Put $s_0 = \frac{1}{2}[\alpha_A(x\cdot y)+T(\alpha_A(y),\alpha_A(y))]$. Thus $s_0 \in [0,1]$ and $\alpha_A(x\cdot y) > s_0 > T(\alpha_A(y),\alpha_A(y)) = \alpha_A(y)$. This implies that $x\cdot y \notin L(A;s_0)$ and $y \in L(A;s_0)$. Thus $L(A;s_0)$ is not a near UP-filter of X. This is a contradiction to the fact that $\emptyset \neq L(A;s_0)$ is a near UP-filter of X. Hence, $\alpha_A(x\cdot y) \leq T(\alpha_A(y),\alpha_A(y))$ for all $x,y \in X$. Therefore, A is an anti-T-fuzzy near UP-filter of X. □

Theorem 5.21. *If A is an anti-T-fuzzy UP-subalgebra (resp., anti-T-fuzzy UP-filter, anti-T-fuzzy UP-ideal, anti-T-fuzzy strongly UP-ideal) of X, then $E(A;\alpha_A(0))$ is a strongly UP-ideal of X.*

Proof. It is straightforward by Theorem 3.25. □

Theorem 5.22. *If a fuzzy set A in X is such that $E(A;\alpha_A(0))$ is a strongly UP-ideal of X and satisfying the imaginable property with respect to T, then A is a T-fuzzy UP-subalgebra of X.*

Proof. Assume that a fuzzy set A in X is such that $E(A;\alpha_A(0))$ is a strongly UP-ideal of X and satisfying the imaginable property with respect to T. Then $\alpha_A(x) = \alpha_A(0)$ for all $x \in X$. Let $x, y \in X$. Then

$$\alpha_A(x \cdot y) = \alpha_A(0) = T(\alpha_A(0), \alpha_A(0)) = T(\alpha_A(x), \alpha_A(y)).$$

Hence, A is an anti-T-fuzzy UP-subalgebra of X. □

6. UP-homomorphisms in UP-algebras with respect to a t-norm

Definition 6.1. *Let f be a function from a nonempty set X to a nonempty set Y. If B is a fuzzy set in Y, then the inverse image of B under f is a fuzzy set $f^{-1}(B)$ in X defined as follows:*

$$(\forall x \in X)(\alpha_{f^{-1}(B)}(x) = \alpha_B(f(x))). \tag{6.1}$$

Definition 6.2. *[5] Let $(X, \cdot, 0_X)$ and $(Y, *, 0_Y)$ be UP-algebras. A mapping f from X to Y is called a UP-homomorphism if*

$$(\forall x, y \in X)(f(x \cdot y) = f(x) * f(y)). \tag{6.2}$$

A UP-homomorphism $f : X \to Y$ is called a UP-epimorphism if f is surjective.

In what follows, let f denote a UP-homomorphism from a UP-algebra $(X, \cdot, 0_X)$ to a UP-algebra $(Y, *, 0_Y)$ unless otherwise specified.

From [5], we have $f(0_X) \geq f(x)$ for all $x \in X$.

Theorem 6.3. *If B is a T-fuzzy UP-subalgebra of Y, then $f^{-1}(B)$ is a T-fuzzy UP-subalgebra of X.*

Proof. Assume that B is a T-fuzzy UP-subalgebra of Y. Let $x, y \in X$. Thus

$$\begin{aligned}
\alpha_{f^{-1}(B)}(x \cdot y) &= \alpha_B(f(x \cdot y)) && ((6.1))\\
&= \alpha_B(f(x) * f(y)) && ((6.2))\\
&\geq T(\alpha_B(f(x)), \alpha_B(f(y))) && (\text{Definition } 3.5\ (1))\\
&= T(\alpha_{f^{-1}(B)}(x), \alpha_{f^{-1}(B)}(y)). && ((6.1))
\end{aligned}$$

Hence, $f^{-1}(B)$ is a T-fuzzy UP-subalgebra of X. □

Theorem 6.4. *If B is a T-fuzzy near UP-filter of Y in which α_B is an order preserving mapping, then $f^{-1}(B)$ is a T-fuzzy near UP-filter of X.*

