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On an Extension of Absolute Summability

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Abstract

In the present paper, a known theorem on absolute summability factors of infinite series has been generalized for $|A,p_n;\delta|_k$ summability by using matrix transformation.

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1. Introduction

Let $\sum a_n$ be an infinite series with its partial sums (s_n) and $A=(a_{nv})$ be a normal matrix; i.e., a lower triangular matrix of nonzero diagonal entries. The series $\sum a_n$ is said to be summable $|A,p_n;\delta|_k$, $k\geq 1$ and $\delta\geq 0$, if (see [8])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|A_n(s)-A_{n-1}(s)|^k<\infty,$$

where (p_n) is a sequence of positive numbers such that

$$P_n=\sum_{v=0}^{n}p_v\rightarrow\infty \quad as \quad n\rightarrow\infty, \quad (P_{-i}=p_{-i}=0, \quad i\geq 1)$$

and $As=(A_n(s))$ is defined by

$$A_n(s)=\sum_{v=0}^{n}a_{nv}s_v, \quad n=0,1,...$$

If we take $a_{nv}=\frac{p_v}{P_n}$, $|A,p_n;\delta|_k$ summability reduces to $|\bar{N},p_n;\delta|_k$ summability (see [4]). For $\delta=0$, $|A,p_n;\delta|_k$ summability reduces to $|A,p_n|_k$ summability (see [17]). Additionally, the series $\sum a_n$ is said to be bounded $[\bar{N},p_n;\delta]_k$, $k\geq 1$ and $\delta\geq 0$, if (see [3])

$$\sum_{v=1}^{n}\left(\frac{P_v}{p_v}\right)^{\delta k}p_v|s_v|^k=O(P_n) \quad as \quad n\rightarrow\infty. \tag{1.1}$$

It should be noted that, for $\delta=0$, $[\bar{N},p_n;\delta]_k$ boundedness is the same as $[\bar{N},p_n]_k$ boundedness (see [1]).

2. Known Results

Some works dealing with absolute summability and absolute matrix summability can be found in [1–3, 5–7, 9–13, 15, 16]. Among them, in [5], Bor has proved a theorem as follows.

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Theorem 2.1. *Let the series $\sum a_n$ be $[\bar{N}, p_n; \delta]_k$ bounded. If the conditions*

$$p_{n+1} = O(p_n) \quad as \quad n \to \infty, \tag{2.1}$$

$$\sum_{n=1}^{m} p_n |\lambda_n| = O(1) \quad as \quad m \to \infty, \tag{2.2}$$

$$P_m |\Delta \lambda_m| = O(p_m |\lambda_m|) \quad as \quad m \to \infty, \tag{2.3}$$

$$\sum_{n=v+1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} = O\left(\left(\frac{P_v}{p_v}\right)^{\delta k} \frac{1}{P_v}\right) \tag{2.4}$$

are satisfied, then the series $\sum a_n P_n \lambda_n$ is summable $|\bar{N}, p_n; \delta|_k$, $k \geq 1$ and $0 \leq \delta < 1/k$.

Lemma 2.2. *[2] If the sequences (λ_n) and (p_n) satisfy the conditions (2.2) and (2.3) of Theorem 2.1, then we have*

$$P_m |\lambda_m| = O(1) \quad as \quad m \to \infty. \tag{2.5}$$

3. Main Result

The goal of the paper is to get a general theorem concerning absolute matrix summability. Now, we should give some notations. Let $A = (a_{nv})$ be a normal matrix, two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ are defined by:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, \ldots \tag{3.1}$$

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \ldots \tag{3.2}$$

and

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{3.3}$$

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{3.4}$$

Theorem 3.1. *Let $A = (a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{n0} = 1, \; n = 0, 1, \ldots, \tag{3.5}$$

$$a_{n-1,v} \geq a_{nv}, \; for \; n \geq v+1, \tag{3.6}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right). \tag{3.7}$$

If the conditions (1.1), (2.1)-(2.3) and

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta_v \hat{a}_{nv}| = O\left(\left(\frac{P_v}{p_v}\right)^{\delta k-1}\right) \quad as \quad m \to \infty, \tag{3.8}$$

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\hat{a}_{n,v+1}| = O\left(\left(\frac{P_v}{p_v}\right)^{\delta k}\right) \quad as \quad m \to \infty \tag{3.9}$$

are satisfied, then the series $\sum a_n P_n \lambda_n$ is summable $|A, p_n; \delta|_k$, $k \geq 1$ and $0 \leq \delta < 1/k$.

