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A Note on Evolution of Quaternionic Curves in the Euclidean Space $\mathbb{R}^4$

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Abstract

In this paper, we investigate the evolution of quaternionic curve in Euclidean 4-space $\mathbb{R}^4$. We obtain evolution equations of the Frenet frame and curvatures. Then we give integrability conditions for the evolutions. Finally we give examples of evolution of curvatures.

Keywords: *Quaternionic curve; evolution; inextensible flow.*
2010 Mathematics Subject Classification: *4C44; 22E15.*

1. Introduction

In differential geometry, although there are many studies about curved spaces or shapes in regardless to time parameter, the recent researches have made a great improvement about the evolution of curved spaces with respect to time. Among them, an envolving curve has arisen in many engineering and physical applications [3, 4, 5, 7, 9, 13, 16]

An envolving curve can be considered as a family of curves parametrized by time. The time evolution of a curve generated by its corresponding flow so we shall also refer to curve evolutions as flows throughout this paper. Firstly, Kwon and Park studied inextensible flows of curves and developable surfaces in Euclidean 3-space [12]. Following them, inextensible flows of curves are studied in many different spaces [8, 10, 11, 15, 18, 19, 20, 21]. Another related work is that of [1], Abdel et al. brought a different approach to time evolution of a curve. They obtained the evolution equations of a generalized space curve. Then, Yıldız et al. studied evolution of generalized space curve in Minkowski Space [22].

Our aim is to study evolution of quaternionic curve in $\mathbb{R}^4$. We give necessary and sufficient conditions for inextensible flows of quaternionic curves in $\mathbb{R}^4$. We express evolution equation of the Frenet frame by matrix equation. Further, we obtain integrability conditions (zero curvature conditions) for the considered model.

2. Preliminaries

A brief summary of the theory of quaternions in the Euclidean space is presented in this section.

The space of quaternions Q is isomorphic to $\mathbb{R}^4$, four-dimensional vector space over the real numbers. There are three operations in Q : addition, scalar multiplication, and quaternion multiplication. The sum of two elements of Q is defined to be their sum as elements of $\mathbb{R}^4$. Similarly, the product of an element of Q by a real number is defined to be the same as the product in $\mathbb{R}^4$.

A real quaternion q is an expression of the form $q = ae_1 + be_2 + ce_3 + de_4$, where a, b, c and d are real numbers, and e_1, e_2, e_3 are quaternionic units which satisfy the non-commutative multiplication rules,

$$\begin{aligned} i)&e_i \times e_i = -e_4, \quad (e_4 = 1,\ 1 \leq i \leq 3) \\ ii)&e_i \times e_j = e_k = -e_j \times e_i, \quad (1 \leq i, j \leq 3), \end{aligned}$$

where (ijk) is an even permutation of (123) in the Euclidean space $\mathbb{R}^4$. A real quaternion can be written as a linear combination of scalar part $S_q = d$ and vectorial part $V_q = ae_1 + be_2 + ce_3$. Using these basic products, the product of two quaternions can be expanded as

$$p \times q = S_pS_q - < V_p, V_q > + S_pV_q + S_qV_p + V_q \wedge V_q$$

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for every $p,q \in Q$, where $<,>$ and $\wedge$ are inner product and cross product on E^3, respectively. The conjugate of the quaternion q is denoted by $\overline{q}$ and defined

$$\overline{q} = S_q - V_q = de_4 - ae_1 - be_2 - ce_3,$$

and is called by "Hamiltonian conjugation". The h-inner product of two quaternions is defined by

$$h(p,q) = \frac{1}{2}(p \times \overline{q} + q \times \overline{p}),$$

where h is the symmetric, non-degenerate, real valued and bilinear form. Thus the definition of the norm for every quaternion can be given. The norm of a real quaternion q is

$$\|q\|^2 = h(q,q) = a^2 + b^2 + c^2 + d^2.$$

Theorem 2.1. *The three-dimensional Euclidean space $\mathbb{R}^3$ is identifed with the space of spatial quaternion $\{\gamma \in Q \,|\, \gamma + \overline{\gamma} = 0\}$ in an obvious manner. Let $I = [0,1]$ be an interval in the real line R and $s \in I$ be the parameter along the smooth curve*

