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Monge hypersurfaces in euclidean 4-space with density

Yoğunluklu öklidyen 4-uzayında monge hiperyüzeyleri

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Yoğunluklu Öklidyen 4-Uzayında Monge Hiperyüzeyleri

Araştırma Makalesi / Research Article

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ÖZ

Bu çalışmada, ilk olarak 4-boyutlu Öklidyen uzayında bir Monge hiperyüzeyinin ortalama ve Gaussian eğriliklerini verdik. Ardından, farklı yoğunluklara sahip E^4 uzayında Monge hiperyüzeylerini çalıştık. Bu bağlamda, α, β, γ ve μ hepsi aynı anda sıfır olmayan sabitler olmak üzere, $e^{\alpha x+\beta y+\gamma z+\mu t}$ (lineer yoğunluk) ve $e^{\alpha x^2+\beta y^2+\gamma z^2+\mu t^2}$ yoğunluklu E^4 uzayında ağırlıklı minimal ve ağırlıklı flat Monge hiperyüzeylerini α, β, γ ve μ sabitlerinin farklı seçimleri yardımıyla elde ettik.

Anahtar Kelimeler:Yoğunluklu manifold, ağırlıklı ortalama eğrilik, ağırlıklı gaussian eğriliği, monge yüzeyleri.

Monge Hypersurfaces in Euclidean 4-Space with Density

ABSTRACT

In the present study, firstly we give the mean and Gaussian curvatures of a Monge hypersurface in 4-dimensional Euclidean space. After this, we study on Monge hypersurfaces in E^4 with different densities. In this context, we obtain the weighted minimal and weighted flat Monge hypersurfaces in E^4 with densities $e^{\alpha x+\beta y+\gamma z+\mu t}$ (linear density) and $e^{\alpha x^2+\beta y^2+\gamma z^2+\mu t^2}$ with the aid of different choices of constants α, β, γ and μ, where α, β, γ and μ are not all zero constants.

Keywords: Manifold with density, weighted mean curvature, weighted gaussian curvature, monge hypersurfaces

1. INTRODUCTION

Minimal and flat surfaces have long been an important topic of study by mathematicians and other scientists. When we focus on the studies on this subject, some of these studies can be given as follows: In the first two decades of 1900s, Moore has studied rotational surfaces and rotational surfaces with constant curvature in four-dimensional space and he has given some relations for them, [1,2]. Moor's studies have examined by Ganchev and Milousheva in Minkowski 4-space and some relations have been expressed, [7]. In [3], complete hypersurfaces in $\mathbb{R}^4$ with constant mean curvature and constant scalar curvature have been classified. In [5,6], authors have studied generalized rotational surfaces and translation surfaces in 4-dimensional Euclidean surfaces and they have investigated curvature properties of these surfaces and they have given some examples for them. Also authors have proved that, a translation surface is flat if and only if it is a hyperplane or a hypercylinder. Moruz and Mounteanu have considered hypersurfaces in $\mathbb{R}^4$ defined as the sum of a curve and a surface whose mean curvature vanishes in [8]. Yoon has investigated the rotational surfaces with finite type Gauss map in Euclidean 4-space. He has proved that, the Gauss map is of finite type if and only if rotatinal surface is a Clifford torus [4]. Dursun and Turgay have studied general rotational surfaces in E^4 whose meridian curves lie in two-dimensional planes and they have found all minimal general rotational surfaces by solving the differential equation that characterizes minimal general rotational surfaces. Also, they have determined all pseudo-umbilical general rotational surfaces in E^4, [9]. Kahraman and Yaylı have studied Bost invariant surfaces with pointwise 1-type Gauss map in E_1^4 and they have generalized rotational surfaces of pointwise 1-type Gauss map in E_2^4 [10,11]. Güler and et al have defined helicoidal hypersurface with the Laplace-Beltrami operator in four space, [12]. Also, Güler and et al have studied Gauss map and the third Laplace-Beltrami operator of the rotational hypersurface in 4-space, [13]. Since, the curvature of a curve and the mean curvature of an n-dimensional hypersurface are important invariants for curves and surfaces, many authors have studied these notions for different types of curves and surfaces for a long time in

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different spaces, such as Euclidean, Minkowski, Galilean and pseudo-Galilean spaces.

Now, let us recall some fundamental notions in Euclidean 4-space.

Let $\vec{x} = (x_1, y_1, z_1, t_1)$, $\vec{y} = (x_2, y_2, z_2, t_2)$ and $\vec{z} = (x_3, y_3, z_3, t_3)$ be three vectors in E^4. Then, the inner product and vector product of these vectors are given by

$$\langle \vec{x}, \vec{y} \rangle = x_1x_2 + y_1y_2 + z_1z_2 + t_1t_2 \tag{1.1}$$

and

$$\vec{x} \times \vec{y} \times \vec{z} = det\begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & y_1 & z_1 & t_1 \\ x_2 & y_2 & z_2 & t_2 \\ x_3 & y_3 & z_3 & t_3 \end{pmatrix}, \tag{1.2}$$

respectively. If

$$X: E^3 \longrightarrow E^4,\ (u_1, u_2, u_3) \longrightarrow X(u_1, u_2, u_3) \tag{1.3}$$
$$= (X_1(u_1, u_2, u_3), X_2(u_1, u_2, u_3), X_3(u_1, u_2, u_3), X_4(u_1, u_2, u_3))$$

is a hypersurface in Euclidean 4-space E^4, then the normal vector field, the matrix forms of the first and second fundamental forms are

$$N = \frac{X_{u_1} \times X_{u_2} \times X_{u_3}}{\|X_{u_1} \times X_{u_2} \times X_{u_3}\|}, \tag{1.4}$$

$$g_{ij} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \tag{1.5}$$

and

$$h_{ij} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}, \tag{1.6}$$

respectively. Here, $g_{ij} = \langle X_{u_i}, X_{u_j} \rangle$, $h_{ij} = \langle X_{u_iu_j}, N \rangle$, $X_{u_i} = \frac{\partial X}{\partial u_i}$, $X_{u_iuj} = \frac{\partial^2 X}{\partial u_iu_j}$, $\{i,j\} \in \{1,2,3\}$.