Proof. Assume that B is a T-fuzzy near UP-filter of Y in which α_B is an order preserving mapping. Then for all $x \in X$,

$$\alpha_{f^{-1}(B)}(0_X) = \alpha_B(f(0_X)) \geq \alpha_B(f(x)) = \alpha_{f^{-1}(B)}(x).$$

Let $x, y \in X$. Thus

$$\begin{aligned}
\alpha_{f^{-1}(B)}(x \cdot y) &= \alpha_B(f(x \cdot y)) && ((6.1))\\
&= \alpha_B(f(x) * f(y)) && ((6.2))\\
&\geq T(\alpha_B(f(y)), \alpha_B(f(y))) && (\text{Definition } 3.5\ (2)\ (ii))\\
&= T(\alpha_{f^{-1}(B)}(y), \alpha_{f^{-1}(B)}(y)). && ((6.1))
\end{aligned}$$

Hence, $f^{-1}(B)$ is a T-fuzzy near UP-filter of X. □

Theorem 6.5. *If B is a T-fuzzy UP-filter of Y in which α_B is an order preserving mapping, then $f^{-1}(B)$ is a T-fuzzy UP-filter of X.*

Proof. Assume that B is a T-fuzzy UP-filter of Y in which α_B is an order preserving mapping. Then for all $x \in X$,

$$\alpha_{f^{-1}(B)}(0_X) = \alpha_B(f(0_X)) \geq \alpha_B(f(x)) = \alpha_{f^{-1}(B)}(x).$$

Let $x, y \in X$. Thus

$$\begin{aligned}
\alpha_{f^{-1}(B)}(y) &= \alpha_B(f(y)) && ((6.1))\\
&\geq T(\alpha_B(f(x) * f(y)), \alpha_B(f(x))) && (\text{Definition } 3.5\ (3)\ (ii))\\
&\geq T(\alpha_B(f(x \cdot y)), \alpha_B(f(x))) && ((6.2))\\
&= T(\alpha_{f^{-1}(B)}(x \cdot y), \alpha_{f^{-1}(B)}(x)). && ((6.1))
\end{aligned}$$

Hence, $f^{-1}(B)$ is a T-fuzzy near UP-filter of X. □

Theorem 6.6. *If B is a T-fuzzy UP-ideal of Y in which α_B is an order preserving mapping, then $f^{-1}(B)$ is a T-fuzzy UP-ideal of X.*

Proof. Assume that B is a T-fuzzy UP-ideal of Y in which α_B is an order preserving mapping. Then for all $x \in X$,

$\alpha_{f^{-1}(B)}(0_X) = \alpha_B(f(0_X)) \geq \alpha_B(f(x)) = \alpha_{f^{-1}(B)}(x)$.

Let $x, y, z \in X$. Thus

$$\begin{aligned}
\alpha_{f^{-1}(B)}(x \cdot z) &= \alpha_B(f(x \cdot z)) && ((6.1))\\
&= \alpha_B(f(x) * f(z)) && ((6.2))\\
&\geq T(\alpha_B(f(x) * (f(y) * f(z))), \alpha_B(f(y))) && (\text{Definition 3.5 (4) (ii)})\\
&= T(\alpha_B(f(x) * f(y \cdot z)), \alpha_B(f(y))) && ((6.2))\\
&= T(\alpha_B(f(x \cdot (y \cdot z)), \alpha_B(f(y))) && ((6.2))\\
&= T(\alpha_{f^{-1}(B)}(x \cdot (y \cdot z)), \alpha_{f^{-1}(B)}(y)). && ((6.1))
\end{aligned}$$

Hence, $f^{-1}(B)$ is a T-fuzzy UP-ideal of X. □

Theorem 6.7. *If B is a T-fuzzy strongly UP-ideal of Y in which α_B is an order preserving mapping, then $f^{-1}(B)$ is a T-fuzzy strongly UP-ideal of X.*

Proof. Assume that B is a T-fuzzy strongly UP-ideal of Y in which α_B is an order preserving mapping. Then for all $x \in X$,

$\alpha_{f^{-1}(B)}(0_X) = \alpha_B(f(0_X)) \geq \alpha_B(f(x)) = \alpha_{f^{-1}(B)}(x)$.

