JOURNAL OF SCIENTIFIC PERSPECTIVES JSP

Journal of Scientific Perspectives

Volume **3,** Issue **4,** Year **2019**, pp. **285-298**
E - ISSN: **2587-3008**
URL: http://ratingacademy.com.tr/ojs/index.php/jsp
DOİ: https://doi.org/10.26900/jsp.3.030
Research Article

ON SOME BULLEN-TYPE INEQUALITIES VIA CONFORMABLE FRACTIONAL INTEGRALS

Musa ÇAKMAK *

* *Assist. Prof. Dr., Hatay Mustafa Kemal University, TURKEY,*
e-mail: enkucukcakmak@gmail.com
ORCİD ID: https://orcid.org/0000-0001-6791-0971

Received: 13 September 2018; Accepted: 21 October 2019

Abstract

In this study, the Author has established a new lemma for $\alpha-$differentiable function and some inequalities of Bullen-type inequalities for conformable fractional integrals. Some applications are also given. Examples are given to show the results.

MSC: 26B25, 26D10

Keywords: Bullen type inequality; Conformable fractional integrals; Hölder's inequality; Power-mean inequality.

1 Introductions

To establish analytic inequalities, one of the most efficient way is the property of convexity of a dedicated function. Notedly, in the theory of higher transcendental functions, there are many significant applications. We can use the integral inequalities in order to study qualitative and quantitative properties of integrals (see [6, 7, 9]). Thing continuing to bewilder us by indicating new inferences, new difficulties and also new open questions are a major mathematical outcome.

The Hermite-Hadamard inequality: Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $\imath_1, \imath_2 \in I$ with $\imath_1 < \imath_2$.

$$\varphi\left(\frac{\imath_1+\imath_2}{2}\right) \leq \frac{1}{\imath_2-\imath_1}\int_{\imath_1}^{\imath_2} \varphi(\varkappa)\, d\varkappa \leq \frac{\varphi(\imath_1)+\varphi(\imath_2)}{2} H \tag{1}$$

If φ is concave, this double inequality hold in the inverse way. See [1, 2, 5, 7] for details.

The Bullen inequality:

$$\frac{1}{\imath_2-\imath_1}\int_{\imath_1}^{\imath_2} \varphi(\varkappa)\, d\varkappa \leq \frac{1}{2}\left[\frac{\varphi(\imath_1)+\varphi(\imath_2)}{2} + \varphi\left(\frac{\imath_1+\imath_2}{2}\right)\right], B \tag{2}$$

provided that $\varphi : [\imath_1, \imath_2] \to \mathbb{R}$ is a convex function on $[\imath_1, \imath_2]$ (see for example [3, 4, 8, 10]) for details.

Lemma 1 *[11]Let $\varphi : I \rightarrow \mathbb{R}$, $I \subset \mathbb{R}$ be a differentiable mapping on I°, and $\imath_1, \imath_2 \in I, \imath_1 < \imath_2$. If $\varphi\prime \in L([\imath_1.\imath_2])$, $t \in [0,1]$ then*

$$\int_0^1 (1-2t)\left[\left(\varphi'\left(t\imath_1 + (1-t)\left(\frac{\imath_1+\imath_2}{2}\right)\right) + \varphi'\left(t\left(\frac{\imath_1+\imath_2}{2}\right) + (1-t)\imath_2\right)\right)\right] dt$$
$$= \frac{4}{\imath_2-\imath_1}\left(\frac{\varphi(\imath_1)+\varphi(\imath_2)}{2} + \varphi\left(\frac{\imath_1+\imath_2}{2}\right) - \frac{2}{\imath_2-\imath_1}\int_{\imath_1}^{\imath_2}\varphi(x)\,dx\right).$$

Here I° denotes the interior of I.

2 "Definition and Properties of Conformable Fractional Derivative and Integral

The following definitions and theorems with respect to conformable fractional derivative and integral were referred in [12]-[17].