$$\gamma : I \subset R \longrightarrow Q, \quad \gamma(s) = \sum_{i=1}^{3} \gamma_i(s) e_i, \quad (1 \le i \le 3),$$

chosen such that the tangent $\gamma'(s) = t$ has unit length $\|t(s)\| = 1$ for all s. This unitarity condition implies;

$$t' \times \bar{t} + t \times \overline{t'} = 0.$$

The last equation implies that t' is orthogonal to t and $t' \times \bar{t}$ is a spatial quaternion. Let $\{t(s), n_1(s), n_2(s)\}$ be the Frenet trihedron in the point $\gamma(s)$ of the quaternionic curve γ. Then Frenet equations are

$$\begin{aligned} t'(s) &= k(s)n_1(s) \\ n_1'(s) &= -k(s)t(s) + r(s)n_2(s) \\ n_2'(s) &= -r(s)n_1(s) \end{aligned}$$

where t is the unit tangent, n_1 is the unit principal normal, n_2 is the unit binormal vector fields, k is the principal curvature and r is the torsion of the quaternionic curve γ, [2].

Theorem 2.2. *The four-dimensional Euclidean space $\mathbb{R}^4$ is identified with the space of unit quaternion. Let*

$$\beta : I \subset R \longrightarrow Q, \quad \beta(s) = \sum_{i=1}^{4} \gamma_i(s) e_i, \quad e_4 = 1$$

be a smooth curve (β) in $\mathbb{R}^4$ defined over the interval I. Let the parameter s be chosen such that the tangent $T = \beta'(s) = \sum_{i=1}^{4} \gamma_i'(s) e_i$ has unit magnitude. Let $\{T, N_1, N_2, N_3\}$ be the Frenet apparatus of the differentiable Euclidean space curve in the Euclidean space $\mathbb{R}^4$. Then the Frenet equations are

$$\begin{aligned} T'(s) &= \kappa N_1(s) \\ N_1'(s) &= -\kappa T(s) + k N_2(s) \\ N_2'(s) &= -k N_1(s) + (r-\kappa) N_3(s) \\ N_3'(s) &= -(r-\kappa) N_2, \end{aligned}$$

where $N_1 = t \times T$, $N_2 = n_1 \times T$, $N_3 = n_2 \times T$ and $K = \|T'(s)\|$, [2].

It is obtained the Frenet formulae in [2] and the apparatus for the curve β by making use of the Frenet formulae for a curve γ in $\mathbb{R}^3$. Moreover, there are relationships between curvatures of the curves β and γ. These relations can be explained that the torsion of β is the principal curvature of the curve γ. Also, the bitorsion of β is $(r-\kappa)$, where r is the torsion of γ and κ is the principal curvature of β. These relations are only determined for quaternions, [2]. Moreover, these formulas can be stated as the follows matrix

$$V_s = QV \tag{2.1}$$

where

$$V = [T, N_1, N_2, N_3]^t, \; Q = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & k & 0 \\ 0 & -k & 0 & (r-\kappa) \\ 0 & 0 & (r-\kappa) & 0 \end{bmatrix} \tag{2.2}$$

For further quaternions concepts see [2, 6, 10, 14, 17, 18].
Unless otherwise stated we assume that

$$\beta : [0,l] \times [0,w) \longrightarrow Q$$

is a one parameter family of smooth quaternionic curves in Q, where l is the arclength of the initial curve. Let u be the curve parametrization variable , $0 \leq u \leq l$ and t be the time parameter. If the metric on the quaternionic curve β is given by $v(u,t) = h\left(\frac{\partial\beta}{\partial u}, \frac{\partial\beta}{\partial u}\right)$, then the arclength variation of $\beta(u,t)$ is

$$s(u,t) = \int_0^u \left\| \frac{\partial\beta}{\partial u} \right\| du = \int_0^u \sqrt{v(u,t)}du.$$

Let s be an arclength parameter then,

$$\frac{\partial}{\partial s} = \frac{1}{\sqrt{v}}\frac{\partial}{\partial u}. \tag{2.3}$$

i.e., $ds = \sqrt{v}du.$

3. Evolution of Quaternionic Curves by Flow in $\mathbb{R}^4$

Let β be a differentiable quaternionic curve. Any flow of the quaternionic curve can be given by

$$\frac{\partial\beta}{\partial t} = f_1 T + f_2 N_1 + f_3 N_2 + f_4 N_3, \tag{3.1}$$

where, f_1, f_2, f_3 and f_4 are scalar speed of the quaternionic curve β.
In E^4,the requirement that a quaternionic curve not be subject to any elongation or compression can be expressed by the condition

$$\frac{\partial}{\partial t}s(u,t) = \int_0^u \frac{\partial\sqrt{v}}{\partial t}du = 0, \;\; u \in [0,l]. \tag{3.2}$$

where $u \in [0,l]$.