Let $x, y, z \in X$. Thus

$$\begin{aligned}
\alpha_{f^{-1}(B)}(x) &= \alpha_B(f(x)) && ((6.1))\\
&\geq T(\alpha_B((f(z) * f(y)) * (f(z) * f(x))), \alpha_B(f(x))) && (\text{Definition 3.5 (5) (ii)})\\
&= T(\alpha_B(f(z \cdot y) * f(z \cdot x)), \alpha_B(f(x))) && ((6.2))\\
&= T(\alpha_B(f((z \cdot y) \cdot (z \cdot x)), \alpha_B(f(x))) && ((6.2))\\
&= T(\alpha_{f^{-1}(B)}((z \cdot y) \cdot (z \cdot x)), \alpha_{f^{-1}(B)}(y)). && ((6.1))
\end{aligned}$$

Hence, $f^{-1}(B)$ is a T-fuzzy strongly UP-ideal of X. □

Theorem 6.8. *If B is an anti-T-fuzzy near UP-filter of Y in which α_B is an anti-order preserving mapping, then $f^{-1}(B)$ is a anti- T-fuzzy near UP-filter of X.*

Proof. Assume that B is an anti-T-fuzzy near UP-filter of Y in which α_B is an anti-order preserving mapping. Then for all $x \in X$,

$\alpha_{f^{-1}(B)}(0_X) = \alpha_B(f(0_X)) \leq \alpha_B(f(x)) = \alpha_{f^{-1}(B)}(x)$.

Let $x, y \in X$. Thus

$$\begin{aligned}
\alpha_{f^{-1}(B)}(x \cdot y) &= \alpha_B(f(x \cdot y)) && ((6.1))\\
&= \alpha_B(f(x) * f(y)) && ((6.2))\\
&\leq T(\alpha_B(f(y)), \alpha_B(f(y))) && (\text{Definition 3.18 (2) (ii)})\\
&= T(\alpha_{f^{-1}(B)}(y), \alpha_{f^{-1}(B)}(y)). && ((6.1))
\end{aligned}$$

Hence, $f^{-1}(B)$ is an anti-T-fuzzy near UP-filter of X. □

Theorem 6.9. *If B is an anti-T-fuzzy UP-subalgebra (resp., anti-T-fuzzy UP-filter, anti-T-fuzzy UP-ideal, anti-T-fuzzy strongly UP-ideal) of Y, then $f^{-1}(B)$ is an anti-T-fuzzy UP-subalgebra of X.*

Proof. Assume that B is an anti-T-fuzzy UP-subalgebra of Y. By Theorem 3.25, we have for all $x \in X$,

$\alpha_{f^{-1}(B)}(x) = \alpha_B(f(x)) = \alpha_B(f(0_X)) = \alpha_{f^{-1}(B)}(0_X)$

Let $x, y \in X$. Thus

$$\begin{aligned}
\alpha_{f^{-1}(B)}(x \cdot y) &= \alpha_{f^{-1}(B)}(0_X) \\
&= \alpha_B(f(0_X)) && ((6.1))\\
&= \alpha_B(f(x) * f(y)) && ((6.2))\\
&\leq T(\alpha_B(f(y)), \alpha_B(f(y))) && (\text{Definition 3.18 (2) (ii)})\\
&= T(\alpha_{f^{-1}(B)}(y), \alpha_{f^{-1}(B)}(y)). && ((6.1))
\end{aligned}$$

Hence, $f^{-1}(B)$ is an anti-T-fuzzy UP-subalgebra of X. □

Definition 6.10. *[14] Let X and Y be any nonempty sets and let $f : X \to Y$ be any function. A fuzzy set A in X is said to be f-invariant if*

$$(\forall x, y \in X)(f(x) = f(y) \Rightarrow \alpha_A(x) = \alpha_A(y)). \quad (6.3)$$