Also, the shape operator of the hypersurface (1.3) is

$$S = (a_{ij}) = (h_{ij}).(g_{ij})^{-1}, \tag{1.7}$$

where $(g_{ij})^{-1}$ is the inverse matrix of (g_{ij}).

With the aid of (1.5)-(1.7), the Gaussian curvature and mean curvature of a hypersurface in E^4 are given by

$$K = \frac{det(h_{ij})}{det(g_{ij})} \tag{1.8}$$

and

$$3H = iz(S), \tag{1.9}$$

respectively.

Furthermore, the notion of weighted manifold which is an important topic for geometers and physicists has been studied by many scientists, recently. Firstly, Gromov has introduced the notion of weighted mean curvature (or φ-mean curvature) of an n-dimensional hypersurface as

$$H_\varphi = H - \frac{1}{(n-1)}\frac{d\varphi}{dN}, \tag{1.10}$$

where H is the mean curvature and N is the unit normal vector field of the surface [14]. A hypersurface is called weighted minimal (or φ-minimal), if its weighted mean curvature vanishes.

Also, Corvin and et al have introduced the notion of generalized weighted Gaussian curvature on a manifold as

$$G_\phi = G - \triangle\varphi, \tag{1.11}$$

where $\triangle$ is the Laplacian operator [15]. A hypersurface is called weighted flat (or φ-flat), if its weighted Gaussian curvature vanishes.

After these definitions, lots of studies have been done by differential geometers about weighted manifolds, for instance [16-25].

2. MONGE HYPERSURFACES IN EUCLIDEAN 4-SPACE

In this section, we obtain the Gaussian and mean curvatures of a Monge hypersurface in Euclidean 4-space, by giving the normal vector field of it.

Let M be a surface in E^4 given by

$$M: X(x,y,z) = (x, y, z, f(x,y,z)). \tag{2.1}$$

Then we call this surface as Monge hypersurface in Euclidean 4-space. For this surface, we have

$$X_x = (1,0,0,f_x),\ X_y = (0,1,0,f_y),\ X_z = (0,0,1,f_z)$$
$$X_{xx} = (0,0,0,f_{xx}),\ X_{xy} = (0,0,0,f_{xy}), \tag{2.2}$$
$$X_{xz} = (0,0,0,f_{xz}), X_{yy} = (0,0,0,f_{yy}),$$
$$X_{yz} = (0,0,0,f_{yz}),\ X_{zz} = (0,0,0,f_{zz}),$$

where $X_i = \frac{\partial X}{\partial i}$, $X_{ij} = \frac{\partial X}{\partial ij}$, $f_i = \frac{\partial f}{\partial i}$, $f_{ij} = \frac{\partial f}{\partial ij}$, $\{i,j\} \in \{x,y\}$. From (2.2),

$$X_x \times X_y \times X_z = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ 1 & 0 & 0 & f_x \\ 0 & 1 & 0 & f_y \\ 0 & 0 & 1 & f_z \end{vmatrix} = (f_x, f_y, f_z, -1) \tag{2.3}$$

and so, from (1.2) and (1.4) the normal vector field of the surface (2.1) is obtained as

$$N = \frac{(f_x, f_y, f_z, -1)}{\sqrt{1 + f_x^2 + f_y^2 + f_z^2}}. \tag{2.4}$$

Also from (1.6), the matrix form of the second fundamental form of the surface (2.1) is

$$(hij) = \frac{1}{\sqrt{1 + f_x^2 + f_y^2 + f_z^2}}\begin{bmatrix} -f_{xx} & -f_{xy} & -f_{xz} \\ -f_{xy} & -f_{yy} & -f_{yz} \\ -f_{xz} & -f_{yz} & -f_{zz} \end{bmatrix} \tag{2.5}$$

and its determinant is

$$det(hij) = \frac{-f_{xx}f_{yy}f_{zz} - 2f_{xy}f_{yz}f_{xz} + f_{xx}f_{yz}^2 + f_{yy}f_{xz}^2 + f_{zz}f_{xy}^2}{(1 + f_x^2 + f_y^2 + f_z^2)^{3/2}}. \tag{2.6}$$

Now, we obtain the matrix of the metric g_{ij}, its determinant and inverse as

$$g_{ij} = \begin{bmatrix} 1 + f_x^2 & f_xf_y & f_xf_z \\ f_xf_y & 1 + f_y^2 & f_yf_z \\ f_xf_z & f_zf_y & 1 + f_z^2 \end{bmatrix}, \tag{2.7}$$

$$det(gij) = (1 + f_x^2)\left[(1 + f_y^2)(1 + f_z^2) - (f_yf_z)^2\right] - f_xf_y\left[f_xf_y(1 + f_z^2) - f_xf_yf_z^2\right]$$

$$+f_x f_z\left[f_x f_z f_y^2-\left(1+f_y^2\right)f_x f_z\right]$$
$$=1+f_x^2+f_y^2+f_z^2 \quad (2.8)$$

and

$$(g_{ij})^{-1}=\frac{1}{1+f_x^2+f_y^2+f_z^2}\begin{bmatrix}1+f_y^2+f_z^2 & -f_x f_y & -f_x f_z\\ -f_x f_y & 1+f_x^2+f_z^2 & -f_y f_z\\ -f_x f_z & -f_z f_y & 1+f_x^2+f_y^2\end{bmatrix}, \quad (2.9)$$

respectively. Hence, using (2.6) and (2.8) in (1.8), we obtain the Gaussian curvature of the surface (2.1) as

$$K=\frac{-f_{xx}f_{zz}f_{yy}-2(f_{xy}f_{yz}f_{xz})+f_{xx}(f_{yz})^2+f_{yy}(f_{xz})^2+f_{zz}(f_{xy})^2}{(1+f_x^2+f_y^2+f_z^2)^{5/2}}. \quad (2.10)$$

