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Why Flc-Frame is Better than Frenet Frame on Polynomial Space Curves?

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Abstract
It is well known that the binormal and normal vectors of Frenet frame rotate around the tangent vector. That is why the Frenet frame is not suitable for some applications such as tube surfaces. However, there is not enough information about why the vectors of the Frenet frame rotate around the tangent vector. In this paper we will deal with this problem. Moreover we show the advantages of Flc-frame over the Frenet frame.

Keywords: *Frenet frame; space curve; adapted frame; polynomial curve.*

AMS Subject Classification (2020): *Primary: 53A04 ; Secondary: 68U05.*

1. Introduction

Recently, the study of the frames along a space curve has arisen some engineering applications [18, 22, 24]. For intance in [19], the authors investigated the Mannheim curves with a new frame called modified orthogonal frame. Despite the fact that Bishop frame (rotation minimizing frame) is more suitable for engineering applications [9], this frame can not be computed analytically. Therefore a number of approximation methods have been proposed for RMF computation [4]. In this paper we will compare the frames which can be computed analytically on polynomial space curves.

The Frenet frame is the most known frame along a space curve [4, 5, 23]. Let $\alpha(t)$ be a regular space curve. The Frenet frame is defined as follows,

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \mathbf{b} = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}, \mathbf{n} = \mathbf{b} \times \mathbf{t}. \tag{1.1}$$

The well-known Frenet formulas are given by,

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{bmatrix} = \|\alpha'(t)\| \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}, \tag{1.2}$$

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where the curvature κ and the torsion τ of the curve are given by

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}, \tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \times \alpha''\|^2}. \tag{1.3}$$

The Frenet frame has inflection points and two type of singular points [5, 11].

Definition 1.1. Let $\alpha(t) : I \to \mathbb{R}^3$ be a space curve. A point $t_0 \in I$ is said to be singular point of order 0 of the curve if $\alpha'(t_0)$ vanishes.

We say that $t_1 \in I$ is a singular point of order 1 if $\alpha''(t_1)$ vanishes.

Definition 1.2. Let $\alpha(t) : I \to \mathbb{R}^3$ be a space curve. A point $t_2 \in I$ is called inflection point if $\alpha'(t_2) \wedge \alpha''(t_2)$ vanishes, namely curvature is zero [16].

Apart from Frenet frame we can define more frame along a space curve [1, 25]. Recently, Dede [15] introduced a new frame along a polynomial space curve, called as Flc-frame. The computation of Flc-frame is easier than the both Frenet and Bishop frames [1, 2]. Moreover they showed that to have a inflection point on Flc frame is less possible than Frenet frame. Discussion of the Flc-frame and its application to the tube surfaces can be found in [15].

Let $\alpha(t)$ be a polynomial space curve of degree n. The Flc-frame is given by

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \mathbf{D}_1 = \frac{\alpha' \wedge \alpha^{(n)}}{\left\|\alpha' \wedge \alpha^{(n)}\right\|}, \mathbf{D}_2 = \mathbf{D}_1 \wedge \mathbf{t}. \tag{1.4}$$

Where the prime $'$ indicates the differentiation with respect to t [15]. If the order of derivative exceeds three, we replace prime by the superscript (n), such as $\alpha'''' = \alpha^{(4)}$. The new vectors $\mathbf{D}_1$ and $\mathbf{D}_2$ are called as binormal-like vector and normal-like vector, respectively.

The local rate of change of the Flc-frame called as the Frenet-like formulas can be expressed in the following form

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{D}_2' \\ \mathbf{D}_1' \end{bmatrix} = \|\alpha'\| \begin{bmatrix} 0 & d_1 & d_2 \\ -d_1 & 0 & d_3 \\ -d_2 & -d_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{D}_2 \\ \mathbf{D}_1 \end{bmatrix}. \tag{1.5}$$