Definition 6.11. *Let f be a function from a nonempty set X to a nonempty set Y. If A is a fuzzy set in X, then the image of A under f is a fuzzy set $f(A)$ in Y defined as follows:*

$$(\forall y \in Y)\left(\alpha_{f(A)}(y) = \begin{cases} \sup\{\alpha_A(x)\}_{x\in f^{-1}(y)} & \text{if } f^{-1}(y) \neq \emptyset, \\ 1 & \text{otherwise} \end{cases}\right). \tag{6.4}$$

Definition 6.12. *[20] A fuzzy set A in X has sup property if for any nonempty subset S of X, there exist $s_0 \in S$ such that $\alpha_A(s_0) = \sup\{\alpha_A(s)\}_{s\in S}$.*

Lemma 6.13. *[16] Assume that f is surjective. Let A be an f-invariant fuzzy set in X with* sup *property. For any $x, y \in Y$, there exist $x_0 \in f^{-1}(x)$ and $y_0 \in f^{-1}(y)$ such that $\alpha_{f(A)}(x) = \alpha_A(x_0), \alpha_{f(A)}(y) = \alpha_A(y_0)$, and $\alpha_{f(A)}(x*y) = \alpha_A(x_0 \cdot y_0)$.*

Theorem 6.14. *Assume that f is surjective. If A is an f-invariant T-fuzzy UP-subalgebra of X with* sup *property, then $f(A)$ is a T-fuzzy UP-subalgebra of Y.*

Proof. Let A be an f-invariant T-fuzzy UP-subalgebra of X with sup property. Let $x, y \in Y$. Since f is surjective, we have $f^{-1}(x), f^{-1}(y)$, and $f^{-1}(x*y)$ are nonempty subsets of X. By Lemma 6.13, there exist $x_0 \in f^{-1}(x)$ and $y_0 \in f^{-1}(y)$ such that $\alpha_{f(A)}(x) = \alpha_A(x_0)$, $\alpha_{f(A)}(y) = \alpha_A(y_0)$, and $\alpha_{f(A)}(x*y) = \alpha_A(x_0 \cdot y_0)$. Thus

$$\begin{aligned} \alpha_{f(A)}(x*y) &= \alpha_A(x_0 \cdot y_0) \\ &\geq T(\alpha_A(x_0), \alpha_A(y_0)) && \text{(Definition 3.5 (1))} \\ &= T(\alpha_{f(A)}(x), \alpha_{f(A)}(y)). \end{aligned}$$

Hence, $f(A)$ is a T-fuzzy UP-subalgebra of Y. □

Theorem 6.15. *Assume that f is surjective. If A is an f-invariant T-fuzzy near UP-filter of X with* sup *property, then $f(A)$ is a T-fuzzy near UP-filter of Y.*

Proof. Let A be an f-invariant T-fuzzy near UP-filter of X with sup property. Since A is a T-fuzzy near UP-filter of X, we have $\alpha_A(0_X) \geq \alpha_A(x)$ for all $x \in X$. Since $f(0_X) = 0_Y$, we have $0_X \in f^{-1}(0_Y)$ and so $f^{-1}(0_Y) \neq \emptyset$. Thus

$$\alpha_{f(A)}(0_Y) = \sup\{\alpha_A(t)\}_{t\in f^{-1}(0_Y)} \geq \alpha_A(0_X).$$

