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Inequalities of Hermite-Hadamard and Bullen Type for Three Times Differentiable AH-Convex Functions

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Abstract

In this paper, after introducing a new integral identity, by using this integral identity we obtain some new general inequalities of the Hermite-Hadamard and Bullen type for functions whose third derivatives in absolute value at certain power are arithmetically-harmonically convex. Some applications to special means of real numbers are also given.

Keywords: *Convex function, AH-convex function, Hermite-Hadamard and Bullen type inequalities.*
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1. Introduction

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)$$

valids for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \varnothing$. This definition is well known in the literature.
Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

holds.

The inequality (1.1) is known in the literature as Hermite-Hadamard integral inequality for convex functions. Moreover, it is known that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the function f. See [3, 5, 8, 9], for the results of the generalizations, improvements and extentions of the famous integral inequality (1.1).

Theorem 1.2. *Suppose that $f : [a,b] \to \mathbb{R}$ is a convex function on $[a,b]$. Then we have the following inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\ &\le \frac{1}{b-a}\int_a^b f(x)dx \\ &\le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \le \frac{f(a)+f(b)}{2} \end{aligned} \tag{1.2}$$

The third inequality in (1.2) is known in the literature as Bullen's inequality.

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Definition 1.3 ([2, 11]). *A function $f : I \subset \mathbb{R} \to (0,\infty)$ is said to be arithmetic-harmonically (AH) convex function if for all $x, y \in I$ and $t \in [0,1]$ the inequality*

$$f(tx+(1-t)y) \leq \frac{f(x)f(y)}{tf(y)+(1-t)f(x)} \tag{1.3}$$

holds. If the inequality (1.2) is reversed then the function $f(x)$ is said to be arithmetic-harmonically (AH) concave function.

Definition 1.4. *(Beta Function) The Beta function denoted by $\beta(a,b)$ is defined by*

$$\beta(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt, \;\; a,b>0.$$

Definition 1.5. *(Incomplete Beta Function) The incomplete beta function is defined by*

$$\beta_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1}dt$$

with $Rep > 0$, $Req > 0$, $0 \leq x \leq 1$.

Readers can find more informations on arithmetic-harmonically convex functions in [1, 2, 4, 6, 7, 11] and references therein.
In [10], Yetgin et al. used to the following lemma in order to establish some integral inequalities of Hermite-Hadamard type for s-convex functions.

Lemma 1.6. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I° such that $f'' \in L[a,b]$, where $a,b \in I^\circ$ with $a<b$, then the following identity holds:*

$$J_n(f,a,b) = \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3}\left[\int_0^1 t(1-t)f''\left(t\left(\tfrac{1+n-k}{n}a+\tfrac{k-1}{n}b\right)+(1-t)\left(\tfrac{n-k}{n}a+\tfrac{k}{n}b\right) \right)dt\right] \tag{1.4}$$

for all $n \in \mathbb{N}$, where

$$J_n(f,a,b) = \sum_{k=1}^{n}\frac{1}{2n}\left[f\left(a+\frac{(k-1)(b-a)}{n}\right)+f\left(a+\frac{k(b-a)}{n}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx.$$

Throughout this paper, for shortness we will use the following notations for special means of two nonnegative numbers a,b with $b>a$:
1. The arithmetic mean

$$A := A(a,b) = \frac{a+b}{2}, \;\; a,b>0,$$

2. The geometric mean

$$G := G(a,b) = \sqrt{ab}, \;\; a,b \geq 0$$

3. The harmonic mean

$$H := H(a,b) = \frac{2ab}{a+b}, \;\; a,b>0,$$

4. The logarithmic mean

$$L := L(a,b) = \begin{cases} \frac{b-a}{\ln b - \ln a}, & a \neq b \\ a, & a=b \end{cases} \; ; \; a,b>0$$

5. The p-logarithmic mean

$$L_p := L_p(a,b) = \begin{cases} \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, & a \neq b, p \in \mathbb{R}\backslash\{-1,0\} \\ a, & a=b \end{cases} \; ; \; a,b>0.$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:

$$H \leq G \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.In addition,

$$A_{n,k} = A_{n,k}(a,b) = \frac{1+n-k}{n}a+\frac{k-1}{n}b, n \in \mathbb{N}, k = 1,2,...,n.$$

2. Main results

In this study, using Hölder integral inequality and the following identity in order to provide inequality for functions whose third derivatives in absolute value at certain power are arithmetic-harmonically-convex functions. In order to establish some integral inequalities of Hermite-Hadamard type for arithmetic-harmonically convex functions, we will use the following lemma.