The curvatures of the Flc-frame are given by

$$d_1 = \frac{\left\langle \alpha' \wedge \alpha'', \alpha' \wedge \alpha^{(n)} \right\rangle}{\|\alpha'\|^3 \left\|\alpha' \wedge \alpha^{(n)}\right\|}, d_2 = \frac{\det[\alpha'', \alpha', \alpha^{(n)}]}{\|\alpha'\|^2 \left\|\alpha' \wedge \alpha^{(n)}\right\|}, \tag{1.6}$$

and

$$d_3 = \frac{\det[\alpha', \alpha'', \alpha^{(n)}] \left\langle \alpha', \alpha^{(n)} \right\rangle}{\|\alpha'\|^2 \left\|\alpha' \wedge \alpha^{(n)}\right\|^2}. \tag{1.7}$$

Corollary 1.1. *If the degree of polynomial space curve is two, then the Flc-frame coincides with the Frenet frame with curvatures $d_1 = \kappa, d_2 = 0$ and $d_3 = \tau = 0$.*

2. Flc-frame vs Frenet frame

There are three main drawbacks associated with the Frenet frame. In this chapter we discuss the drawbacks of the Frenet frame. Moreover we explain why the Flc-frame is better than Frenet frame from points of these drawbacks. As an application we consider tube surfaces.

- Singular point of order 1.

One of the most important advantages of the Flc-frame over the Frenet frame is that when the second derivative of the curve vanishes the Frenet frame behaves erratically. This is why the rotation minimizing frame (RMF) is widely used in surface modeling such as tube (pipe) surfaces.

Theorem 2.1. *Let $\alpha(t)$ be a polynomial space curve of degree n. The Flc-frame doesn't have singular point of order 1.*

Proof. From point of Definition 1.1, since the n-th derivative of polynomial space curve of degree n never vanishes, the Flc-frame doesn't have singular point of order 1. □

Here's an example about this case.

Example 2.1. Assume that a curve $\alpha(t)$ is given by

$$\alpha(t) = (t, \frac{t^4}{12} - \frac{t^3}{6}, (t-1)^3).$$

It follows that $\alpha''(t) = (0, t^2 - t, 6t - 6)$ therefore the point $t = 1$ is a singular point of order 1. When $t = 1$ the

Figure 1. The normal and binormal vectors of Frenet frame (Left) and the normal-like and binormal-like vectors of Flc-frame (Right) along the curve $t \in (-2, 2)$.

binormal vectors of the Frenet frame suddenly exhibits 180 degree rotation (highlighted by an arrow in Figure 1). The Figure 1 compares the behaviour of the binormal (black) and the normal (red) vectors of the Frenet frame with the binormal-like (black) and the normal-like (red) vectors of the Flc-frame.

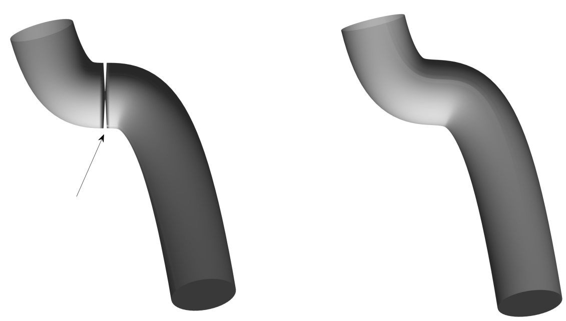


Figure 2. The tube surfaces generated by Frenet frame (Left) and the Flc-frame (Right) $t \in (-2, 2)$, $v \in (-4, 4)$.

As a result, the sudden rotation of normal and binormal vectors of the Frenet frame causes deformation in tube surface. The tube (pipe) surfaces with radius $r = 0.8$ generated by the Frenet frame and Flc-frame are illustrated in Figure 2.

- Inflection points; at the points where the curvature κ is zero, namely $\|\alpha' \wedge \alpha''\| = 0$.

In the case of Flc-frame it corresponds to $\|\alpha' \wedge \alpha^n\| = 0$. Dede [15] showed that to have a inflection point on Flc-frame is less possible than the Frenet frame. However since Flc frame permit analytical computation, it has inflection points when $\|\alpha'(t_2) \wedge \alpha^n(t_2)\| = 0$ at the point t_2.

