On an Extension of Absolute Summability

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Abstract

In the present paper, a known theorem on absolute summability factors of infinite series has been generalized for $|A,p_n;\delta|_k$ summability by using matrix transformation.

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1. Introduction

Let $\sum a_n$ be an infinite series with its partial sums $(s_n)$ and $A = (a_{nv})$ be a normal matrix; i.e., a lower triangular matrix of nonzero diagonal entries. The series $\sum a_n$ is said to be summable $|A,p_n;\delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [8])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k+1-k} |A_n(s) - A_{n-1}(s)|^k < \infty,$$

where $(p_n)$ is a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \rightarrow \infty \text{ as } n \rightarrow \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1)$$

and $A_n = (A_n(s))$ is defined by

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v, \quad n = 0, 1, \ldots$$

If we take $a_{mv} = \frac{p_v}{p_n}$, $|A,p_n;\delta|_k$ summability reduces to $|\tilde{N},p_n;\delta|_k$ summability (see [4]). For $\delta = 0$, $|A,p_n;\delta|_k$ summability reduces to $|A,p_n|_k$ summability (see [17]). Additionally, the series $\sum a_n$ is said to be bounded $|\tilde{N},p_n;\delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [3])

$$\sum_{v=1}^{n} \left(\frac{P_v}{p_v}\right)^{\delta} |p_v s_v|^k = O(p_n) \quad \text{as } n \rightarrow \infty. \quad (1.1)$$

It should be noted that, for $\delta = 0$, $|\tilde{N},p_n;\delta|_k$ boundedness is the same as $|\tilde{N},p_n|_k$ boundedness (see [1]).

2. Known Results

Some works dealing with absolute summability and absolute matrix summability can be found in [1–3, 5–7, 9–13, 15, 16]. Among them, in [5], Bor has proved a theorem as follows.

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Theorem 2.1. Let the series $\sum a_n$ be $[\hat{N}, p_n; \delta]_k$ bounded. If the conditions

$$p_{n+1} = O(p_n) \quad \text{as} \quad n \to \infty,$$  \hspace{1cm} (2.1)

$$\sum_{n=1}^{m} p_n |\lambda_n| = O(1) \quad \text{as} \quad m \to \infty,$$ \hspace{1cm} (2.2)

$$P_m |\Delta \lambda_m| = O(p_m |\lambda_m|) \quad \text{as} \quad m \to \infty,$$ \hspace{1cm} (2.3)

are satisfied, then the series $\sum a_n p_n \lambda_n$ is summable $[\hat{N}, p_n; \delta]_k$, $k \geq 1$ and $0 \leq \delta < 1/k$.

Lemma 2.2. [2] If the sequences $(\lambda_n)$ and $(p_n)$ satisfy the conditions (2.2) and (2.3) of Theorem 2.1, then we have

$$P_m |\lambda_m| = O(1) \quad \text{as} \quad m \to \infty.$$ \hspace{1cm} (2.5)

3. Main Result

The goal of the paper is to get a general theorem concerning absolute matrix summability. Now, we should give some notations. Let $A = (a_{mn})$ be a normal matrix, two lower semimatrices $\hat{A} = (\hat{a}_{mn})$ and $\hat{A} = (\hat{a}_{mn})$ are defined by:

$$\hat{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ...$$ \hspace{1cm} (3.1)

$$\hat{a}_{00} = \hat{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \hat{a}_{nv} - \hat{a}_{n-1,v}, \quad n = 1, 2, ...$$ \hspace{1cm} (3.2)

and

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \hat{a}_{mv} a_v$$ \hspace{1cm} (3.3)

$$\hat{A}_n(s) = \sum_{v=0}^{n} \hat{a}_{mv} a_v.$$ \hspace{1cm} (3.4)

Theorem 3.1. Let $A = (a_{mn})$ be a positive normal matrix such that

$$a_{n0} = 1, \quad n = 0, 1, ...,$$ \hspace{1cm} (3.5)

$$a_{n-1,v} \geq a_{mv}, \quad \text{for} \quad n \geq v + 1,$$ \hspace{1cm} (3.6)

$$a_{mn} = O \left( \frac{p_n}{p_m} \right).$$ \hspace{1cm} (3.7)

If the conditions (1.1), (2.1)-(2.3) and

$$\sum_{n=v+1}^{m+1} \left( \frac{p_n}{p_v} \right) \delta^{k-1} |\Delta \hat{a}_{nv}| = O \left( \left( \frac{p_v}{p_n} \right) \delta^{k-1} \right) \quad \text{as} \quad m \to \infty,$$ \hspace{1cm} (3.8)

$$\sum_{n=v+1}^{m+1} \left( \frac{p_n}{p_v} \right) \delta^{k} |\hat{a}_{n,v+1}| = O \left( \left( \frac{p_v}{p_n} \right) \delta^{k} \right) \quad \text{as} \quad m \to \infty,$$ \hspace{1cm} (3.9)

are satisfied, then the series $\sum a_n p_n \lambda_n$ is summable $[\hat{A}, p_n; \delta]_k$, $k \geq 1$ and $0 \leq \delta < 1/k$.