Lemma 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a three times differentiable mapping on I° such that $f''' \in L[a,b]$, where $a,b \in I^\circ$ with $a<b$, then the following identity holds:*

$$J_n(f,a,b) = \sum_{k=1}^{n} \frac{(b-a)^3}{12n^4}\left[\int_0^1 t^2(3-2t) f'''\left(tA_{n,k}+(1-t)A_{n,k+1}\right)dt\right]$$

for all $n \in \mathbb{N}$, where

$$J_n(f,a,b) = \sum_{k=1}^{n} \frac{1}{2n}\left[f\left(a+\frac{(k-1)(b-a)}{n}\right)+f\left(a+\frac{k(b-a)}{n}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx.$$

Proof. If we take partial integration of the right hand-side of the identity (1.4) in Lemma 1.6 as follows

$$\begin{aligned} u &= f''\left(t\left(\frac{1+n-k}{n}a+\frac{k-1}{n}b\right)+(1-t)\left(\frac{n-k}{n}a+\frac{k}{n}b\right)\right) \\ du &= f'''\left(t\left(\frac{1+n-k}{n}a+\frac{k-1}{n}b\right)+(1-t)\left(\frac{n-k}{n}a+\frac{k}{n}b\right)\right)\frac{a-b}{n}dt \\ dv &= t(1-t)dt \implies v=\frac{t^2}{2}-\frac{t^3}{3}, \end{aligned}$$

we get

$$J_n(f,a,b) = \sum_{k=1}^{n} \frac{(b-a)^3}{12n^4}\left[\int_0^1 t^2(3-2t) f'''\left(tA_{n,k}+(1-t)A_{n,k+1}\right)dt\right].$$

□

Theorem 2.2. *Let $f : I \subset (0,\infty) \to (0,\infty)$ be a three times differentiable mapping on I°, $n \in \mathbb{N}$ and $a,b \in I^\circ$ with $a<b$ such that $|f'''|^q \in L_1[a,b]$ is an arithmetic-harmonically convex function on the interval $[a,b]$ for some fixed $q>1$, then the following inequalities hold:*
i) If $\left|f'''(A_{n,k+1})\right|^q - \left|f'''(A_{n,k})\right|^q \neq 0$, then

$$|J_n(f,a,b)| \leq \sum_{k=1}^{n} \frac{(b-a)^3}{4n^4}\left(\frac{3}{2}\right)^{2+\frac{1}{p}} \beta_{\frac{2}{3}}^{\frac{1}{q}}(2p+1,p+1) \times \frac{G^2\left(\left|f'''(A_{n,k})\right|,\left|f'''(A_{n,k+1})\right|\right)}{L^{\frac{1}{q}}\left(\left|f'''(A_{n,k})\right|^q,\left|f'''(A_{n,k+1})\right|^q\right)}, \tag{2.1}$$

ii) If $\left|f'''(A_{n,k+1})\right|^q - \left|f'''(A_{n,k})\right|^q = 0$, then

$$|J_n(f,a,b)| \leq \sum_{k=1}^{n} \frac{(b-a)^3}{4n^4}\left(\frac{3}{2}\right)^{2+\frac{1}{p}} \beta_{\frac{2}{3}}^{\frac{1}{q}}(2p+1,p+1)\left|f'''(A_{n,k+1})\right|. \tag{2.2}$$

where $\beta_x(p,q)$ is the incomplete Beta function and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. i) Let $\left|f'''(A_{n,k+1})\right|^q - \left|f'''(A_{n,k})\right|^q \neq 0$. From the Lemma 2.1 and the properties of modulus, we write

$$|J_n(f,a,b)| \leq \sum_{k=1}^{n} \frac{(b-a)^3}{12n^4}\left[\int_0^1 t^2(3-2t) f'''\left(tA_{n,k}+(1-t)A_{n,k+1}\right)dt\right]. \tag{2.3}$$