Definition 2 *(Conformable fractional derivative) Given a function $\varphi : [0,\infty) \rightarrow \mathbb{R}$. Then the "conformable fractional derivative" of φ of order α is defined by*

$$D_\alpha(\varphi)(t) = \lim_{\varepsilon \rightarrow 0} \frac{\varphi\left(t + \varepsilon t^{1-\alpha}\right) - \varphi(t)}{\varepsilon}$$

for all $t > 0$, $\alpha \in (0,1]$. If φ is $\alpha-$differentiable in some $(0,\imath_1)$, $\alpha > 0$, $\lim_{t\rightarrow 0^+}\varphi^{(\alpha)}(t)$ exist, then define

$$\varphi^{(\alpha)}(0) = \lim_{t\rightarrow 0^+}\varphi^{(\alpha)}(t).$$

We can write $\varphi^{(\alpha)}(t)$ for $D_\alpha(\varphi)(t)$ to denote the conformable fractional derivatives of φ of order α.In addition, if the conformable fractional derivative of φ of order α exists, then we simply say φ is $\alpha-$differentiable.

Theorem 3 *Let $\alpha \in (0,1]$ and φ, Λ be $\alpha-$differentiable at a point $t > 0$. Then,*

1) $D_\alpha(a\varphi + b\Lambda) = aD_\alpha(\varphi) + bD_\alpha(\Lambda)$, for all $\imath_1, \imath_2 \in \mathbb{R}$,
2) $D_\alpha(\lambda) = 0$, for all constant functions $\varphi(t) = \lambda$,
3) $D_\alpha(\varphi\Lambda) = \varphi D_\alpha(\Lambda) + \Lambda D_\alpha(\varphi)$,
4) $D_\alpha\left(\frac{\varphi}{\Lambda}\right) = \frac{D_\alpha(\varphi)\Lambda + D_\alpha(\Lambda)\varphi}{\Lambda^2}$,
5) If φ is differentiable, then

$$D_\alpha(\varphi)(t) = t^{1-\alpha}\frac{df}{dt}(t).$$

Also,

a) $D_\alpha(1) = 0$
b) $D_\alpha(e^{at}) = at^{1-\alpha}e^{at}$, $\imath_1 \in \mathbb{R}$
c) $D_\alpha(\sin(at)) = at^{1-\alpha}\cos(at)$, $\imath_1 \in \mathbb{R}$
d) $D_\alpha(\cos(at)) = -at^{1-\alpha}\sin(at)$, $\imath_1 \in \mathbb{R}$

e) $D_\alpha\left(\frac{1}{\alpha}t^\alpha\right)=1$
f) $D_\alpha\left(\sin\left(\frac{t^\alpha}{\alpha}\right)\right)=\cos\left(\frac{t^\alpha}{\alpha}\right)$
g) $D_\alpha\left(\cos\left(\frac{t^\alpha}{\alpha}\right)\right)=-\sin\left(\frac{t^\alpha}{\alpha}\right)$
h) $D_\alpha\left(e^{\frac{t^\alpha}{\alpha}}\right)=e^{\frac{t^\alpha}{\alpha}}$.

Theorem 4 *(Mean value theorem for conformable fractional differentiable functions). Let $\alpha\in(0,1]$ and $\varphi:[0,\infty)\to\mathbb{R}$ be a continuous on $[\imath_1,\imath_2]$ and an $\alpha-$fractional differentiable mapping on $(\imath_1,\imath_2)$ with $0\le\imath_1<\imath_2$. Then, there exist $c\in(\imath_1,\imath_2)$, such that*

$$D_\alpha(\varphi)(c)=\frac{\varphi(\imath_2)-\varphi(\imath_1)}{\frac{\imath_2}{\alpha}-\frac{\imath_1}{\alpha}}.$$

Definition 5 *(Conformable fractional integral). Let $\alpha\in(0,1]$ and $0\le\imath_1<\imath_2$. A fucntion $\varphi:[0,\infty)\to\mathbb{R}$ is $\alpha-$fractional integrable on $[\imath_1,\imath_2]$ if the integral*

$$\int_{\imath_1}^{\imath_2}\varphi(x)\,d_\alpha x:=\int_{\imath_1}^{\imath_2}\varphi(x)\,x^{\alpha-1}dx,$$

exists and is finite. All $\alpha-$fractional integrable on $[\imath_1,\imath_2]$ is indicated by $L_\alpha^1([\imath_1,\imath_2])$.