Let we take $(a_{ij})=(hij)\times(g_{ij})^{-1}$. Then, since

$$(\alpha_{ij})= \quad (2.11)$$
$$\frac{1}{(1+f_x^2+f_y^2+f_z^2)^{3/2}}\begin{bmatrix}-f_{xx} & -f_{xy} & -f_{xz}\\ -f_{xy} & -f_{yy} & -f_{yz}\\ -f_{xz} & -f_{yz} & -f_{zz}\end{bmatrix}.\begin{bmatrix}1+f_y^2+f_z^2 & -f_x f_y & -f_x f_z\\ -f_x f_y & 1+f_x^2+f_z^2 & -f_y f_z\\ -f_x f_z & -f_z f_y & 1+f_x^2+f_y^2\end{bmatrix}$$

from (1.9), we obtain the mean curvature of the surface (2.1) as

$$H=\frac{\left\{\begin{matrix}-f_{xx}(1+f_y^2+f_z^2)-f_{yy}(1+f_x^2+f_z^2)-f_{zz}(1+f_x^2+f_y^2)\\ +2(f_{xy}f_x f_y+f_{xz}f_x f_z+f_{yz}f_y f_z)\end{matrix}\right\}}{3(1+f_x^2+f_y^2+f_z)^{3/2}}. \quad (2.12)$$

3. MONGE HYPERSURFACES IN E^4 WITH LINEAR DENSITY

In the first subsection of this section, we investigate the weighted minimal Monge hypersurfaces in Euclidean 4-space with linear density $e^{\alpha x+\beta y+\gamma z+\mu t}$ and in the second subsection of this section, we investigate the weighted flat Monge hypersurfaces in E^4 with this density.

3.1. Weighted Minimal Monge Hypersurfaces in E^4 with Linear Density

Let M be a Monge hypersurface given by (2.1) in Euclidean 4-space with linear density $e^{\alpha x+\beta y+\gamma z+\mu t}$, where α,β,γ and μ are not all zero constants. Then from (1.10), the weighted mean curvature of this surface is obtained as

$$H_\varphi=\frac{\left\{\begin{matrix}-f_{xx}(1+f_y^2+f_z^2)-f_{yy}(1+f_x^2+f_z^2)\\ -f_{zz}(1+f_x^2+f_y^2)+\\ 2(f_{xy}f_x f_y+f_{xz}f_x f_z+f_{yz}f_y f_z)-\\ (\alpha f_x+\beta f_y+\gamma f_z-\mu)(1+f_x^2+f_y^2+f_z^2)\end{matrix}\right\}}{3(1+f_x^2+f_y^2+f_z^2)^{3/2}}. \quad (3.1)$$

So, we have

Proposition 1. *Let $M: X(x,y,z)=(x,y,z,f(x,y,z))$ be a Monge hypersurface in Euclidean 4-space with linear density $e^{\alpha x+\beta y+\gamma z+\mu t}$, where α,β,γ and μ are not all zero constants. Then, this surface is weighted minimal if and only if*

$$2(f_{xy}f_x f_y+f_{xz}f_x f_z+f_{yz}f_y f_z)=$$
$$f_{xx}(1+f_y^2+f_z^2)+f_{yy}(1+f_x^2+f_z^2)+f_{zz}(1+f_x^2+f_y^2)$$
$$+(\alpha f_x+\beta f_y+\gamma f_z-\mu)(1+f_x^2+f_y^2+f_z^2) \quad (3.2)$$

satisfies.

Now, let we take

$$f(x,y,z)=h(x)+g(y)+m(z),$$

where h,g and m are C^2 −differentiable functions. Thus, we have

$$f_x=h'(x),\ f_y=g'(y),\ f_z=m'(z),$$
$$f_{xx}=h''(x),\ f_{xy}=0,\ f_{xz}=0, \quad (3.3)$$
$$f_{yy}=g''(y),\ f_{yz}=0,\ f_{zz}=m''(z).$$

Using (3.3) in (3.1), the weighted mean curvature of the surface (2.1) is obtained as

$$H_\varphi=\frac{\left\{\begin{matrix}-h''(x)(1+g'(y)^2+m'(z)^2)\\ -g''(y)(1+h'(x)^2+m'(z)^2)\\ -m''(z)(1+h'(x)^2+g'(y)^2)-\\ (\alpha h'(x)+\beta g'(y)+\gamma m'(z)-\mu)(1+h'(x)^2+g'(y)^2+m'(z)^2)\end{matrix}\right\}}{3(1+h'(x)^2+g'(y)^2+m'(z)^2)^{3/2}}. \quad (3.4)$$

Proposition 2. *Let $M: X(x,y,z)=(x,y,z,h(x)+g(y)+m(z))$ be a Monge hypersurface in Euclidean 4-space with linear density $e^{\alpha x+\beta y+\gamma z+\mu t}$, where α,β,γ and μ are not all zero constants. Then, this surface is weighted minimal if and only if*

$$0=h''(x)(1+g'(y)^2+m'(z)^2)+$$
$$g''(y)(1+h'(x)^2+m'(z)^2)+ \quad (3.5)$$
$$m''(z)(1+h'(x)^2+g'(y)^2)+$$
$$(\alpha h'(x)+\beta g'(y)+\gamma m'(z)-\mu)(1+h'(x)^2+g'(y)^2+m'(z)^2) \quad (3.5)$$

satisfies.

Next, we'll obtain the weighted minimal Monge hypersurfaces in E^4 with density $e^{\alpha x+\beta y+\gamma z+\mu t}$ for different choices of the not all zero constants α,β,γ and μ.

We note that, throughout this study we consider k_i and λ_i, $i\in\mathbb{N}^+$, are real constants.

Case 1. Let the density be $e^{\alpha x}$:

In this case, let us consider the Monge hypersurface

$$M: X(x,y,z)=(x,y,z,h(x)+g(y)+m(z))$$

in Euclidean 4-space with linear density $e^{\alpha x}$. Then, this surface is weighted minimal if and only if

$$0=h''(x)(1+g'(y)^2+m'(z)^2)+$$
$$g''(y)(1+h'(x)^2+m'(z)^2)+ \quad (3.6)$$
$$m''(z)(1+h'(x)^2+g'(y)^2)+$$
$$\alpha h'(x)(1+h'(x)^2+g'(y)^2+m'(z)^2)$$

satisfies. Here, by obtaining some special solutions for the equation (3.6), we'll construct the weighted minimal Monge hypersurfaces in E^4 with linear density $e^{\alpha x}$.