Since $|f'''|^q$ is an AH-convex function on the interval $[a,b]$, the inequality

$$\left|f'''\left(tA_{n,k}+(1-t)A_{n,k+1}\right)\right|^q \leq \frac{\left|f'''(A_{n,k})\right|^q\left|f'''(A_{n,k+1})\right|^q}{t\left|f'''(A_{n,k+1})\right|^q+(1-t)\left|f'''(A_{n,k})\right|^q} \tag{2.4}$$

holds. By applying the Hölder integral inequality and the inequality (2.4) on (2.3), we get

$$\begin{aligned}
&|J_n(f,a,b)| \\
\leq\ & \sum_{k=1}^{n}\frac{(b-a)^3}{12n^4}\left(\int_0^1\left[t^2(3-2t)\right]^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'''\left(tA_{n,k}+(1-t)A_{n,k+1}\right)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \sum_{k=1}^{n}\frac{(b-a)^3}{12n^4}\left(\int_0^1 t^{2p}(3-2t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\frac{\left|f'''\left(A_{n,k}\right)\right|^q\left|f'''\left(A_{n,k+1}\right)\right|^q}{t\left|f'''\left(A_{n,k+1}\right)\right|^q+(1-t)\left|f'''\left(A_{n,k}\right)\right|^q}\right)^{\frac{1}{q}} \qquad (2.5)\\
=\ & \sum_{k=1}^{n}\frac{(b-a)^3}{12n^4}3\left(\int_0^1 t^{2p}\left(1-\frac{2t}{3}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\frac{\left|f'''\left(A_{n,k}\right)\right|^q\left|f'''\left(A_{n,k+1}\right)\right|^q}{t\left|f'''\left(A_{n,k+1}\right)\right|^q+(1-t)\left|f'''\left(A_{n,k}\right)\right|^q}\right)^{\frac{1}{q}} \\
=\ & \sum_{k=1}^{n}\frac{(b-a)^3}{12n^4}3\left(\frac{3}{2}\right)^{\frac{1}{p}}\left(\frac{3}{2}\right)^2\left(\int_0^{\frac{2}{3}} x^{2p}(1-x)^p dt\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1\frac{\left|f'''\left(A_{n,k}\right)\right|^q\left|f'''\left(A_{n,k+1}\right)\right|^q}{t\left|f'''\left(A_{n,k+1}\right)\right|^q+(1-t)\left|f'''\left(A_{n,k}\right)\right|^q}\right)^{\frac{1}{q}} \\
=\ & \sum_{k=1}^{n}\frac{(b-a)^3}{4n^4}\left(\frac{3}{2}\right)^{2+\frac{1}{p}}\beta_{\frac{2}{3}}^{\frac{1}{q}}(2p+1,p+1)\frac{G^2\left(\left|f'''\left(A_{n,k}\right)\right|,\left|f'''\left(A_{n,k+1}\right)\right|\right)}{L^{\frac{1}{q}}\left(\left|f'''\left(A_{n,k}\right)\right|^q,\left|f'''\left(A_{n,k+1}\right)\right|^q\right)}, \qquad (2.6)
\end{aligned}$$

where

$$\begin{aligned}
\int_0^1 t^{2p}\left(1-\frac{2t}{3}\right)^p dt &= 3\left(\frac{3}{2}\right)^{\frac{1}{p}}\left(\frac{3}{2}\right)^2\left(\int_0^{\frac{2}{3}} x^{2p}(1-x)^p dt\right)^{\frac{1}{p}} \\
\int_0^1\frac{dt}{t\left|f'''\left(A_{n,k+1}\right)\right|^q+(1-t)\left|f'''\left(A_{n,k}\right)\right|^q} &= L^{-1}\left(\left|f'''\left(A_{n,k}\right)\right|^q,\left|f'''\left(A_{n,k+1}\right)\right|^q\right).
\end{aligned}$$

ii) Let $\left|f'''\left(A_{n,k+1}\right)\right|^q-\left|f'''\left(A_{n,k}\right)\right|^q=0$. Then, substituting $\left|f'''\left(A_{n,k+1}\right)\right|^q=\left|f'''\left(A_{n,k}\right)\right|^q$ in the inequality (2.5), we have

$$|J_n(f,a,b)|\leq\sum_{k=1}^{n}\frac{(b-a)^3}{4n^4}\left(\frac{3}{2}\right)^{2+\frac{1}{p}}\beta_{\frac{2}{3}}^{\frac{1}{q}}(2p+1,p+1)\left|f'''\left(A_{n,k+1}\right)\right|.$$