Proof of Theorem 3.1. Let (M_n) be the sequence of A-transform of the series $\sum a_n P_n \lambda_n$. Then, by (3.3) and (3.4), we have

$$\bar{\Delta} M_n = \sum_{v=1}^{n} \hat{a}_{nv} a_v P_v \lambda_v.$$

Operating Abel's transformation for above sum, we get

$$\begin{aligned}
\bar{\Delta}M_n &= \sum_{v=1}^{n} \hat{a}_{nv} a_v P_v \lambda_v \\
&= \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv} P_v \lambda_v) \sum_{r=1}^{v} a_r + \hat{a}_{nn} P_n \lambda_n \sum_{v=1}^{n} a_v \\
&= \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv} P_v \lambda_v) s_v + \hat{a}_{nn} P_n \lambda_n s_n \\
&= a_{nn} P_n \lambda_n s_n + \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv}) P_v \lambda_v s_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} P_v \Delta\lambda_v s_v \\
&\quad - \sum_{v=1}^{n-1} \hat{a}_{n,v+1} p_{v+1} \lambda_{v+1} s_v \\
&= M_{n,1} + M_{n,2} + M_{n,3} + M_{n,4}.
\end{aligned}$$

To prove Theorem 3.1, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |M_{n,r}|^k < \infty, \quad for \quad r = 1,2,3,4.$$

On account of (3.7), (2.5), (1.1), (2.3) and (2.2), we achieve

$$\begin{aligned}
\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |M_{n,1}|^k &= \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} a_{nn}^k P_n^k |\lambda_n|^k |s_n|^k \\
&= O(1) \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} (P_n|\lambda_n|)^{k-1} p_n |\lambda_n| \, |s_n|^k \\
&= O(1) \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} p_n |\lambda_n| |s_n|^k \\
&= O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| \sum_{r=1}^{n} \left(\frac{P_r}{p_r}\right)^{\delta k} p_r \, |s_r|^k + O(1) \, |\lambda_m| \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} p_n \, |s_n|^k \\
&= O(1) \sum_{n=1}^{m-1} P_n \, |\Delta\lambda_n| + O(1) P_m \, |\lambda_m| \\
&= O(1) \sum_{n=1}^{m-1} p_n |\lambda_n| + O(1) P_m \, |\lambda_m| \\
&= O(1) \quad as \quad m \to \infty.
\end{aligned}$$

Now, operating Hölder's inequality, we obtain

$$\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |M_{n,2}|^k &\leq \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \, P_v \, |\lambda_v| \, |s_v|\right)^k \\
&\leq \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \, P_v^k \, |\lambda_v|^k \, |s_v|^k\right) \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})|\right)^{k-1}.
\end{aligned}$$

By virtue of (3.2) and (3.1), we have

$$\Delta_v(\hat{a}_{nv}) = \hat{a}_{nv} - \hat{a}_{n,v+1} = \bar{a}_{nv} - \bar{a}_{n-1,v} - \bar{a}_{n,v+1} + \bar{a}_{n-1,v+1} = a_{nv} - a_{n-1,v}.$$

Above equality implies that

$$\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| = \sum_{v=1}^{n-1} (a_{n-1,v} - a_{nv}) \leq a_{nn},$$

by using (3.6), (3.1) and (3.5).