Definition 3.1. *Let $\beta(u,t)$ be a quaternionic curve in $\mathbb{R}^4$. The flow $\frac{\partial\beta}{\partial t}$ is said to be inextensible if*

$$\frac{\partial}{\partial t}\left\| \frac{\partial\beta}{\partial u} \right\| = 0.$$

Before deriving the necessary and sufficient condition for inextensible quaternionic curve flow, we need the following lemma.

Lemma 3.2. *Let $\frac{\partial\beta}{\partial t}$ be a smooth flow of β in E^4. Then, the evolution equation of v is*

$$\frac{\partial v}{\partial t} = v_t = 2v\left(\frac{\partial f_1}{\partial s} - f_2\kappa\right). \tag{3.3}$$

Proof. By the direct computation, we have

$$\begin{aligned}
\frac{\partial v}{\partial t} &= \frac{\partial}{\partial t}h\left(\frac{\partial\beta}{\partial u}, \frac{\partial\beta}{\partial u}\right) \\
&= 2h\left(\frac{\partial\beta}{\partial u}, \frac{\partial}{\partial u}\left(f_1 T + f_2 N_1 + f_3 N_2 + f_4 N_3\right)\right) \\
&= 2vh\left(T, \lambda T + \sum_{i=1}^{3} A_i N_i\right),
\end{aligned}$$

where

$$\begin{aligned}
\lambda &= \left(\frac{\partial f_1}{\partial s} - f_2\kappa\right), \\
A_1 &= f_1\kappa + \frac{\partial f_2}{\partial s} - f_3 k, \\
A_2 &= f_2 k + \frac{\partial f_3}{\partial s} - f_4(r-\kappa), \\
A_3 &= f_3(r-\kappa) + \frac{\partial f_4}{\partial s}.
\end{aligned} \tag{3.4}$$

Then,

$$\frac{\partial v}{\partial t} = 2v\lambda.$$

$\square$

Theorem 3.3. *The flow of a quaternionic curve is inextensible if and only if*

$$\frac{\partial f_1}{\partial s} = f_2\kappa. \tag{3.5}$$

Proof. Suppose that $\frac{\partial\beta}{\partial t}$ is inextensible. From equations (3.2) and (3.3) it follows that

$$\frac{\partial}{\partial t}s(u,t)=\int_0^u \frac{v_t}{2\sqrt{v}}du=0,\ \ u\in[0,l].$$

This clearly forces

$$\frac{\partial f_1}{\partial s}-f_2\kappa=0\Rightarrow\frac{\partial f_1}{\partial s}=f_2\kappa.$$

On the contrary, assume that $\frac{\partial f_1}{\partial s}=f_2\kappa$. By applying $\frac{\partial f_1}{\partial s}$ into (3.3), we get $v_t=0$ then $s_t=0$. This means that the flow is inextensbile. □

Theorem 3.4. *Let $\frac{\partial\beta}{\partial t}$ be a smooth flow of β, then, the following statements hold:*
i) Evolution of the elements of Frenet frame can be given by

$$V_t=MV$$

where

$$M=\begin{bmatrix}0 & A_1 & A_2 & A_3\\ -A_1 & 0 & B_2 & B_4\\ -A_2 & -B_2 & 0 & C_3\\ -A_3 & -B_3 & -C_3 & 0\end{bmatrix},\tag{3.6}$$

$$A_i=f_{i+1,s}+k_if_i-f_{i+2}k_{i+1};\ \ i=1,2,3.$$

$$B_j=\frac{1}{k_1}\left(A_{j,s}+k_jA_{j-1}-k_{j+1}A_{j+1}\right);\ \ j=2,3.$$

$$C_3=\frac{1}{k_2}\left(B_{3,s}+k_1A_3+k_3B_2\right)$$

$$k_1=\kappa,k_2=k,k_3=(r-\kappa),k_4=0.$$

ii)Evolution equations of the curvatures are

$$\begin{aligned}\kappa_t&=A_{1,s}-\lambda\kappa-kA_2\\ k_t&=B_{2,s}-k\lambda-(r-\kappa)B_3+\kappa A_2\\ (r-\kappa)_t&=C_{3,s}-\lambda(r-\kappa)+kB_3\end{aligned}\tag{3.7}$$