Let $y \in Y$. Since f is surjective, we have $f^{-1}(y) \neq \emptyset$. Since A is a T-fuzzy near UP-filter of X, we have $\alpha_A(0_X) \geq \alpha_A(t)$ for all $t \in f^{-1}(y)$. Thus $\alpha_A(0_X)$ is an upper bound of $\{\alpha_A(t)\}_{t\in f^{-1}(y)}$ and so

$$\alpha_A(0_X) \geq \sup\{\alpha_A(t)\}_{t\in f^{-1}(y)} = \alpha_{f(A)}(y).$$

Thus $\alpha_{f(A)}(0_Y) \geq \alpha_{f(A)}(y)$. Let $x, y \in Y$. By Lemma 6.13, there exist $x_0 \in f^{-1}(x)$ and $y_0 \in f^{-1}(y)$ such that $\alpha_{f(A)}(x) = \alpha_A(x_0), \alpha_{f(A)}(y) = \alpha_A(y_0)$, and $\alpha_{f(A)}(x*y) = \alpha_A(x_0 \cdot y_0)$. Thus

$$\begin{aligned} \alpha_{f(A)}(x*y) &= \alpha_A(x_0 \cdot y_0) \\ &\geq T(\alpha_A(y_0), \alpha_A(y_0)) && \text{(Definition 3.5 (2) (ii))} \\ &= T(\alpha_{f(A)}(y), \alpha_{f(A)}(y)). \end{aligned}$$

Hence, $f(A)$ is a T-fuzzy near UP-filter of Y. □

Theorem 6.16. *Assume that f is surjective. If A is an f-invariant T-fuzzy UP-filter of X with* sup *property, then $f(A)$ is a T-fuzzy UP-filter of Y.*

Proof. Let A be an f-invariant T-fuzzy near UP-filter of X with sup property. Since A is a T-fuzzy UP-filter of X, we have $\alpha_A(0_X) \geq \alpha_A(x)$ for all $x \in X$. Since $f(0_X) = 0_Y$, we have $0_X \in f^{-}(0_Y)$ and so $f^{-1}(0_Y) \neq \emptyset$. Thus

$$\alpha_{f(A)}(0_Y) = \sup\{\alpha_A(t)\}_{t\in f^{-1}(0_Y)} \geq \alpha_A(0_X).$$

Let $y \in Y$. Since f is surjective, we have $f^{-1}(y) \neq \emptyset$. Since A is a T-fuzzy UP-filter of X, we have $\alpha_A(0_X) \geq \alpha_A(t)$ for all $t \in f^{-1}(y)$. Thus $\alpha_A(0_X)$ is an upper bound of $\{\alpha_A(t)\}_{t\in f^{-1}(y)}$ and so

$$\alpha_A(0_X) \geq \sup\{\alpha_A(t)\}_{t\in f^{-1}(y)} = \alpha_{f(A)}(y).$$

Thus $\alpha_{f(A)}(0_Y) \geq \alpha_{f(A)}(y)$. Let $x, y \in Y$. By Lemma 6.13, there exist $x_0 \in f^{-1}(x)$, $y_0 \in f^{-1}(y)$ such that $\alpha_{f(A)}(x) = \alpha_A(x_0)$, $\alpha_{f(A)}(y) = \alpha_A(y_0)$ and $\alpha_{f(A)}(x*y) = \alpha_A(x_0 \cdot y_0)$. Thus

$$\begin{aligned} \alpha_{f(A)}(y) &= \alpha_A(y_0) \\ &\geq T(\alpha_A(x_0 \cdot y_0), \alpha_A(x_0)) && \text{(Definition 3.5 (3) (ii))} \\ &= T(\alpha_{f(A)}(x*y), \alpha_{f(A)}(x)). \end{aligned}$$

Hence, $f(A)$ is a T-fuzzy UP-filter of Y. □

Theorem 6.17. *Assume that f is surjective. If A is an f-invariant T-fuzzy UP-ideal of X with* sup *property, then $f(A)$ is a T-fuzzy UP-ideal of Y.*

Proof. Let A be an f-invariant T-fuzzy UP-ideal of X with sup property. Since A is a T-fuzzy UP-ideal of X, we have $\alpha_A(0_X) \geq \alpha_A(x)$ for all $x \in X$. Since $f(0_X) = 0_Y$, we have $0_X \in f^{-}(0_Y)$ and so $f^{-1}(0_Y) \neq \emptyset$. Thus

$$\alpha_{f(A)}(0_Y) = \sup\{\alpha_A(t)\}_{t \in f^{-1}(0_Y)} \geq \alpha_A(0_X).$$