Also by (3.7), (3.8) and (2.5), we get

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|M_{n,2}|^k &\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} a_{nn}^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|P_v^k|\lambda_v|^k|s_v|^k\right)\\ &= O(1)\sum_{v=1}^{m}P_v^k|\lambda_v|^k|s_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\Delta_v(\hat{a}_{nv})|\\ &= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}(P_v|\lambda_v|)^{k-1}p_v|\lambda_v||s_v|^k\\ &= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}p_v|\lambda_v||s_v|^k = O(1) \quad as \quad m\to\infty,\end{aligned}$$

as in $M_{n,1}$.
Now, we have

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|M_{n,3}|^k &\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|P_v|\Delta\lambda_v||s_v|\right)^k\\ &\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|^k|s_v|^kP_v|\Delta\lambda_v|\right)\left(\sum_{v=1}^{n-1}P_v|\Delta\lambda_v|\right)^{k-1}.\end{aligned}$$

Here, the conditions (2.3) and (2.2) imply

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|M_{n,3}|^k = O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|^{k-1}|\hat{a}_{n,v+1}||s_v|^kP_v|\Delta\lambda_v|\right).$$

By (3.2), (3.1), (3.5) and (3.6), it is obvious that

$$\begin{aligned}\hat{a}_{n,v+1}=\bar{a}_{n,v+1}-\bar{a}_{n-1,v+1} &= \sum_{i=v+1}^{n}a_{ni}-\sum_{i=v+1}^{n-1}a_{n-1,i}\\ &= \sum_{i=0}^{n}a_{ni}-\sum_{i=0}^{v}a_{ni}-\sum_{i=0}^{n-1}a_{n-1,i}+\sum_{i=0}^{v}a_{n-1,i}\\ &= 1-\sum_{i=0}^{v}a_{ni}-1+\sum_{i=0}^{v}a_{n-1,i}\\ &= \sum_{i=0}^{v}(a_{n-1,i}-a_{ni})\geq 0.\end{aligned}$$

So, we can write

$$\left|\hat{a}_{n,v+1}\right| = \bar{a}_{n,v+1}-\bar{a}_{n-1,v+1} = a_{nn}+\sum_{i=v+1}^{n-1}(a_{ni}-a_{n-1,i})\leq a_{nn} \tag{3.10}$$

by (3.2), (3.1) and (3.6). Thence, from (3.10), (3.7), (3.9) and (2.3), we get

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|M_{n,3}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}a_{nn}^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||s_v|^kP_v|\Delta\lambda_v|\right)\\ &= O(1)\sum_{v=1}^{m}|s_v|^kP_v|\Delta\lambda_v|\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,v+1}|\\ &= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}p_v|\lambda_v||s_v|^k = O(1) \quad as \quad m\to\infty,\end{aligned}$$

as in $M_{n,1}$.
Finally, using (2.1) and operating Hölder's inequality, we obtain

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|M_{n,4}|^k &\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}||s_v|p_{v+1}\right)^k\\ &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|^k|s_v|^kp_v|\lambda_v|\right)\left(\sum_{v=1}^{n-1}p_v|\lambda_v|\right)^{k-1}\\ &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|^{k-1}|\hat{a}_{n,v+1}||s_v|^kp_v|\lambda_v|\right)\left(\sum_{v=1}^{n-1}p_v|\lambda_v|\right)^{k-1}.\end{aligned}$$

Now, using (3.10) and (2.2), we get

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|M_{n,4}|^k = O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} a_{nn}^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||s_v|^k p_v|\lambda_v|\right).$$

Then, using (3.7) and (3.9), we get

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|M_{n,4}|^k &= O(1)\sum_{v=1}^{m} p_v|\lambda_v||s_v|^k \sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,v+1}| \\ &= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k} p_v|\lambda_v|\,|s_v|^k \\ &= O(1) \quad as \quad m\to\infty,\end{aligned}$$

as in $M_{n,1}$. Therefore the proof of Theorem 3.1 is completed. □

If we take $a_{nv} = p_v/P_n$ in Theorem 3.1, we get Theorem 2.1. Additionally, if we take $\delta = 0$ in Theorem 3.1, then we deduce a known theorem on $|A, p_n|_k$ summability of infinite series (see [14]).

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