Proof. By taking derivative of (3.1) with respect to u, we get

$$\beta_{tu}=\sqrt{v}\beta_{ts}=\sqrt{v}\left(\lambda T+\sum_{i=1}^{3}A_iN_i\right).\tag{3.8}$$

Since $\beta_u=\sqrt{v}\beta_s=\sqrt{v}T$, by taking derivative of β_u with respect to t, we have

$$\beta_{ut}=\sqrt{v}\left(\frac{v_t}{2v}T+T_t\right)\tag{3.9}$$

By using (3.4), (3.8), (3.9) and $\beta_{tu}=\beta_{ut}$,we have

$$T_t=\sum_{i=2}^{3}A_iN_i\tag{3.10}$$

By taking derivative of (3.10) with respect to u, then we get

$$\begin{aligned}T_{tu}=\sqrt{v}\big(&(-\kappa A_1)T+\left(A_{1,s}-kA_2\right)N_1+\left(kA_1+A_{2,s}-(r-\kappa)A_3\right)N_2\\ &+\left((r-\kappa)A_2+A_{3,s}\right)N_3\big)\end{aligned}\tag{3.11}$$

and by taking derivative of T_u with respect to t, we have

$$T_{ut}=\sqrt{v}\left(\left(\frac{v_t}{2v}\kappa+\kappa_t\right)N_1+\kappa N_{1,t}\right).\tag{3.12}$$

By using (3.11), (3.12) and $T_{tu}=T_{ut}$, we get

$$\begin{aligned}\kappa_t&=A_{1,s}-\lambda\kappa-kA_2\\ N_{1,t}&=-A_1T+\sum_{i=2}^{3}B_iN_i\\ B_2&=\frac{1}{\kappa}\left(A_{2,s}+kA_1-(r-\kappa)A_3\right)\\ B_3&=\frac{1}{\kappa}\left(A_{3,s}+(r-\kappa)A_2\right)\end{aligned}\tag{3.13}$$

Next, by taking derivative of (3.13) with respect to u, then we get

$$N_{1,tu} = \sqrt{v}\left(\left(-A_{1,s}\right)T + \left(-\kappa A_1 - kB_2\right)N_1 + \left(B_{2,s} - (r-\kappa)B_3\right)N_2 \right. \\ \left. + \left((r-\kappa)B_2 + B_{3,s}\right)N_3 \tag{3.14}$$

and by taking derivative of $N_{1,u}$ with respect to t, we have

$$N_{1,ut} = \sqrt{v}\left(-(\lambda\kappa + \kappa_t)T - (\kappa A_1)N_1 \right. \\ + (\lambda k + k_t - \kappa A_2)N_2 \\ \left. + kN_{2,t} - \kappa A_3 N_3\right). \tag{3.15}$$

By using (3.14), (3.15) and $N_{1,tu} = N_{1,ut}$, we get

$$\begin{aligned} k_t &= B_{2,s} - \lambda k - (r-\kappa)B_3 + \kappa A_2 \\ N_{2,t} &= -A_2 T - B_2 N_1 + C_3 N_3 \\ C_3 &= \frac{1}{k}\left(\kappa A_3 + (r-\kappa)B_2 + B_{3,s}\right). \end{aligned} \tag{3.16}$$