Let $y \in Y$. Since f is surjective, we have $f^{-1}(y) \neq \emptyset$. Since A is a T-fuzzy UP-ideal of X, we have $\alpha_A(0_X) \geq \alpha_A(t)$ for all $t \in f^{-1}(y)$. Thus $\alpha_A(0_X)$ is an upper bound of $\{\alpha_A(t)\}_{t \in f^{-1}(y)}$ and so

$$\alpha_A(0_X) \geq \sup\{\alpha_A(t)\}_{t \in f^{-1}(y)} = \alpha_{f(A)}(y).$$

Thus $\alpha_{f(A)}(0_Y) \geq \alpha_{f(A)}(y)$. Let $x, y, z \in Y$. By Lemma 6.13, there exist $x_0 \in f^{-1}(x)$, $y_0 \in f^{-1}(y)$ and $z_0 \in f^{-1}(z)$ such that $\alpha_{f(A)}(x*(y*z)) = \alpha_A(x_0 \cdot (y_0 * z_0))$, $\alpha_{f(A)}(y) = \alpha_A(y_0)$ and $\alpha_{f(A)}(x*z) = \alpha_A(x_0 \cdot z_0)$. Thus

$$\begin{aligned}
\alpha_{f(A)}(x*z) &= \alpha_A(x_0 \cdot z_0) \\
&\geq T(\alpha_A(x_0 \cdot (y_0 \cdot z_0)), \alpha_A(y_0)) && \text{(Definition 3.5 (4) (ii))} \\
&= T(\alpha_{f(A)}(x*(y*z)), \alpha_{f(A)}(y)).
\end{aligned}$$

Hence, $f(A)$ is a T-fuzzy UP-ideal of Y. □

Theorem 6.18. *Assume that f is surjective. If A is an f-invariant T-fuzzy strongly UP-ideal of X with* sup *property, then $f(A)$ is a T-fuzzy strongly UP-ideal of Y.*

Proof. Let A be an f-invariant T-fuzzy strongly UP-ideal of X with sup property. Since A is a T-fuzzy strongly UP-ideal of X, we have $\alpha_A(0_X) \geq \alpha_A(x)$ for all $x \in X$. Since $f(0_X) = 0_Y$, we have $0_X \in f^{-}(0_Y)$ and so $f^{-1}(0_Y) \neq \emptyset$. Thus

$$\alpha_{f(A)}(0_Y) = \sup\{\alpha_A(t)\}_{t \in f^{-1}(0_Y)} \geq \alpha_A(0_X).$$

Let $y \in Y$. Since f is surjective, we have $f^{-1}(y) \neq \emptyset$. Since A is a T-fuzzy strongly UP-ideal of X, we have $\alpha_A(0_X) \geq \alpha_A(t)$ for all $t \in f^{-1}(y)$. Thus $\alpha_A(0_X)$ is an upper bound of $\{\alpha_A(t)\}_{t \in f^{-1}(y)}$ and so

$$\alpha_A(0_X) \geq \sup\{\alpha_A(t)\}_{t \in f^{-1}(y)} = \alpha_{f(A)}(y).$$

Thus $\alpha_{f(A)}(0_Y) \geq \alpha_{f(A)}(y)$. Let $x, y, z \in Y$. By Lemma 6.13, there exist $x_0 \in f^{-1}(x)$, $y_0 \in f^{-1}(y)$ and $z_0 \in f^{-1}(z)$ such that $\alpha_{f(A)}((z*y)*(z*x)) = \alpha_A((z_0 \cdot y_0) \cdot (z_0 * x_0))$, $\alpha_{f(A)}(y) = \alpha_A(y_0)$ and $\alpha_{f(A)}(x) = \alpha_A(x_0)$. Thus

$$\begin{aligned}
\alpha_{f(A)}(x) &= \alpha_A(x_0) \\
&\geq T(\alpha_A((z_0 \cdot y_0) \cdot (z_0 \cdot x_0)), \alpha_A(y_0)) && \text{(Definition 3.5 (5) (ii))} \\
&= T(\alpha_{f(A)}((z*y)*(z*x)), \alpha_{f(A)}(y)).
\end{aligned}$$