Firstly, let us take the functions $g(y)$ and $m(z)$ are linear, i.e. $g(y)=k_1y+k_2, m(z)=k_3z+k_4$.

Then, the equation (3.6) becomes

$$h''(x)(1+(k_1)^2+(k_3)^2)=$$
$$-\alpha h'(x)(1+(h')^2+(k_1)^2+(k_3)^2). \quad (3.7)$$

From (3.7),

$$\frac{h''(1+(k_1)^2+(k_3)^2)}{\alpha h'(1+(h')^2+(k_1)^2+(k_3)^2)}=-1$$

$$\Rightarrow \frac{h''}{h'} - \frac{h''h'}{(h')^2 + 1 + (k_1)^2 + (k_3)^2} = -\alpha$$

$$\Rightarrow \left(ln|h'| - \frac{1}{2} ln|(h')^2 + 1 + (k_1)^2 + (k_3)^2|\right)' = -\alpha$$

$$\Rightarrow ln\left|\frac{h'}{\sqrt{(h')^2 + 1 + (k_1)^2 + (k_3)^2}}\right| = -\alpha x + \lambda_1$$

$$\Rightarrow \frac{h'}{\sqrt{(h')^2 + 1 + (k_1)^2 + (k_3)^2}} = e^{-\alpha x + \lambda_1}$$

$$\Rightarrow h' = e^{-\alpha x + \lambda_1}\sqrt{(h')^2 + 1 + (k_1)^2 + (k_3)^2}$$

$$\Rightarrow (h')^2 = e^{-2\alpha x + 2\lambda_1}((h')^2 + 1 + (k_1)^2 + (k_3)^2)$$

$$\Rightarrow \left(1 - e^{-2\alpha x + 2\lambda_1}\right)(h')^2 = e^{-2\alpha x + 2\lambda_1}(1 + (k_1)^2 + (k_3)^2)$$

$$\Rightarrow h' = \frac{\sqrt{1 + (k_1)^2 + (k_3)^2}.\, e^{-\alpha x + \lambda_1}}{\sqrt{1 - (e^{-\alpha x + \lambda_1})^2}}$$

$$\Rightarrow h(x) = \frac{\sqrt{(1 + (k_1)^2 + (k_3)^2)} arctan\left(e^{-\lambda_1}\sqrt{e^{2\alpha x} - e^{2\lambda_1}}\right)}{\alpha} + \lambda_2. \quad (3.8)$$

Thus,

$$f(x,y,z) = \frac{\sqrt{(1+(k_1)^2+(k_3)^2)} arctan\left(e^{-\lambda_1}\sqrt{e^{2\alpha x} - e^{2\lambda_1}}\right)}{\alpha} + k_1 y + k_3 z + k_2 + k_4 + \lambda_2$$

is a solution of (3.7).

So, we can give the following Theorem:

Theorem 1. *The weighted minimal Monge hypersurface in Euclidean 4-space with linear density* $e^{\alpha x}$ *for* $(\alpha \neq 0) \in \mathbb{R}$ *can be parametrized by*

$$X(x,y,z) = (x, y, z, k_1 y + k_3 z + k + \frac{\sqrt{(1+(k_1)^2+(k_3)^2)} arctan\left(e^{-\lambda_1}\sqrt{e^{2\alpha x} - e^{2\lambda_1}}\right)}{\alpha}), \quad (3.9)$$

where $k = k_2 + k_4 + \lambda_2$.

Secondly, taking the functions $h(x)$ and $m(z)$ are linear, i.e. $h(x) = k_5 x + k_6, m(z) = k_3 z + k_4$, from (3.6), we have

$$g''(y)(1 + (k_5)^2 + (k_3)^2) = -\alpha k_5 (1 + (g'(y))^2 + (k_5)^2 + (k_3)^2). \quad (3.10)$$

Solving this equation, we reach that

$$g'' = -\alpha k_5 \left(1 + \frac{(g')^2}{1 + (k_5)^2 + (k_3)^2}\right)$$

$$\Rightarrow \frac{g''}{1 + \frac{(g')^2}{1+(k_5)^2+(k_3)^2}} = -\alpha k_5$$

$$\Rightarrow \frac{\frac{g''}{\sqrt{1+(k_5)^2+(k_3)^2}}}{1 + \frac{(g')^2}{1+(k_5)^2+(k_3)^2}} = \frac{-\alpha k_5}{\sqrt{1 + (k_5)^2 + (k_3)^2}}$$

$$\Rightarrow arctan\left(\frac{g'}{\sqrt{1 + (k_5)^2 + (k_3)^2}}\right) = \frac{-\alpha k_5 y}{\sqrt{1+(k_5)^2+(k_3)^2}} + \lambda_3$$

$$\Rightarrow g' = \sqrt{1 + (k_5)^2 + (k_3)^2} tan\left(\frac{-\alpha k_5 y}{\sqrt{1 + (k_5)^2 + (k_3)^2}} + \lambda_3\right)$$

$$\Rightarrow g(y) = \frac{ln(cos(\frac{\alpha k_5 y}{\sqrt{1+(k_5)^2+(k_3)^2}} + \lambda_3))(1 + (k_5)^2 + (k_3)^2)}{\alpha k_5} + \lambda_4. \quad (3.11)$$

Hence, we have

Theorem 2. *The weighted minimal Monge hypersurface in Euclidean 4-space with linear density* $e^{\alpha x}$ *for* $(\alpha \neq 0) \in \mathbb{R}$ *can be parametrized by*

$$X(x,y,z) = (x, y, z, k_5 x + k_3 z + k + \frac{ln(cos(\frac{\alpha k_5 y}{\sqrt{1+(k_5)^2+(k_3)^2}} + \lambda_3))(1 + (k_5)^2 + (k_3)^2)}{\alpha k_5}), \quad (3.12)$$

where $k = k_4 + k_6 + \lambda_4$.