Proof of Theorem 3.1. Let $(M_n)$ be the sequence of $A$-transform of the series $\sum a_n p_n \lambda_n$. Then, by (3.3) and (3.4), we have

$$\hat{M}_n = \sum_{v=1}^{n} \hat{a}_{mv} p_v \lambda_v.$$
Operating Abel’s transformation for above sum, we get

\[
\tilde{\Delta} M_n = \sum_{v=1}^{n} \hat{a}_m a_v \lambda_v
\]

\[
= \sum_{v=1}^{n-1} \Delta_v (\hat{a}_m P_v \lambda_v) \sum_{r=1}^{n} a_r + \hat{a}_m P_n \lambda_n \sum_{v=1}^{n} a_v
\]

\[
= \sum_{v=1}^{n-1} \Delta_v (\hat{a}_m P_v \lambda_v) s_v + \hat{a}_m P_n \lambda_n s_n
\]

\[
= \alpha_m P_n \lambda_n s_n + \sum_{v=1}^{n-1} \Delta_v (\hat{a}_m P_v \lambda_v) s_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} P_v \bar{\Delta}_V s_v
\]

\[
- \sum_{v=1}^{n-1} \hat{a}_{n,v+1} P_v + 1 \lambda_v + 1 s_v
\]

\[
= M_{n,1} + M_{n,2} + M_{n,3} + M_{n,4}.
\]

To prove Theorem 3.1, it is sufficient to show that

\[
\sum_{n=1}^{\infty} \left( \frac{P_n}{P_m} \right) \delta^{k+1} |M_{n,r}|^k < \infty, \quad \text{for} \quad r = 1, 2, 3, 4.
\]

On account of (3.7), (2.5), (1.1), (2.3) and (2.2), we achieve

\[
\sum_{n=1}^{m+1} \left( \frac{P_n}{P_m} \right) \delta^{k+1} |M_{n,1}|^k = \sum_{n=1}^{m+1} \left( \frac{P_n}{P_m} \right) \delta^{k+1} a_m P_n |\lambda_m| |s_n|^k
\]

\[
= O(1) \sum_{n=1}^{m+1} \left( \frac{P_n}{P_m} \right) \delta^{k+1} (P_n |\lambda_m|)^{k-1} P_n |\lambda_m| |s_n|^k
\]

\[
= O(1) \sum_{n=1}^{m+1} \frac{P_n}{P_m} \delta^{k+1} |\lambda_m| |s_n|^k
\]

\[
= O(1) \sum_{n=1}^{m+1} P_n |\Delta \lambda_m| + O(1) |\lambda_m|
\]

\[
= O(1) \sum_{n=1}^{m+1} P_n |\lambda_m| + O(1) |\lambda_m|
\]

\[
= O(1) \quad \text{as} \quad m \to \infty.
\]

Now, operating Hölder’s inequality, we obtain

\[
\sum_{n=2}^{m+1} \left( \frac{P_n}{P_m} \right) \delta^{k+1} |M_{n,2}|^k \leq \sum_{n=2}^{m+1} \left( \frac{P_n}{P_m} \right) \delta^{k+1} \left( \sum_{r=1}^{n-1} \Delta_r (\hat{a}_m) P_r |\lambda_r| |s_r| \right)^k
\]

\[
\leq \sum_{n=2}^{m+1} \left( \frac{P_n}{P_m} \right) \delta^{k+1} \left( \sum_{r=1}^{n-1} \Delta_r (\hat{a}_m) P_r |\lambda_r|^k |s_r|^k \right) \left( \sum_{r=1}^{n-1} |\Delta_r (\hat{a}_m)| \right)^{k-1}.
\]

By virtue of (3.2) and (3.1), we have

\[
\Delta_r (\hat{a}_m) = \hat{a}_m - \hat{a}_{n,r+1} = \hat{a}_m - \hat{a}_{n-1,r} - \hat{a}_{n,r+1} + \hat{a}_{n-1,r+1} = a_m - a_{n-1,1}.
\]

Above equality implies that

\[
\sum_{v=1}^{n-1} |\Delta_v (\hat{a}_m)| = \sum_{v=1}^{n-1} (a_{n-1,1} - a_m) \leq a_m.
\]

by using (3.6), (3.1) and (3.5).
Also by (3.7), (3.8) and (2.5), we get
\[
\sum_{n=2}^{m+1} \left( \frac{P_n}{P_m} \right)^{\delta_{k+1}} |M_{n,2}|^k \leq \sum_{n=2}^{m+1} \left( \frac{P_n}{P_m} \right)^{\delta_{k+1}} \left( \sum_{i=1}^{n-1} |\Delta_i(\hat{a}_m)| \right)^k
\]
\[
= O(1) \sum_{i=1}^{m} \left( \frac{P_i}{P_m} \right)^{\delta_{k+1}} \left( \sum_{i=1}^{n-1} |\Delta_i(\hat{a}_m)| \right)^k.
\]

as in \( M_{n,1} \).