This completes the proof of theorem. □

Corollary 2.3. *By choosing $n=1$ in Theorem 2.2, we obtain the following inequalities:*
i) If $\left|f'''\left(A_{1,k+1}\right)\right|-\left|f'''\left(A_{1,k}\right)\right|\neq 0$ for $k=1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)^3}{4}\left(\frac{3}{2}\right)^{2+\frac{1}{p}}\beta_{\frac{2}{3}}^{\frac{1}{q}}(2p+1,p+1)\frac{G^2\left|f'''(a)\right|,\left|f'''(b)\right|}{L^{\frac{1}{q}}\left(\left|f'''(a)\right|^q,\left|f'''(b)\right|^q\right)},$$

ii) If $\left|f'''\left(A_{1,k+1}\right)\right|-\left|f'''\left(A_{1,k}\right)\right|=0$ for $k=1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)^3}{4}\left(\frac{3}{2}\right)^{2+\frac{1}{p}}\beta_{\frac{2}{3}}^{\frac{1}{q}}(2p+1,p+1)\left|f'''(b)\right|.$$

Corollary 2.4. *By choosing $n=2$ in Theorem 2.2, we obtain the following Bullen type inequalities:*
i) If $\left|f'''\left(A_{2,k+1}\right)\right|-\left|f'''\left(A_{2,k}\right)\right|\neq 0$ for $k=1,2$, then

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{(b-a)^3}{64}\left(\frac{3}{2}\right)^{2+\frac{1}{p}}\beta_{\frac{2}{3}}^{\frac{1}{q}}(2p+1,p+1) \\
&\times\left[\frac{G^2\left(\left|f'''(a)\right|,\left|f'''\left(\frac{a+b}{2}\right)\right|\right)}{L^{\frac{1}{q}}\left(\left|f'''(a)\right|^q,\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right)}+\frac{G^2\left(\left|f'''\left(\frac{a+b}{2}\right)\right|,\left|f'''(b)\right|\right)}{L^{\frac{1}{q}}\left(\left|f'''\left(\frac{a+b}{2}\right)\right|,\left|f'''(b)\right|\right)}\right],
\end{aligned}$$

ii) If $\left|f'''\left(A_{2,k+1}\right)\right|-\left|f'''\left(A_{2,k}\right)\right|=0$ for $k=1,2$, then

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{(b-a)^3}{64}\left(\frac{3}{2}\right)^{2+\frac{1}{p}}\beta_{\frac{2}{3}}^{\frac{1}{q}}(2p+1,p+1)\left[\left|f'''\left(\frac{a+b}{2}\right)\right|+\left|f'''(b)\right|\right].
\end{aligned}$$

Theorem 2.5. *Let $f : I \subset (0,\infty) \to (0,\infty)$ be a three times differentiable mapping on I°, $n \in \mathbb{N}$ and $a,b \in I^{\circ}$ with $a < b$ such that $|f'''|^{q} \in L_1[a,b]$ is an AH-convex function on the interval $[a,b]$ for some fixed $q \geq 1$, then the following inequalities hold:*
i) If $\left|f'''\left(A_{n,k+1}\right)\right|^{q} - \left|f'''\left(A_{n,k}\right)\right|^{q} \neq 0$, then

$$\begin{aligned} |J_n(f,a,b)| &\leq \sum_{k=1}^{n} \frac{(b-a)^3}{12n^4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \frac{\left|f'''\left(A_{n,k}\right)\right|\left|f'''\left(A_{n,k+1}\right)\right|}{\left(\left|f'''\left(A_{n,k+1}\right)\right|^{q} - \left|f'''\left(A_{n,k}\right)\right|^{q}\right)^{\frac{3}{q}}} \\ &\quad \times \left[\frac{5}{3}A\left(\left|f'''\left(A_{n,k+1}\right)\right|^{2q}, \left|f'''\left(A_{n,k}\right)\right|^{2q}\right) - \frac{11}{3}G^2\left|f'''\left(A_{n,k}\right)\right|^{q}, \left|f'''\left(A_{n,k+1}\right)\right|^{q} \right. \\ &\quad \left. + \frac{\left(3\left|f'''\left(A_{n,k+1}\right)\right|^{q} - \left|f'''\left(A_{n,k}\right)\right|^{q}\right)\left|f'''\left(A_{n,k}\right)\right|^{2q}}{L\left(\left|f'''\left(A_{n,k}\right)\right|^{q}, \left|f'''\left(A_{n,k+1}\right)\right|^{q}\right)}\right]^{\frac{1}{q}}, \end{aligned} \tag{2.7}$$