And now, taking the functions $h(x)$ and $g(y)$ are linear, i.e. $h(x) = k_5 x + k_6, g(y) = k_1 y + k_2$, from (3.6), we have

$$m''(z) = -\alpha k_5 \left(1 + \frac{m'(z)^2}{1+(k_5)^2+(k_1)^2}\right). \quad (3.13)$$

Solving (3.13) with the same procedure as above, we have

$$m(z) = \frac{ln(cos(\frac{\alpha k_5 z}{\sqrt{1+(k_5)^2+(k_1)^2}} + \lambda_5))(1 + (k_5)^2 + (k_1)^2)}{\alpha k_5} + \lambda_6. \quad (3.14)$$

So, we get

Theorem 3. *The weighted minimal Monge hypersurface in Euclidean 4-space with linear density* $e^{\alpha x}$ *for* $(\alpha \neq 0) \in \mathbb{R}$ *can be parametrized by*

$$X(x,y,z) = (x, y, z, k_5 x + k_1 y + k + \frac{ln(cos(\frac{\alpha k_5 z}{\sqrt{1+(k_5)^2+(k_1)^2}} + \lambda_5))(1 + (k_5)^2 + (k_1)^2)}{\alpha k_5}), \quad (3.15)$$

where $k = k_6 + k_2 + \lambda_6$.

Case 2. Let the density be $e^{\beta y}$:

In this case, let us consider the Monge hypersurface

$$M: X(x,y,z) = \left(x, y, z, h(x) + g(y) + m(z)\right)$$

in Euclidean 4-space with linear density $e^{\beta y}$. Then, this surface is weighted minimal if and only if

$$\begin{aligned} 0 = {} & h''(x)(1 + g'(y)^2 + m'(z)^2) \\ & + g''(y)(1 + h'(x)^2 + m'(z)^2) \\ & + m''(z)(1 + h'(x)^2 + g'(y)^2) \\ & + \beta g'(y)(1 + h'(x)^2 + g'(y)^2 + m'(z)^2) \end{aligned} \quad (3.16)$$

satisfies. With the same procedure as first case, one can obtain the following Theorem:

Theorem 4. *The weighted minimal Monge hypersurface in Euclidean 4-space with linear density* $e^{\beta y}$ *for* $(\beta \neq 0) \in \mathbb{R}$ *can be parametrized by*

$$X(x,y,z) = (x, y, z, k_5 x + k_1 y + k + \frac{ln(cos(\frac{\beta k_1 z}{\sqrt{1+(k_5)^2+(k_1)^2}} + \lambda_7))(1 + (k_5)^2 + (k_1)^2)}{\beta k_1}), \quad (3.17)$$

$$X(x,y,z) = (x, y, z, k_1 y + k_3 z + l + \frac{ln(cos(\frac{\beta k_1 x}{\sqrt{1+(k_3)^2+(k_1)^2}} + \lambda_9))(1 + (k_3)^2 + (k_1)^2)}{\beta k_1}) \quad (3.18)$$

or

$$X(x,y,z)=(x,y,z,k_5x+k_3z+n+\frac{\sqrt{(1+(k_5)^2+(k_3)^2)}\arctan\left(e^{-\lambda_{11}}\sqrt{e^{2\beta y}-e^{2\lambda_{11}}}\right)}{\beta}), \quad (3.19)$$

where $k=k_6+k_2+\lambda_8$, $l=k_2+k_4+\lambda_{10}$ *and* $n=k_6+k_4+\lambda_{12}$.

Case 3. Let the density be $e^{\gamma z}$:

In this case, let us consider the Monge hypersurface

$$M: X(x,y,z)=\big(x,y,z,h(x)+g(y)+m(z)\big)$$

in Euclidean 4-space with linear density $e^{\gamma z}$. Then, this surface is weighted minimal if and only if

$$\begin{aligned}0&=h''(x)(1+g'(y)^2+m'(z)^2)\\&+g''(y)(1+h'(x)^2+m'(z)^2)\\&+m''(z)(1+h'(x)^2+g'(y)^2)\\&+\gamma m'(z)(1+h'(x)^2+g'(y)^2+m'(z)^2)\end{aligned} \quad (3.20)$$

satisfies. Hence, from (3.20) we have

Theorem 5. *The weighted minimal Monge hypersurface in Euclidean 4-space with linear density* $e^{\gamma z}$ *for* $(\gamma\neq 0)\in\mathbb{R}$ *can be parametrized by*

$$X(x,y,z)=(x,y,z,k_1y+k_3z+k+\frac{\ln(\cos(\frac{\gamma k_3 x}{\sqrt{1+(k_3)^2+(k_1)^2}}+\lambda_{13}))(1+(k_3)^2+(k_1)^2)}{\gamma k_3}), \quad (3.21)$$

$$X(x,y,z)=(x,y,z,k_5x+k_3z+l+\frac{\ln(\cos(\frac{\gamma k_3 y}{\sqrt{1+(k_3)^2+(k_1)^2}}+\lambda_{15}))(1+(k_3)^2+(k_5)^2)}{\gamma k_3}) \quad (3.22)$$

or

$$X(x,y,z)=(x,y,z,k_5x+k_1y+n+\frac{\sqrt{(1+(k_5)^2+(k_1)^2)}\arctan\left(e^{-\lambda_{17}}\sqrt{e^{2\gamma z}-e^{2\lambda_{17}}}\right)}{\gamma}), \quad (3.23)$$

where $k=k_2+k_4+\lambda_{14}$, $l=k_6+k_4+\lambda_{16}$ *and* $n=k_6+k_2+\lambda_{18}$.