Example 2.2. In this example, we would like to deal with the inflection points. Let us consider a curve given by

$$\alpha(t) = (t^3, t^3, t^2 - 2t). \tag{2.1}$$

By using the derivatives of the curve, we have

$$\alpha'(t) \wedge \alpha''(t) = (-6t^2 + 12t, 6t^2 - 12t, 0),$$

and

$$\alpha'(t) \wedge \alpha^n(t) = (12 - 12t, 12t - 12, 0).$$

Observe that the Frenet frame has two inflection points at $t = 0$ and $t = 2$ whereas the Flc frame has one at the point $t = 1$.

Note that it is all about the degree of a curve. Since the degree of $\alpha'(t) \wedge \alpha^n(t)$ is less than $\alpha'(t) \wedge \alpha''(t)$, it has fewer possible roots. The Figure 3 compares the behaviour of the vectors of Frenet frame with the Flc-frame. Similar

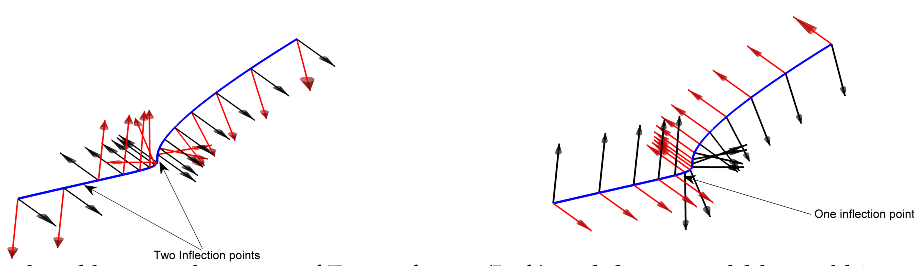


Figure 3. The normal and binormal vectors of Frenet frame (Left) and the normal-like and binormal-like vectors of Flc-frame (Right) along the curve $t \in (-2, 2)$.

to the case of singular point of order 1, the vectors of the Frenet frame suddenly exhibits 180 degree rotation at the inflection points.

Figure 4. The tube surfaces generated by Frenet frame (left) and the Flc-frame (right) $t \in (-2, 2)$, $v \in (-4, 4)$.

The tube (pipe) surfaces with radius $r = 1.9$ generated by the Frenet frame and Flc-frame are illustrated in Figure 4.

The solution of this problem is not hard. The following theorem and algorithm demonstrate a good way to solve this problem.

Theorem 2.2. *Let $\alpha(t)$ be a polynomial space curve of degree 3. The Flc-frame has just one inflection point which never coincidences with the inflection points of the Frenet frame.*

Proof. **:** Three-dimensional cubic polynomial curve is of the form

$$\alpha(t) = (\sum_{i=0}^{3} a_i t^i, \sum_{i=0}^{3} b_i t^i, \sum_{i=0}^{3} c_i t^i),$$

which we represent by its polynomial coefficients, a_i,b_i and c_i.

The inflection point of the Flc-frame is

$$\alpha' \wedge \alpha^n = \begin{cases} 12(b_2c_3 - b_3c_2)t + 6(b_1c_3 - b_3c_1) = 0 \\ 12(-a_2c_3 - a_3c_2)t + 6(a_3c_1 - a_1c_3) = 0 \\ 12(a_2b_3 - a_3b_2)t + 6(a_1b_3 - a_3b_1) = 0 \end{cases}$$

The solution of the above system of equations is obtained as

$$t = \frac{b_3c_1 - b_1c_3}{2b_2c_3 - 2b_3c_2}, c_3 = \frac{a_1b_3c_2 - a_2b_3c_1 - a_3b_1c_2 + a_3b_2c_1}{b_2a_1 - a_2b_1}.$$

□

The nice result is that this point is not the inflection point of the Frenet frame. For matlab program, we can write an easy algorithm to construct tube surface as follows

$$\begin{array}{cccc} & begin & & \\ & for\ i = 1\ to\ j\ do & & \\ if\quad \|\alpha' \wedge \alpha^n\| > 0 & \text{then } Flc - frame & \text{(use Flc-frame for tube)} \\ else & Frenet \text{ frame} & \text{(use Frenet frame for tube)} \\ break & & \text{(to stop to use of Frenet frame)} \\ & end & \\ & for\ k = i+1\ to\ j\ do & \\ continue & \text{use } -D_1 \text{ and } -D_2 & \text{(to avoid 180 degree rotation)} \\ & end & \\ & end & \end{array}$$