ii) If $\left|f'''\left(A_{n,k+1}\right)\right|^{q} - \left|f'''\left(A_{n,k}\right)\right|^{q} = 0$, then

$$|J_n(f,a,b)| \leq \sum_{k=1}^{n} \frac{(b-a)^3}{24n^4}\left|f'''\left(A_{n,k+1}\right)\right|. \tag{2.8}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. i) Let $\left|f'''\left(A_{n,k+1}\right)\right|^{q} - \left|f'''\left(A_{n,k}\right)\right|^{q} \neq 0$. From the Lemma 2.1 and the properties of modulus,

$$|J_n(f,a,b)| \leq \sum_{k=1}^{n} \frac{(b-a)^3}{12n^4}\left[\int_0^1 t^2(3-2t) f'''\left(tA_{n,k} + (1-t)A_{n,k+1}\right) dt\right]. \tag{2.9}$$

Since $|f'''|^{q}$ is an AH-convex function on the interval $[a,b]$, the inequality

$$\left|f'''\left(tA_{n,k} + (1-t)A_{n,k+1}\right)\right|^{q} \leq \frac{\left|f'''\left(A_{n,k}\right)\right|^{q}\left|f'''\left(A_{n,k+1}\right)\right|^{q}}{t\left|f'''\left(A_{n,k+1}\right)\right|^{q} + (1-t)\left|f'''\left(A_{n,k}\right)\right|^{q}}$$

holds. If we use this inequality in (2.9) and the power-mean integral inequality, we get

$$\begin{aligned} &|J_n(f,a,b)| \\ \leq\ & \sum_{k=1}^{n} \frac{(b-a)^3}{12n^4}\left(\int_0^1 \left|t^2(3-2t)\right| dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|t^2(3-2t)\right|\left|f'''\left(tA_{n,k} + (1-t)A_{n,k+1}\right)\right|^{q} dt\right)^{\frac{1}{q}} \\ \leq\ & \sum_{k=1}^{n} \frac{(b-a)^3}{12n^4}\left(\int_0^1 t^2(3-2t)\, dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{t^2(3-2t)\left|f'''\left(A_{n,k}\right)\right|^{q}\left|f'''\left(A_{n,k+1}\right)\right|^{q}}{t\left|f'''\left(A_{n,k+1}\right)\right|^{q} + (1-t)\left|f'''\left(A_{n,k}\right)\right|^{q}} dt\right)^{\frac{1}{q}} \\ =\ & \sum_{k=1}^{n} \frac{(b-a)^3}{12n^4}\left|f'''\left(A_{n,k}\right)\right|\left|f'''\left(A_{n,k+1}\right)\right|\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{t^2(3-2t)\, dt}{t\left|f'''\left(A_{n,k+1}\right)\right|^{q} + (1-t)\left|f'''\left(A_{n,k}\right)\right|^{q}}\right)^{\frac{1}{q}} \\ =\ & \sum_{k=1}^{n} \frac{(b-a)^3}{12n^4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \frac{\left|f'''\left(A_{n,k}\right)\right|\left|f'''\left(A_{n,k+1}\right)\right|}{\left(\left|f'''\left(A_{n,k+1}\right)\right|^{q} - \left|f'''\left(A_{n,k}\right)\right|^{q}\right)^{\frac{3}{q}}} \\ & \times \left[\frac{5}{3}A\left(\left|f'''\left(A_{n,k+1}\right)\right|^{2q}, \left|f'''\left(A_{n,k}\right)\right|^{2q}\right) - \frac{11}{3}G^2\left|f'''\left(A_{n,k}\right)\right|^{q}, \left|f'''\left(A_{n,k+1}\right)\right|^{q}\right. \\ & \left. + \frac{\left(3\left|f'''\left(A_{n,k+1}\right)\right|^{q} - \left|f'''\left(A_{n,k}\right)\right|^{q}\right)\left|f'''\left(A_{n,k}\right)\right|^{2q}}{L\left(\left|f'''\left(A_{n,k}\right)\right|^{q}, \left|f'''\left(A_{n,k+1}\right)\right|^{q}\right)}\right]^{\frac{1}{q}}, \end{aligned} \tag{2.10}$$