Hence, $f(A)$ is a T-fuzzy strongly UP-ideal of Y. □

Definition 6.19. *Let f be a function from a nonempty set X to a nonempty set Y. If A is a fuzzy set in X, then the image of A under f is a fuzzy set $\overline{f(A)}$ in Y defined as follows:*

$$(\forall y \in Y)\left(\alpha_{\overline{f(A)}}(y) = \begin{cases} \inf\{\alpha_A(x)\}_{x \in f^{-1}(y)} & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise} \end{cases}\right). \tag{6.5}$$

Definition 6.20. *[20] A fuzzy set A in X has* inf *property if for any nonempty subset S of X, there exist $s_0 \in S$ such that $\alpha_A(s_0) = \inf\{\alpha_A(s)\}_{s \in S}$.*

Lemma 6.21. *[8] Assume that f is surjective. Let A be an f-invariant fuzzy set in X with* inf *property. For any $x, y \in Y$, there exist $x_0 \in f^{-1}(x)$ and $y_0 \in f^{-1}(y)$ such that $\alpha_{\overline{f(A)}}(x) = \alpha_A(x_0)$, $\alpha_{\overline{f(A)}}(y) = \alpha_A(y_0)$, and $\alpha_{\overline{f(A)}}(x*y) = \alpha_A(x_0 \cdot y_0)$.*

Theorem 6.22. *Assume that f is surjective. If A is an f-invariant anti-T-fuzzy UP-subalgebra of X with* inf *property, then $\overline{f(A)}$ is an anti-T-fuzzy UP-subalgebra of Y.*

Proof. Let A be an f-invariant anti-T-fuzzy UP-subalgebra of X with inf property. By Theorem 3.25, we have $\alpha_A(x_0) = \alpha_A(0_X)$ for all $x_0 \in X$. Since $f(0_X) = 0_Y$, we have $0_X \in f^{-}(0_Y)$ and so $f^{-1}(0_Y) \neq \emptyset$. Let $x \in Y$. Since f is surjective, we have $f^{-1}(x)$ are nonempty subsets of X. By Lemma 6.21, there exist $x_0 \in f^{-1}(x)$, $y_0 \in f^{-1}(y)$ and $x_0 \cdot y_0 \in f^{-1}(x*y)$ such that $\alpha_{\overline{f(A)}}(x) = \alpha_A(x_0)$, $\alpha_{\overline{f(A)}}(y) = \alpha_A(y_0)$ and $\alpha_{\overline{f(A)}}(x*y) = \alpha_A(x_0 \cdot y_0)$. Thus

$$\begin{aligned}
\alpha_{\overline{f(A)}}(x*y) &= \alpha_A(x_0 \cdot y_0) \\
&= \alpha_A(0_X) \\
&= T(\alpha_A(0_X), \alpha_A(0_X)) && \text{(Lemma 3.19 (1))} \\
&= T(\alpha_A(x_0), \alpha_A(y_0)) \\
&= T(\alpha_{\overline{f(A)}}(x), \alpha_{\overline{f(A)}}(y)).
\end{aligned}$$

Hence, $\overline{f(A)}$ is an anti-T-fuzzy UP-subalgebra of Y. □

Theorem 6.23. *Assume that f is surjective. If A is an f-invariant anti-T-fuzzy near UP-filter of X with* inf *property, then $\overline{f(A)}$ is an anti-T-fuzzy near UP-filter of Y.*

Proof. Let A be an f-invariant anti-T-fuzzy near UP-filter of X with sup property. Since A is an anti-T-fuzzy near UP-filter of X, we have $\alpha_A(0_X) \leq \alpha_A(x)$ for all $x \in X$. Since $f(0_X) = 0_Y$, we have $0_X \in f^{-}(0_Y)$ and so $f^{-1}(0_Y) \neq \emptyset$. Thus

$$\alpha_{\overline{f(A)}}(0_Y) = \inf\{\alpha_A(t)\}_{t\in f^{-1}(0_Y)} \leq \alpha_A(0_X).$$