Then, by taking derivative of $N_{2,t}$ with respect to u, we have

$$N_{2,tu} = \sqrt{v}\left(\left(-A_{2,s} + \kappa B_2\right)T - \left(\kappa A_2 + B_{2,s}\right)N_1 \right. \\ \left. - (kB_2 + (r-\kappa)C_3)N_2 + C_{3,s}N_3 \tag{3.17}$$

and by taking derivative of $N_{2,u}$ with respect to t, we have

$$N_{2,ut} = \sqrt{v}((\kappa A_1)T - (\lambda k + k_t)N_1 - kB_2 N_2 \\ + (\lambda(r-\kappa) - kB_3 + (r-\kappa)_t)N_3 + (r-\kappa)N_{3,t}. \tag{3.18}$$

Since $N_{2,tu} = N_{2,ut}$ and from (3.17), (3.18), we get

$$\begin{aligned} (r-\kappa)_t &= C_{3,s} - \lambda(r-\kappa) + kB_3, \\ N_{3,t} &= -A_3 T - B_3 N_1 - C_3 N_2. \end{aligned} \tag{3.19}$$

The obtained equations can be given in a matrix form as

$$V_t = M \cdot V$$

where

$$M = \begin{bmatrix} 0 & A_1 & A_2 & A_3 \\ -A_1 & 0 & B_2 & B_4 \\ -A_2 & -B_2 & 0 & C_3 \\ -A_3 & -B_3 & -C_3 & 0 \end{bmatrix}.$$

From first equation of (3.13), (3.16) and (3.19), we get

$$\begin{aligned} \kappa_t &= A_{1,s} - \lambda\kappa - kA_2 \\ k_t &= B_{2,s} - \lambda k - (r-\kappa)B_3 + \kappa A_2 \\ (r-\kappa)_t &= C_{3,s} - \lambda(r-\kappa) + kB_3 \end{aligned}$$

□

Corollary 1. *If the flow of $\beta(u,t)$ is inextensible, then evolution equations of curvatures (3.7) are*

$$\begin{aligned} \kappa_t &= A_{1,s} - kA_2 \\ k_t &= B_{2,s} - (r-\kappa)B_3 + \kappa A_2 \\ (r-\kappa)_t &= C_{3,s} + kB_3 \end{aligned}$$

Proof. If the flow of $\beta(u,t)$ be inextensible, then $v_t = 0$, moreover $\lambda = 0$. By applying $\lambda = 0$ into (3.7), the the lemma holds. □

Theorem 3.5. *The flow of $\beta(u,t)$ is inextensible if and only if the following condition (zero curvature condition) holds*

$$Q_t - M_s + [Q,M] = 0 \tag{3.20}$$

where $[Q,M] = QM - MQ$ is the Lie bracket.

Proof. In order to prove the theorem, there is need some calculations. Considering the equations (2.2) and (3.6). By taking derivative of $V_u = \sqrt{v}QV$ with respect to t, we obtain

$$V_{ut} = \sqrt{v}\left(\frac{v_t}{2v}Q + Q_t + QM\right)V \tag{3.21}$$

and taking derivative of V_t with respect to u, we have

$$V_{tu} = \sqrt{v}\left(M_s + MQ\right)V. \tag{3.22}$$

From (3.21) and (3.22), we obtain

$$V_{ut} - V_{tu} = \sqrt{v}\left(\frac{v_t}{2v}Q + Q_t - M_s + [Q,M]\right)V.$$

First, if the flow is an inextensible, then $v_t = 0$ and $\partial/\partial u$ and $\partial/\partial t$ are commutative, hence

$$Q_t - M_s + [Q,M] = 0.$$

Conversely, suppose the integrability condition is satisfied, i.e.,

$$Q_t - M_s + [Q,M] = 0.$$

From (2.2) and (3.6), we get

$$[Q,M] = \begin{bmatrix} 0 & kA_2 & A_{2,s} & A_{3,s} \\ -kA_2 & 0 & -\kappa A_2 + (r-\kappa)B_3 & B_{3,s} \\ -A_{2,s} & \kappa A_2 - (r-\kappa)B_3 & 0 & -kB_3 \\ -A_{3,s} & -B_{3,s} & kB_3 & 0 \end{bmatrix} \tag{3.23}$$