where A is the arithmetic mean, G is the geometric mean and $\int_0^1 t^2(3-2t)\, dt = \frac{1}{2}$.
ii) Let $\left|f'''\left(A_{n,k+1}\right)\right|^{q} - \left|f'''\left(A_{n,k}\right)\right|^{q} = 0$. Then, substituting $\left|f'''\left(A_{n,k+1}\right)\right|^{q} = \left|f'''\left(A_{n,k}\right)\right|^{q}$ in the inequality (2.10), we obtain

$$\begin{aligned} |J_n(f,a,b)| &\leq \sum_{k=1}^{n} \frac{(b-a)^3}{12n^4}\left(\int_0^1 t^2(3-2t)\, dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2(3-2t)\left|f'''\left(A_{n,k+1}\right)\right|^{q} dt\right)^{\frac{1}{q}} \\ &= \sum_{k=1}^{n} \frac{(b-a)^3}{24n^4}\left|f'''\left(A_{n,k+1}\right)\right|. \end{aligned}$$

This completes the proof of theorem. ☐

Corollary 2.6. *By choosing $n = 1$ in Theorem 2.5, we obtain the following inequalities:*

i) If $\left|f'''\left(A_{n,k+1}\right)\right|^q-\left|f'''\left(A_{n,k}\right)\right|^q\neq 0$ *for* $k=1$, *then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ &\frac{(b-a)^3}{12}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\frac{G^2\left(\left|f'''(a)\right|,\left|f'''(b)\right|\right)}{\left(\left|f'''(b)\right|^q-\left|f'''(a)\right|^q\right)^{\frac{2}{q}}}\left[\frac{5}{3}A\left(\left|f'''(a)\right|^{2q},\left|f'''(b)\right|^{2q}\right)-\right.\\ &\left.\frac{11}{3}G^2\left|f'''(a)\right|^q,\left|f'''(b)\right|^q+\frac{\left(3\left|f'''(b)\right|^q-\left|f'''(a)\right|^q\right)\left|f'''(a)\right|^{2q}}{L\left(\left|f'''(a)\right|^q,\left|f'''(b)\right|^q\right)}\right]^{\frac{1}{q}},\end{aligned}$$

ii) If $\left|f'''\left(A_{n,k+1}\right)\right|^q-\left|f'''\left(A_{n,k}\right)\right|^q=0$ *for* $k=1$, *then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)^3}{24}\left|f'''(b)\right|.$$

Corollary 2.7. *By choosing* $n=2$ *in Theorem 2.5, we obtain the following Bullen type inequalities:*
i) If $\left|f'''\left(A_{n,k+1}\right)\right|^q-\left|f'''\left(A_{n,k}\right)\right|^q\neq 0$ *for* $k=1,2$, *then*