Case 4. Let the density be $e^{\mu t}$:

Here, let us consider the Monge hypersurface

$$M: X(x,y,z)=\big(x,y,z,h(x)+g(y)+m(z)\big)$$

in Euclidean 4-space with linear density $e^{\mu t}$. Then, this surface is weighted minimal if and only if

$$\begin{aligned}0&=h''(x)(1+g'(y)^2+m'(z)^2)\\&+g''(y)(1+h'(x)^2+m'(z)^2)\\&+m''(z)(1+h'(x)^2+g'(y)^2)\\&-\mu(1+h'(x)^2+g'(y)^2+m'(z)^2)\end{aligned} \quad (3.24)$$

satisfies. Thus,

Theorem 6. *The weighted minimal Monge hypersurface in Euclidean 4-space with linear density* $e^{\mu t}$ *for* $(\mu\neq 0)\in R$ *can be parametrized by*

$$X(x,y,z)=(x,y,z,k_1y+k_3z+k-\frac{\ln(\cos(\frac{x\mu}{\sqrt{1+(k_3)^2+(k_1)^2}}+\lambda_{19}))(1+(k_3)^2+(k_1)^2)}{\mu}, \quad (3.25)$$

$$X(x,y,z)=(x,y,z,k_5x+k_3z+l-\frac{\ln(\cos(\frac{y\mu}{\sqrt{1+(k_5)^2+(k_1)^2}}+\lambda_{21}))(1+(k_5)^2+(k_1)^2)}{\mu}) \quad (3.26)$$

or

$$X(x,y,z)=(x,y,z,k_5x+k_1y+n-\frac{\ln(\cos(\frac{z\mu}{\sqrt{1+(k_3)^2+(k_5)^2}}+\lambda_{23}))(1+(k_3)^2+(k_5)^2)}{\mu}), \quad (3.27)$$

where $k=k_2+k_4+\lambda_{20}$, $l=k_6+k_4+\lambda_{22}$ *and* $n=k_6+k_2+\lambda_{24}$.

3.2. Weighted Flat Monge Hypersurfaces in E^4 with Linear Density

From (1.11), the weighted Gaussian curvature of the Monge hypersurface in Euclidean 4-space with linear density $e^{\alpha x+\beta y+\gamma z+\mu t}$ is obtained as

$$K_\varphi=-\frac{h''(x)g''(y)m''(z)}{(1+h'(x)^2+g'(y)^2+m'(z)^2)^{\frac{3}{2}}}. \quad (3.28)$$

So from (3.28), we can state the following theorems:

Theorem 7. *Let* $M: X(x,y,z)=\big(x,y,z,h(x)+g(y)+m(z)\big)$ *be a Monge hypersurface in Euclidean 4-space with linear density* $e^{\alpha x+\beta y+\gamma z+\mu t}$, *where* α,β,γ *and* μ *are not all zero constants. If one of the functions* $h(x)$, $g(y)$ *and* $m(z)$ *is linear, then* M *is weighted flat.*

Theorem 8. *If* $M: X(x,y,z)=\big(x,y,z,h(x)+g(y)+m(z)\big)$ *is a Monge hypersurface in Euclidean 4-space with linear density* $e^{\alpha x+\beta y+\gamma z+\mu t}$, *where* α,β,γ *and* μ *are not all zero constants, then its weighted Gaussian curvature cannot be constant except for zero.*

4. MONGE HYPERSURFACES IN E^4 WITH DENSITY $e^{\alpha x^2+\beta y^2+\gamma z^2+\mu t^2}$

In this section, we obtain the weighted minimal Monge hypersurfaces and give a characterization for the constancy of weighted Gaussian curvature of Monge hypersurfaces in E^4 with density $e^{\alpha x^2+\beta y^2+\gamma z^2+\mu t^2}$.

4.1. Weighted Minimal Monge Hypersurfaces in E^4 with Density $e^{\alpha x^2+\beta y^2+\gamma z^2+\mu t^2}$

From (1.10) and (2.4), the weighted mean curvature of the Monge hypersurface

$$M: X(x,y,z)=\big(x,y,z,f(x,y,z)\big)$$

in E^4 with density $e^{\alpha x^2+\beta y^2+\gamma z^2+\mu t^2}$ is obtained as

$$H_\varphi=\frac{\left\{\begin{array}{c}-f_{xx}(1+f_y^2+f_z^2)-f_{yy}(1+f_x^2+f_z^2)\\+f_{zz}(1+f_x^2+f_y^2)\\2(f_{xy}f_xf_y+f_{xz}f_xf_z+f_{yz}f_yf_z)-\\2(\alpha xf_x+\beta yf_y+\gamma zf_z-\mu f)(1+f_x^2+f_y^2+f_z^2)\end{array}\right\}}{3(1+f_x^2+f_y^2+f_z^2)^{3/2}}. \quad (4.1)$$

Thus, we get

Proposition 3. *Let* $M: X(x,y,z)=\big(x,y,z,f(x,y,z)\big)$ *be a Monge hypersurface in Euclidean 4-space with density* $e^{\alpha x^2+\beta y^2+\gamma z^2+\mu t^2}$, *where* α,β,γ *and* μ *are not all zero* *constants. Then, this surface is weighted minimal if and only if*

$$2\left(f_{xy}f_xf_y + f_{xz}f_xf_z + f_{yz}f_yf_z\right)= f_{xx}\left(1+f_y^2+f_z^2\right)+f_{yy}(1+f_x^2+f_z^2)+f_{zz}\left(1+f_x^2+f_y^2\right) +\left(\alpha x f_x+\beta y f_y+\gamma z f_z-\mu f\right)\left(1+f_x^2+f_y^2+f_z^2\right) \quad (4.2)$$

satisfies.

Here, if we take $f(x,y,z)=h(x)+g(y)+m(z)$, where h, g and m are C^2 −differentiable functions, then using (3.3) in (4.2), the weighted mean curvature of the Monge hypersurface

$$M: X(x,y,z)=\left(x,y,z,h(x)+g(y)+m(z)\right)$$

is obtained as

$$H_\varphi=\frac{\left\{\begin{array}{c}-h''(x)\left(1+g'(y)^2+m'(z)^2\right)\\ -g''(y)\left(1+h'(x)^2+m'(z)^2\right)\\ -m''(z)\left(1+h'(x)^2+g'(y)^2\right)\\ -2\left(\alpha x h'(x)+\beta y g'(y)+\gamma z m'(z)\right)-\\ \mu(h(x)+g(y)+m(z))\left(1+h'(x)^2+g'(y)^2+m'(z)^2\right)\end{array}\right\}}{3(1+h'(x)^2+g'(y)^2+m'(z)^2)^{3/2}}. \quad (4.3)$$

Proposition 4. *Let* $M: X(x,y,z)=\left(x,y,z,h(x)+g(y)+m(z)\right)$ *be a Monge hypersurface in Euclidean 4-space with density* $e^{\alpha x^2+\beta y^2+\gamma z^2+\mu t^2}$, *where* α,β,γ *and* μ *are not all zero constants. Then, this surface is weighted minimal if and only if*

$$0=h''(x)(1+g'(y)^2+m'(z)^2)+ g''(y)(1+h'(x)^2+m'(z)^2)+ m''(z)(1+h'(x)^2+g'(y)^2)+ 2\left(\alpha x h'(x)+\beta y g'(y)+\gamma z m'(z)\right)- \mu(h(x)+g(y)+m(z))(1+h'(x)^2+g'(y)^2+m'(z)^2) \quad (4.4)$$

satisfies.