Let $y \in Y$. Since f is surjective, we have $f^{-1}(y) \neq \emptyset$. Since A is an anti-T-fuzzy near UP-filter of X, we have $\alpha_A(0_X) \leq \alpha_A(t)$ for all $t \in f^{-1}(y)$. Thus $\alpha_A(0_X)$ is a lower bound of $\{\alpha_A(t)\}_{t\in f^{-1}(y)}$ and so

$$\alpha_A(0_X) \leq \inf\{\alpha_A(t)\}_{t\in f^{-1}(y)} = \alpha_{\overline{f(A)}}(y).$$

Thus $\alpha_{\overline{f(A)}}(0_Y) \leq \alpha_{\overline{f(A)}}(y)$. Let $x, y \in Y$. By Lemma 6.21, there exist $x_0 \in f^{-1}(x)$, $y_0 \in f^{-1}(y)$ such that $\alpha_{\overline{f(A)}}(x) = \alpha_A(x_0)$, $\alpha_{\overline{f(A)}}(y) = \alpha_A(y_0)$ and $\alpha_{\overline{f(A)}}(x*y) = \alpha_A(x_0 \cdot y_0)$. Thus

$$\begin{aligned}\alpha_{\overline{f(A)}}(x*y) &= \alpha_A(x_0 \cdot y_0)\\ &\leq T(\alpha_A(y_0), \alpha_A(y_0)) && \text{(Definition 3.18 (2) (ii))}\\ &= T(\alpha_{\overline{f(A)}}(y), \alpha_{\overline{f(A)}}(y)).\end{aligned}$$

Hence, $\overline{f(A)}$ is an anti-T-fuzzy near UP-filter of Y. □

7. Conclusions

In this paper, we have introduced the notions of T-fuzzy UP-subalgebras, T-fuzzy near UP-filters, T-fuzzy UP-filters, T-fuzzy UP-ideals, and T-fuzzy strongly UP-ideals and also introduced the notions of anti-T-fuzzy UP-subalgebras, anti-T-fuzzy near UP-filters, anti-T-fuzzy UP-filters, anti-T-fuzzy UP-ideals, and anti-T-fuzzy strongly UP-ideals of UP-algebras, and investigated some of their important properties. Then we have the diagram of generalization of fuzzy sets with respect to a t-norm in UP-algebras (see Figure 7.1) below.

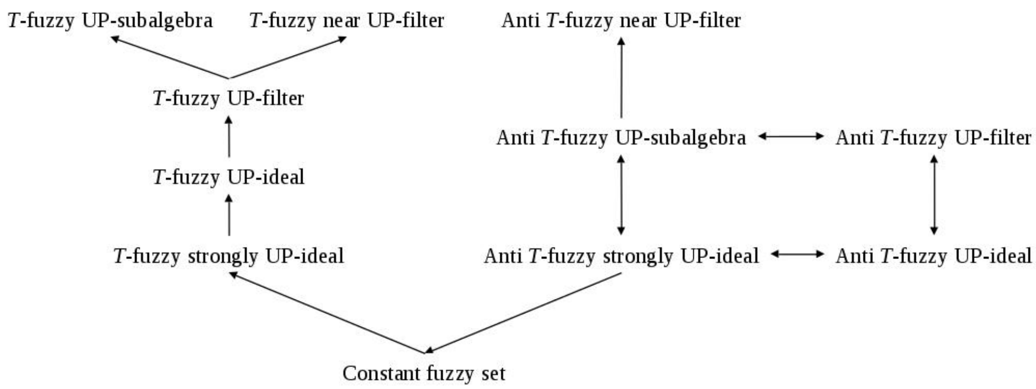

Figure 7.1: fuzzy sets with respect to a t-norm in UP-algebras

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