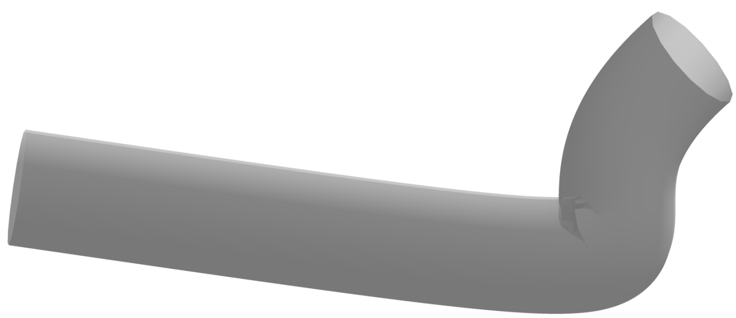


Figure 5. The tube surface generated by the above algorithm $t \in (-2, 2)$, $v \in (-4, 4)$.

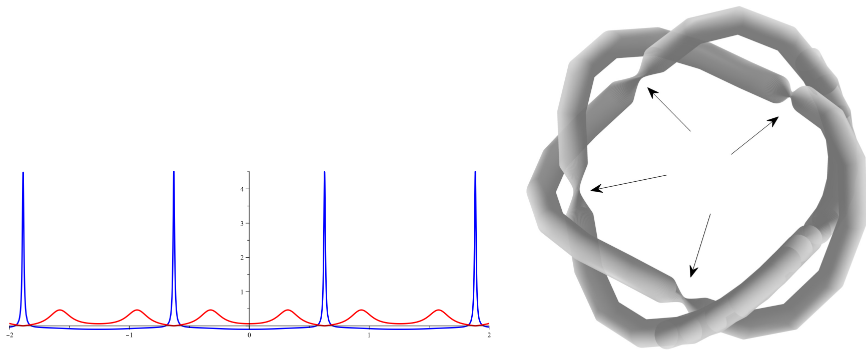
With this algorithm, the tube surface generated by the curve given in Equation 2.1 is shown in Figure 5. The following case is the most interesting. Because currently there is not exact description for this error. Let's begin with the most convenient one.

- At the points where the curvature of curve is small and the absolute value of the torsion is large.

Sometimes, despite the fact that where the Frenet frame doesn't have singular point of order 1 and inflection point, interestingly the normal and binormal vectors still exhibit rotation around the tangent vector, but not 180 degree. There are some instances in the literature to explain why the Frenet frame behaves badly. In this section we focus on this problem, and review some recently published comments that are used to explain the unpredictable behavior of the Frenet frame.

In [8] the authors have tried to explain what causes abnormal behavior of the normal and binormal vectors of the Frenet frame. They realized that the small curvature and large absolute value of torsion produce so much twisting in the tube. The Figure 6 shows that this is a highly convincing explanation. Let us consider a curve given by

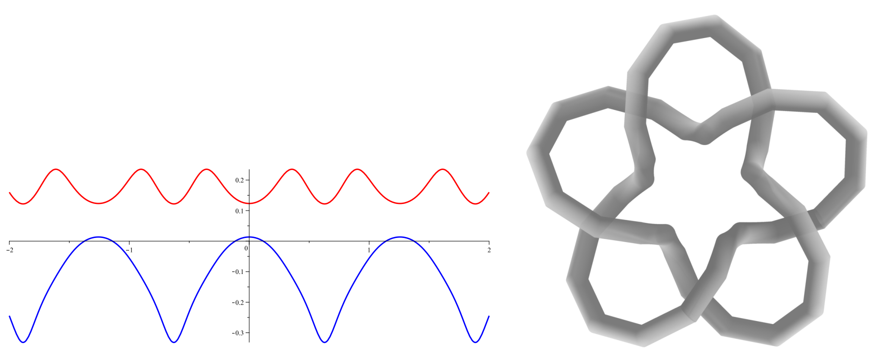
$$\alpha(t) = (8 + \cos(5t)\cos(2t), (8 + \cos(5t))\sin(2t), 5\sin(5t)). \tag{2.2}$$



(a) The curvature (Red) and the torsion (Blue) of the curve $\alpha(t)$ $t \in (-2, 2)$.