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ &\frac{(b-a)^3}{192}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\frac{G^2\left(\left|f'''(a)\right|,\left|f'''\left(\frac{a+b}{2}\right)\right|\right)}{\left(\left|f'''\left(\frac{a+b}{2}\right)\right|^q-\left|f'''(a)\right|^q\right)^{\frac{3}{q}}}\\ &\times\left[\frac{5}{3}A\left(\left|f'''\left(\frac{a+b}{2}\right)\right|^{2q},\left|f'''(a)\right|^{2q}\right)-\frac{11}{3}G^2\left(\left|f'''(a)\right|^q,\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right)\right.\\ &\left.+\frac{\left(3\left|f'''\left(\frac{a+b}{2}\right)\right|^q-\left|f'''(a)\right|^q\right)\left|f'''(a)\right|^{2q}}{L\left(\left|f'''(a)\right|^q,\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right)}\right]^{\frac{1}{q}}\\ &+\frac{(b-a)^3}{192}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\frac{G^2\left(\left|f'''(a)\right|,\left|f'''\left(\frac{a+b}{2}\right)\right|\right)}{\left(\left|f'''\left(\frac{a+b}{2}\right)\right|^q-\left|f'''(a)\right|^q\right)^{\frac{3}{q}}}\\ &\times\left[\frac{5}{3}A\left(\left|f'''\left(\frac{a+b}{2}\right)\right|^{2q},\left|f'''(b)\right|^{2q}\right)-\frac{11}{3}G^2\left(\left|f'''\left(\frac{a+b}{2}\right)\right|^q,\left|f'''(b)\right|^q\right)\right.\\ &\left.+\frac{\left(3\left|f'''(b)\right|^q-\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right)\left|f'''\left(\frac{a+b}{2}\right)\right|^{2q}}{L\left(\left|f'''\left(\frac{a+b}{2}\right)\right|^q,\left|f'''(b)\right|^q\right)}\right]^{\frac{1}{q}},\end{aligned}$$

ii) If $\left|f'''\left(A_{n,k+1}\right)\right|^q-\left|f'''\left(A_{n,k}\right)\right|^q=0$ *for* $k=1,2$, *then*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)^2}{384}\left[\left|f'''\left(\frac{a+b}{2}\right)\right|+\left|f'''(b)\right|\right].$$

Corollary 2.8. *If we take* $q=1$ *in the inequality (2.7), we get*

$$\begin{aligned}\left|J_n(f,a,b)\right|\ \leq\ &\sum_{k=1}^{n}\frac{(b-a)^3}{12n^4}\frac{\left|f'''\left(A_{n,k}\right)\right|\left|f'''\left(A_{n,k+1}\right)\right|}{\left(\left|f'''\left(A_{n,k+1}\right)\right|-\left|f'''\left(A_{n,k}\right)\right|\right)^3}\\ &\times\left[\frac{5}{3}A\left(\left|f'''\left(A_{n,k+1}\right)\right|^2,\left|f'''\left(A_{n,k}\right)\right|^2\right)-\frac{11}{2}G^2\left|f'''\left(A_{n,k}\right)\right|,\left|f'''\left(A_{n,k+1}\right)\right|\right.\\ &\left.+\frac{\left(3\left|f'''\left(A_{n,k+1}\right)\right|-\left|f'''\left(A_{n,k}\right)\right|\right)\left|f'''\left(A_{n,k}\right)\right|^2}{L\left(\left|f'''\left(A_{n,k}\right)\right|,\left|f'''\left(A_{n,k+1}\right)\right|\right)}\right].\end{aligned}$$

3. Some applications for special means

If $p\in(-1,0)$ then the function $f(x)=x^p, x>0$ is an arithmetic harmonically-convex [2]. Using this function we obtain following propositions:

Proposition 3.1. *Let* $a,b\in(0,\infty)$ *with* $a<b$, $q>1$ *and* $m\in(-1,0)$. *Then, we have the following inequality:*

$$\begin{aligned}&\prod_{s=1}^{3}\left(\frac{q^3}{m+sq}\right)\left|\sum_{k=1}^{n}\frac{1}{n}A\left(\left(A_{n,k}\right)^{\frac{m}{q}+3},\left(A_{n,k+1}\right)^{\frac{m}{q}+3}\right)-L_{\frac{m}{q}+3}^{\frac{m}{q}+3}(a,b)\right|\\ \leq\ &\sum_{k=1}^{n}\frac{(b-a)^3}{4n^4}\left(\frac{3}{2}\right)^{2+\frac{1}{p}}\beta_{\frac{2}{3}}^{\frac{1}{q}}(2p+1,p+1)\frac{G^{\frac{2m}{q}}\left(\left(A_{n,k}\right),\left(A_{n,k+1}\right)\right)}{L^{\frac{1}{q}}\left(\left(A_{n,k}\right)^m,\left(A_{n,k+1}\right)^m\right)}.\end{aligned}$$

Proof. The assertion follows from the inequality (2.1) in the Theorem 2.2. Let

$$f(x)=\prod_{s=1}^{3}\left(\frac{q^3}{m+sq}\right)x^{\frac{m}{q}+3},\quad x\in(0,\infty).$$