By taking derivative of Q with respect to t and derivative of M with respect to s and using (3.7), we get

$$Q_t - M_s = \begin{bmatrix} 0 & -\lambda\kappa - kA_2 & -A_{2,s} & -A_{3,s} \\ \lambda\kappa + kA_2 & 0 & -\lambda k + \kappa A_2 - (r-\kappa)B_3 & -B_{3,s} \\ A_{2,s} & \lambda k - \kappa A_2 + (r-\kappa)B_3 & 0 & -\lambda(r-\kappa) + kB_3 \\ A_{3,s} & B_{3,s} & \lambda(r-\kappa) - kB_3 & 0 \end{bmatrix} \tag{3.24}$$

. From the equation (3.23) and (3.24) we obtain

$$\begin{bmatrix} 0 & -\lambda\kappa & 0 & 0 \\ \lambda\kappa & 0 & -\lambda k & 0 \\ 0 & \lambda k & 0 & -\lambda(r-\kappa) \\ 0 & 0 & \lambda(r-\kappa) & 0 \end{bmatrix} = 0.$$

From last equation it can be seen that $\lambda = 0$ i.e., $v = constant$, so the flow is an inextensible. □

Corollary 2. *Let the curve flow be inextensible. If the Q and M are abelian, then*

$$A_2 = B_3 = 0.$$

Proof. Assume that Q and M are abelian, so $[Q,M] = 0$, then (3.20) as follows

$$M_s - Q_t = 0. \tag{3.25}$$

Since the flow is inextensible, then

$$M_s - Q_t = \begin{bmatrix} 0 & -kA_2 & -A_{2,s} & -A_{3,s} \\ kA_2 & 0 & -\kappa A_2 + (r-\kappa)B_3 & B_{3,s} \\ -A_{2,s} & \kappa A_2 - (r-\kappa)B_3 & 0 & kB_3 \\ -A_{3,s} & -B_{3,s} & -kB_3 & 0 \end{bmatrix}. \tag{3.26}$$

By using (3.25) and (3.26), we get

$$A_2 = B_3 = 0.$$

□

Example 3.6. *Let quaternionic curve $\beta(s)$ be*

$$\beta(s) = \left(\cos\sqrt{\frac{2}{3}}s, \sin\sqrt{\frac{2}{3}}s, \cos\sqrt{\frac{1}{3}}s, \sin\sqrt{\frac{1}{3}}s\right)$$

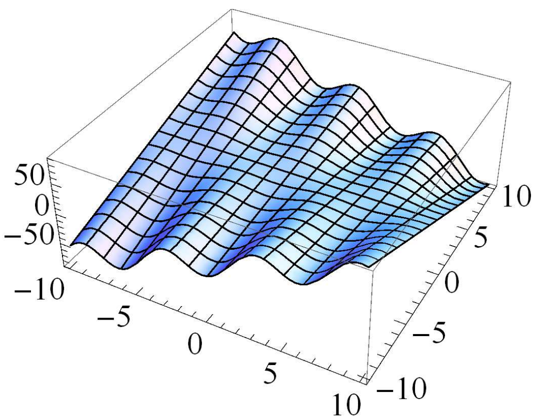
for all $s \in I$. Curvatures of are as follows:

$$\kappa = \frac{\sqrt{5}}{3}, k = \frac{\sqrt{2}}{3\sqrt{5}}, (r-\kappa) = \sqrt{\frac{2}{5}}.$$

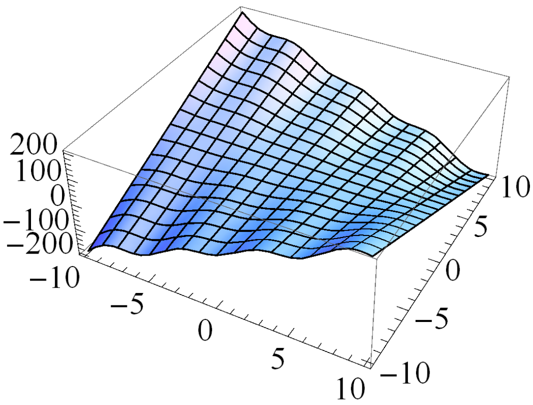
If $f_1 = s^2 \cos\left(s^2\right),\ f_2 = s \sin(s),\ f_3 = s^2, f_2 = s$, *then graphs of evolution of the curvatures in domain*