(b) The tube surface generated by Frenet frame of the curve $\alpha(t)$ $t \in (-2, 2)$, $v \in (-4, 4)$.

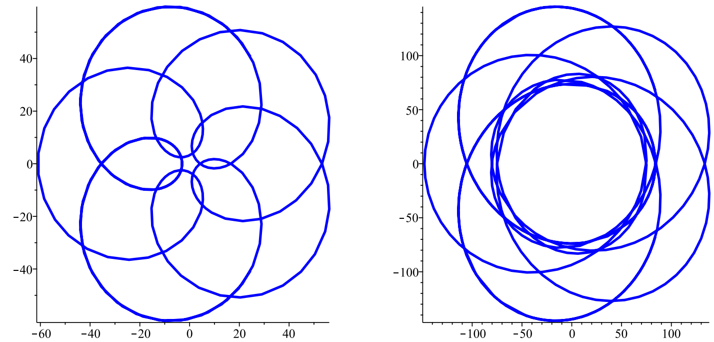
Figure 6.



(a) the curvature (Red) and the torsion (Blue) of the curve $\beta(t)$ $t \in (-2, 2)$.

(b) The tube surface generated by Frenet frame of the curve $\beta(t)$ $t \in (-2, 2)$, $v \in (-4, 4)$.

Figure 7.



(a) The second derivative of the curve $\alpha(t)$.

(b) The second derivative of the curve $\beta(t)$.

Figure 8. The top view from z-axis

In [8] the authors pointed out that at the plot of the tube shows that this increased twisting occurs in four different places where simultaneously the curvature is small and the absolute value of the torsion is large. However the small curvature and large torsion is a relative concept. For instance, let us consider a curve given by

$$\beta(t) = (8 + 3\cos(5t)\cos(2t), (8 + 3\cos(5t))\sin(2t), 5\sin(5t)). \tag{2.3}$$

The Figure 7 shows that despite the fact that the graph of the curvature and torsion is similar, the tube surface generated by the curve $\beta(t)$ doesn't have any deformation on it.

In addition, a different approach has been given for this case in [14]. The author claimed that when the the second derivative of the curve becomes very small, the Frenet frame behaves erratically which causes twisting in the tube.

Now let us plot the graph of the second derivative of the curves given in (2.2) and (2.3). In Figure 8, observe that the curve $\alpha''(t)$ approaches to zero at the four points, $\beta''(t)$ is not. Note that this explanation also shows that why the Flc-frame is better than the Frenet frame? Because we use highest order derivative instead of second order derivatives of the curve to construct the Flc-frame. The following example shows these advantages.

Example 2.3. Let us consider a curve given by

$$\alpha(t) = (t, t^3 - t^2 + 3t, t^3). \tag{2.4}$$

Figure 9. The tube surfaces generated by Frenet frame (left) and the Flc-frame (right) $t \in (-2, 2)$, $v \in (-4, 4)$.

It is easy to see that although this curve doesn't have neither singular point of order 1 nor inflection point, the Figure 9 shows that the tube surface generated by the Frenet frame is deformed.

3. Conclusion

In this paper we investigated three drawbacks of the Frenet frame and compared the Frenet frame with a new frame called as Flc-frame. Moreover, we tried to explain what causes the last drawback of the Frenet frame. Where as the Frenet frame, the Flc-frame has just one drawback, for which we constructed an easy algorithm. As a result, this new frame not only decreases the singular points, but it also decreases the undesirable rotation around the tangent vector of the curve which is a advantage in computer graphics and related fields.

- Whereas the Frenet frame, the Flc-frame does not have singular point of order 1.
- To have a inflection point on Flc frame is less possible than the Frenet frame.
- Whereas the Frenet frame, the normal and binormal vectors of the Flc-frame does not exhibit rotation around the tangent vector.

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Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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