Now, we have
\[
\sum_{n=2}^{m+1} \left( \frac{P_n}{P_m} \right)^{\delta_{k+1}} |M_{n,3}|^k \leq \sum_{n=2}^{m+1} \left( \frac{P_n}{P_m} \right)^{\delta_{k+1}} \left( \sum_{i=1}^{n-1} |\Delta_i(\hat{a}_m)| \right)^k
\]
\[
\leq \sum_{n=2}^{m+1} \left( \frac{P_n}{P_m} \right)^{\delta_{k+1}} \left( \sum_{i=1}^{n-1} \left| \Delta_i(\hat{a}_m) \right|^k \right)^k.
\]

Here, the conditions (2.3) and (2.2) imply
\[
\sum_{n=2}^{m+1} \left( \frac{P_n}{P_m} \right)^{\delta_{k+1}} |M_{n,3}|^k = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{P_m} \right)^{\delta_{k+1}} \left( \sum_{i=1}^{n-1} \left| \Delta_i(\hat{a}_m) \right|^k \right)^k.
\]

By (3.2), (3.1), (3.5) and (3.6), it is obvious that
\[
\hat{a}_{n+1} = \hat{a}_{n+1} - \hat{a}_{n-1} + 1 = \sum_{i=0}^{n-1} a_{n-1,i} - \sum_{i=0}^{n-1} a_{n-1,i} - 1 \geq 0.
\]

So, we can write
\[
|\hat{a}_{n+1}| = \hat{a}_{n+1} - \hat{a}_{n-1} + 1 = a_{n+1} + \sum_{i=0}^{n-1} (a_{n-1,i} - a_{n-1,i}) \leq a_{n+1}
\]

(3.10)

by (3.2), (3.1) and (3.6). Hence, from (3.10), (3.7), (3.9) and (2.3), we get
\[
\sum_{n=2}^{m+1} \left( \frac{P_n}{P_m} \right)^{\delta_{k+1}} |M_{n,4}|^k \leq \sum_{n=2}^{m+1} \left( \frac{P_n}{P_m} \right)^{\delta_{k+1}} \left( \sum_{i=1}^{n-1} |\Delta_i(\hat{a}_m)| \right)^k
\]
\[
= O(1) \sum_{i=1}^{m} \left( \frac{P_i}{P_m} \right)^{\delta_{k+1}} \left( \sum_{i=1}^{n-1} |\Delta_i(\hat{a}_m)| \right)^k.
\]

as in \( M_{n,1} \).

Finally, using (2.1) and operating Hölder’s inequality, we obtain
\[
\sum_{n=2}^{m+1} \left( \frac{P_n}{P_m} \right)^{\delta_{k+1}} |M_{n,4}|^k \leq \sum_{n=2}^{m+1} \left( \frac{P_n}{P_m} \right)^{\delta_{k+1}} \left( \sum_{i=1}^{n-1} |\Delta_i(\hat{a}_m)| \right)^k
\]
\[
= O(1) \sum_{i=1}^{m} \left( \frac{P_i}{P_m} \right)^{\delta_{k+1}} \left( \sum_{i=1}^{n-1} |\Delta_i(\hat{a}_m)| \right)^k.
\]
Now, using (3.10) and (2.2), we get

\[ m + 1 \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right) \delta k + 1 - 1 |M_{n,k}|^k = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right) \delta k - 1 \left( \sum_{n=1}^{m-1} |d_{n+1}||s_{n+1}|^k p_n |a_{n+1}| \right). \]

Then, using (3.7) and (3.9), we get

\[ m + 1 \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right) \delta k + 1 - 1 |M_{n,k}|^k = O(1) \sum_{n=2}^{m+1} p_n |a_{n+1}| |s_{n+1}|^k \]

as in $M_{n,1}$. Therefore the proof of Theorem 3.1 is completed.

If we take $a_{m,v} = p_v / p_n$ in Theorem 3.1, we get Theorem 2.1. Additionally, if we take $\delta = 0$ in Theorem 3.1, then we deduce a known theorem on $|A, p_n|^k$ summability of infinite series (see [14]).

References