$$D : \begin{cases} -10 < u < 10 \\ -10 < t < 10 \end{cases}$$

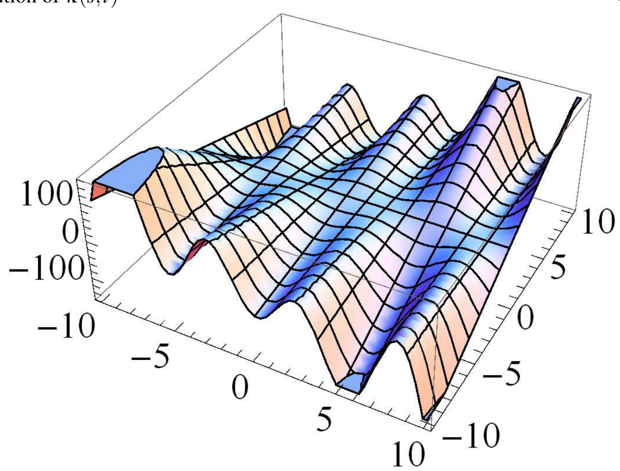
are given in figure 3.1.



(a) The evolution of $\kappa(s,t)$



(b) The evolution of $k(s,t)$



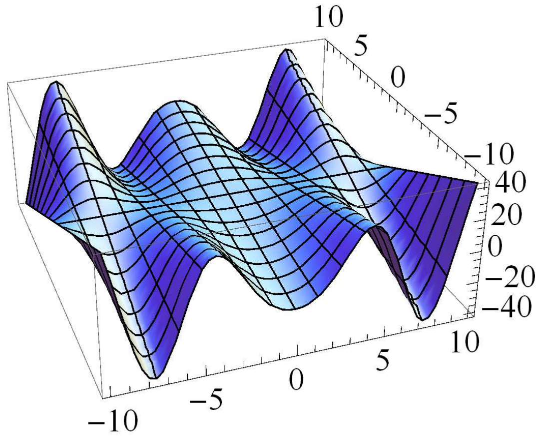
(c) The evolution of $(r-\kappa)(s,t)$

Figure 3.1

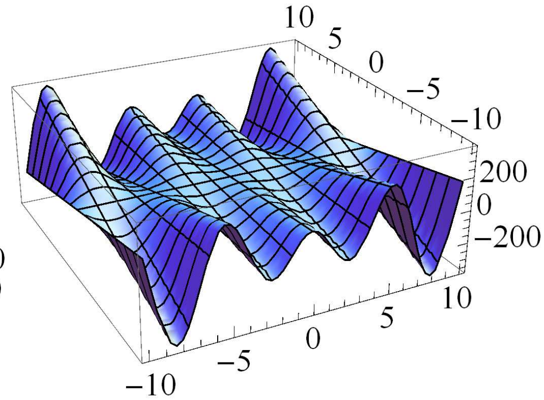
If $f_1 = s\cos(s),\ f_2 = s\sin(s),\ f_3 = s\cos(s),\ f_2 = s\sin(s)$, *then graphs of evolution of the curvatures in domain*

$$D : \begin{cases} -5 < u < 5 \\ -5 < t < 5 \end{cases}$$

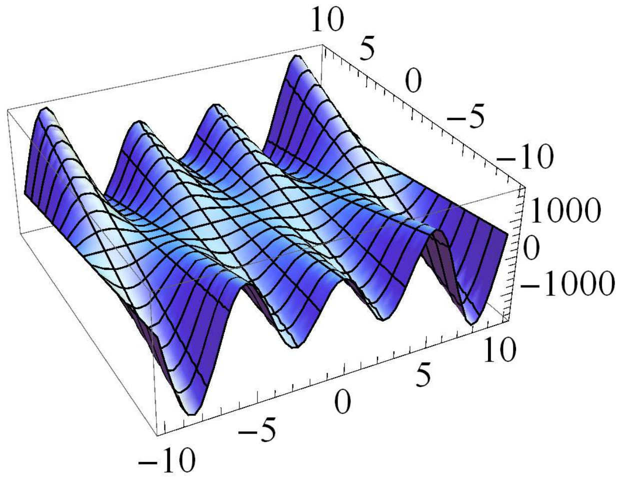
are given in figure 3.2



(a) The evolution of $\kappa(s,t)$



(b) The evolution of $k(s,t)$



(c) The evolution of $(r-\kappa)(s,t)$

Figure 3.2

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