Now, we'll obtain the weighted minimal Monge hypersurfaces in E^4 with density $e^{\alpha x^2+\beta y^2+\gamma z^2+\mu t^2}$ for different choices of the not all zero constants α,β,γ and μ.

<u>**Case 1**. Let the density be $e^{\alpha x^2}$:</u>

In this case, let us consider the Monge hypersurface

$$M: X(x,y,z)=\left(x,y,z,h(x)+g(y)+m(z)\right)$$

in Euclidean 4-space with density $e^{\alpha x^2}$. Then, this surface is weighted minimal if and only if

$$0=h''(x)(1+g'(y)^2+m'(z)^2)+ g''(y)(1+h'(x)^2+m'(z)^2)+ m''(z)(1+h'(x)^2+g'(y)^2)+ 2\alpha x h'(x)(1+h'(x)^2+g'(y)^2+m'(z)^2) \quad (4.5)$$

satisfies. Here, by obtaining some special solutions for the equation (4.5), we'll construct the weighted minimal Monge hypersurfaces in E^4 with density $e^{\alpha x^2}$.

Firstly, let us take the functions $g(y)$ and $m(z)$ are linear, i.e. $g(y)=k_1y+k_2, m(z)=k_3z+k_4$. Then, the equation (4.5) becomes

$$h''(1+(k_1)^2+(k_3)^2)= -2\alpha x h'(1+(h')^2+(k_1)^2+(k_3)^2). \quad (4.6)$$

From (4.6), we have

$$\frac{h''(1+(k_1)^2+(k_3)^2)}{h'(1+(h')^2+(k_1)^2+(k_3)^2)}=-2\alpha x$$

$$\Rightarrow \frac{h''}{h'}-\frac{h''h'}{(h')^2+1+(k_1)^2+(k_3)^2}=-2\alpha x$$

$$\Rightarrow \left(\ln|h'|-\frac{1}{2}\ln|h'^2+1+(k_1)^2+(k_3)^2|\right)'=-2\alpha x$$

$$\Rightarrow \ln\left|\frac{h'}{\sqrt{(h')^2+1+(k_1)^2+(k_3)^2}}\right|=-\alpha x^2+\lambda_{25}$$

$$\Rightarrow \frac{h'}{\sqrt{(h')^2+1+(k_1)^2+(k_3)^2}}=e^{-\alpha x^2+\lambda_{25}}$$

$$\Rightarrow h'=e^{-\alpha x^2+\lambda_{25}}\sqrt{(h')^2+1+(k_1)^2+(k_3)^2}$$

$$\Rightarrow h'=e^{-\alpha x^2+\lambda_{25}}\sqrt{(h')^2+1+(k_1)^2+(k_3)^2}$$

$$\Rightarrow (h')^2=e^{-2\alpha x^2+2\lambda_{25}}((h')^2+1+(k_1)^2+(k_3)^2)$$

$$\Rightarrow h'=\frac{\sqrt{1+(k_1)^2+(k_3)^2}.\,e^{-\alpha x^2+\lambda_{25}}}{\sqrt{1-\left(e^{-\alpha x^2+\lambda_{25}}\right)^2}}$$

$$\Rightarrow h=\int \frac{\sqrt{1+(k_1)^2+(k_3)^2}.e^{-\alpha x^2+\lambda_{25}}}{\sqrt{1-\left(e^{-\alpha x^2+\lambda_{25}}\right)^2}}dx\,. \quad (4.7)$$

Thus,

$$f(x,y,z)=\int \frac{\sqrt{1+(k_1)^2+(k_3)^2}.e^{-\alpha x^2+e^{-\alpha x^2+\lambda_{25}}}}{\sqrt{1-\left(e^{-\alpha x^2+n}\right)^2}}dx+ k_1y+k_3z+k_2+k_4$$

is a solution of (4.6).

So, we can give the following Theorem:

Theorem 9. *The weighted minimal Monge hypersurface in Euclidean 4-space with density* $e^{\alpha x^2}$ *for* $(\alpha\neq 0)\in\mathbb{R}$ *can be parametrized by*

$$X(x,y,z)=\left(x,y,z,\int \frac{\sqrt{1+(k_1)^2+(k_3)^2}.e^{-\alpha x^2+\lambda_{25}}}{\sqrt{1-\left(e^{-\alpha x^2+\lambda_{25}}\right)^2}}dx +k_1y+k_3z+k_2+k_4\right). \quad (4.8)$$

Secondly, taking the functions $h(x)$ and $m(z)$ are linear, i.e. $h(x)=k_5x+k_6, m(z)=k_3z+k_4$, from (4.5), we have

$$g''(1+(k_5)^2+(k_3)^2)= -2\alpha x k_5(1+(g')^2+(k_5)^2+(k_3)^2). \quad (4.9)$$

The equation (4.9) satisfies for $k_5=0$ and $g''(y)=0$.

Similarly, taking the functions $h(x)$ and $g(y)$ are linear, i.e. $h(x)=k_5x+k_6, g(y)=k_1y+k_2$, from (4.5), we have

$$m''(1+(k_5)^2+(k_1)^2)= -2\alpha x k_5(1+(m')^2+(k_5)^2+(k_1)^2). \quad (4.10)$$

The equation (4.10) satisfies for $k_5=0$ and $m''(z)=0$.