Then $|f'''(x)|^q=x^m$ is an arithmetic harmonically-convex on $(0,\infty)$ and the result follows directly from Theorem 2.2. □

Corollary 3.2. *If we take $n=1$ in Proposition 3.1, we get the following inequality:*

$$\prod_{s=1}^{3}\left(\frac{q^3}{m+sq}\right)\left|A\left(a^{\frac{m}{q}+3},b^{\frac{m}{q}+3}\right)-L_{\frac{m}{q}+3}^{\frac{m}{q}+3}(a,b)\right|\le\frac{(b-a)^3}{4}\left(\frac{3}{2}\right)^{2+\frac{1}{p}}\beta_{\frac{2}{3}}^{\frac{1}{q}}(2p+1,p+1)\frac{G^{\frac{2m}{q}}(a,b)}{L^{\frac{1}{q}}(a^m,b^m)}.$$

Proposition 3.3. *Let $a,b\in(0,\infty)$ with $a<b$, $q\ge1$ and $m\in(-1,0)$. Then, we have the following inequality:*

$$\begin{aligned}&\prod_{s=1}^{3}\left(\frac{q^3}{m+sq}\right)\left|\sum_{k=1}^{n}\frac{1}{n}A\left((A_{n,k})^{\frac{m}{q}+3},(A_{n,k+1})^{\frac{m}{q}+3}\right)-L_{\frac{m}{q}+3}^{\frac{m}{q}+3}(a,b)\right|\\
\le&\sum_{k=1}^{n}\frac{(b-a)^3}{12n^4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\frac{G^{\frac{2m}{q}}(a,b)}{(b^m-a^m)^{\frac{3}{q}}}\left[\frac{5}{3}A\left((A_{n,k+1})^{2m},(A_{n,k})^{2m}\right)\right.\\
&\left.-\frac{11}{3}G^{2m}\left((A_{n,k}),(A_{n,k+1})\right)+\frac{\left[3(A_{n,k+1})^m-(A_{n,k})^m\right](A_{n,k})^{2m}}{L\left((A_{n,k})^m,(A_{n,k+1})^m\right)}\right]^{\frac{1}{q}}.\end{aligned}$$

Proof. The assertion follows from the inequality (2.7) in the Theorem 2.5. Let

$$f(x)=\prod_{s=1}^{3}\left(\frac{q^3}{m+sq}\right)x^{\frac{m}{q}+3},\quad x\in(0,\infty).$$

Then $|f'''(x)|^q=x^m$ is an arithmetic harmonically-convex on $(0,\infty)$ and the result follows directly from Theorem 2.5. □

Corollary 3.4. *If we take $n=1$ in Proposition 3.3, we get the following inequality:*

$$\begin{aligned}\prod_{s=1}^{3}\left(\frac{q^3}{m+sq}\right)\left|A\left(a^{\frac{m}{q}+3},b^{\frac{m}{q}+3}\right)-L_{\frac{m}{q}+3}^{\frac{m}{q}+3}(a,b)\right|&\le\frac{(b-a)^3}{12}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\frac{G^{\frac{2m}{q}}(a,b)}{(b^m-a^m)^{\frac{3}{q}}}\left[\frac{5}{3}A\left(a^{2m},b^{2m}\right)\right.\\
&\left.-\frac{11}{3}G^{2m}(a,b)+\frac{[3b^m-a^m]a^{2m}}{L(a^m,b^m)}\right]^{\frac{1}{q}}.\end{aligned}$$

Corollary 3.5. *If we take $q=1$ in Corollary 3.4, we get the following inequality:*

$$\begin{aligned}&\prod_{s=1}^{3}\left(\frac{q^3}{m+sq}\right)\left|A\left(a^{m+3},b^{m+3}\right)-L_{m+3}^{m+3}(a,b)\right|\\
\le&\frac{(b-a)^3}{12}\frac{G^{2m}(a,b)}{(b^m-a^m)^3}\left[\frac{5}{3}A\left(a^{2m},b^{2m}\right)-\frac{11}{3}G^{2m}(a,b)+\frac{[3b^m-a^m]a^{2m}}{L(a^m,b^m)}\right].\end{aligned}$$

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