So, we get

Theorem 10. *The weighted minimal Monge hypersurface in Euclidean 4-space with linear density* $e^{\alpha x^2}$ *for* $(\alpha\neq 0)\in\mathbb{R}$ *can be parametrized by*

$$X(x,y,z) = (x,y,z,k_1y + k_3z + k), \quad (4.11)$$

where $k = k_2 + k_4 + k_6$.

Case 2. Let the density be $e^{\beta y^2}$:

In this case, let us consider the Monge hypersurface

$$M: X(x,y,z) = \big(x,y,z,h(x) + g(y) + m(z)\big)$$

in Euclidean 4-space with linear density $e^{\beta y^2}$. Then, this surface is weighted minimal if and only if

$$\begin{aligned} 0 = {} & h''(x)(1 + g'(y)^2 + m'(z)^2) + \\ & + g''(y)(1 + h'(x)^2 + m'(z)^2) \\ & + m''(z)(1 + h'(x)^2 + g'(y)^2) \\ & + 2\beta y g'(y)(1 + h'(x)^2 + g'(y)^2 + m'(z)^2) \end{aligned} \quad (4.12)$$

satisfies. With the same procedure as first case, one can obtain the following Theorem:

Theorem 11. *The weighted minimal Monge hypersurface in Euclidean 4-space with linear density* $e^{\beta y^2}$ *for* $(\beta \neq 0) \in \mathbb{R}$ *can be parametrized by*

$$X(x,y,z) = (x,y,z,k_5x + k_3z + k_4 + k_6 + \int \frac{\sqrt{1+(k_5)^2+(k_3)^2 . e^{-\beta y^2 + \lambda_{26}}}}{\sqrt{1-\left(e^{-\beta y^2+\lambda_{26}}\right)^2}} dy) \quad (4.13)$$

or

$$X(x,y,z) = (x,y,z,k_5x + k_3z + k), \quad (4.14)$$

where $k = k_2 + k_4 + k_6$.

Case 3. Let the density be $e^{\gamma z^2}$:

Here, let us consider the Monge hypersurface

$$M: X(x,y,z) = \big(x,y,z,h(x) + g(y) + m(z)\big)$$

in Euclidean 4-space with linear density $e^{\gamma z^2}$. Then, this surface is weighted minimal if and only if

$$\begin{aligned} 0 = {} & h''(x)(1 + g'(y)^2 + m'(z)^2) \\ & + g''(y)(1 + h'(x)^2 + m'(z)^2) \\ & + m''(z)(1 + h'(x)^2 + g'(y)^2) \\ & + 2\gamma m'(z)(1 + h'(x)^2 + g'(y)^2 + m'(z)^2) \end{aligned} \quad (4.15)$$

satisfies. Hence, we have

Theorem 12. *The weighted minimal Monge hypersurface in Euclidean 4-space with linear density* $e^{\gamma z^2}$ *for* $(\gamma \neq 0) \in \mathbb{R}$ *can be parametrized by*

$$X(x,y,z) = (x,y,z,k_5x + k_1y + k_2 + k_6 + \int \frac{\sqrt{1+(k_5)^2+(k_1)^2 . e^{-\gamma z^2 + \lambda_{27}}}}{\sqrt{1-\left(e^{-\gamma z^2+\lambda_{27}}\right)^2}} dx) \quad (4.16)$$

or

$$X(x,y,z) = (x,y,z,k_5x + k_1y + k), \quad (4.17)$$

where $k = k_2 + k_4 + k_6$.

Case 4. Let the density be $e^{\mu t^2}$:

In this case, let us consider the Monge hypersurface

$$M: X(x,y,z) = \big(x,y,z,h(x) + g(y) + m(z)\big)$$

in Euclidean 4-space with linear density $e^{\mu t^2}$. Then, this surface is weighted minimal if and only if

$$\begin{aligned} & 2\mu(h(x) + g(y) + m(z))(1 + h'(x)^2 + g'(y)^2 + m'(z)^2) \\ & = h''(x)(1 + g'(y)^2 + m'(z)^2) \\ & + g''(y)(1 + h'(x)^2 + m'(z)^2) \\ & + m''(z)(1 + h'(x)^2 + g'(y)^2) \end{aligned} \quad (4.18)$$

satisfies.

4.2. The Constancy of Weighted Gaussian Curvature of Monge Hypersurfaces in E^4 with Density $e^{\alpha x^2+\beta y^2+\gamma z^2+\mu t^2}$

From (1.11), the weighted Gaussian curvature of the Monge hypersurface in Euclidean 4-space with density $e^{\alpha x^2+\beta y^2+\gamma z^2+\mu t^2}$ is obtained as

$$K_\varphi = \frac{\left(\begin{array}{c} h''(x)g''(y)m''(z) - \\ 2(\alpha+\beta+\gamma+\mu)\left(1+h'(x)^2+g'(y)^2+m'(z)^2\right)^{3/2} \end{array}\right)}{\left(1+h'(x)^2+g'(y)^2+m'(z)^2\right)^{3/2}}. \quad (4.19)$$

So from (4.19), we can state the following Theorem:

Theorem 13. *Let* $M: X(x,y,z) = \big(x,y,z,h(x) + g(y) + m(z)\big)$ *be a Monge hypersurface in Euclidean 4-space with density* $e^{\alpha x^2+\beta y^2+\gamma z^2+\mu t^2}$, *where* α, β, γ *and* μ *are not all zero constants. If one of the functions* $h(x)$, $g(y)$ *and* $m(z)$ *is linear and* $\alpha + \beta + \gamma + \mu = \frac{-r}{2}$, *then the weighted Gaussian curvature of* M *is constant* r.

5. CONCLUSION

Surface theory has an important place in 4-dimensional spaces as in 3-dimensional spaces. So, in the peresent study, we consider the Monge hypersurfaces in Euclidean 4-space with different densities and obtain the weighted minimal and weighted flat Monge hypersurfaces in this space. We think that, the results which are obtained in this study are important for differential geometers who are dealing with weighted surfaces and in the near future, the results which are stated in this study can be handled in different four or higher dimensional spaces.

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