Remark 6

$$I_\alpha^{\imath_1}(\varphi)(t)=I_1^{\imath_1}\left(t^{\alpha-1}\varphi\right)=\int_{\imath_1}^{t}\frac{\varphi(x)}{x^{1-\alpha}}dx,$$

where the integral is the usual Riemann improper integral and $\alpha\in(0,1]$.

Theorem 7 *Let $\varphi:(\imath_1,\imath_2)\to\mathbb{R}$ be differentiable and $\alpha\in(0,1]$. Then, for all $t>\imath_1$ we have*

$$I_\alpha^{\imath_1}D_\alpha^{\imath_1}(\varphi)(t)=\varphi(t)-\varphi(\imath_1).$$

Theorem 8 *(Integration by parts). Let $\varphi,\Lambda:[\imath_1,\imath_2]\to\mathbb{R}$ be two functions such that φg is differentiable. Then,*

$$\begin{aligned}&\int_{\imath_1}^{\imath_2}\varphi(x)\,D_\alpha^{\imath_1}(\Lambda)(x)\,d_\alpha x\\ =\;&\varphi g|_{\imath_1}^{\imath_2}-\int_{\imath_1}^{\imath_2}\Lambda(x)\,D_\alpha^{\imath_1}(\varphi)(x)\,d_\alpha x.\end{aligned}$$

Theorem 9 *Assume that $\varphi:[\imath_1,\infty)\to\mathbb{R}$ such that $\varphi^{(n)}(t)$ is continuous and $\alpha\in(n,n+1]$. Then, for all $t>\imath_1$ we have*

$$D_\alpha^{\imath_1}(\varphi)(t)\,I_\alpha^{\imath_1}=\varphi(t).$$

Theorem 10 *Let $\alpha\in(0,1]$ and $\varphi:[\imath_1,\imath_2]\to\mathbb{R}$ be a continuous on $[\imath_1,\imath_2]$ with $0\le\imath_1<\imath_2$. Then,*

$$\left|I_\alpha^{\imath_1}(\varphi)(x)\right|\le I_\alpha^{\imath_1}|\varphi|(x).$$

Many studies in the literature on integral inequalities related to conformable fractional integration have been performed by many researchers. For more details and properties concerning the conformable integral operators, we refer, for example,to the works [18]-[21]. In this paper, we establish the Bullen type inequalities for conformable fractional integral and we will investigate some integral inequalities connected with Bullen-type inequalities for conformable fractional integral. The results presented here would provide generalizations of those given. In this study, some new Identity and Bullen type integral inequalities for differentiable functions are established, and are applied to produce some inequalities of special means."

3 Bullen Type Inequalities for Conformable Fractional Integral.

By using the following lemma, we will give some integral inequalities connected with Bullen-type inequalities for conformable fractional integral.

Lemma 11 *Let $\alpha \in (0,1]$ and $\varphi : I \subset \mathbb{R}^{+} \to \mathbb{R}$ be an $\alpha-$fractional differentiable function on $(\imath_1, \imath_2)$ with $0 \leq \imath_1 < \imath_2$. If $D_\alpha(\varphi)$ be an $\alpha-$fractional integrable function on $[\imath_1, \imath_2]$, then the following identity for conformable fractional integral holds:*

$$\begin{aligned} & \frac{1}{4}\int_0^1 (1-2t^\alpha)\left[D_\alpha(\varphi)\left(t^\alpha \imath_1^\alpha + (1-t^\alpha)\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)\right. \\ & \left. + D_\alpha(\varphi)\left(t^\alpha\frac{\imath_1^\alpha+\imath_2^\alpha}{2} + (1-t^\alpha)\imath_2^\alpha\right)\right] d_\alpha t \\ = & \frac{\alpha}{\imath_2^\alpha - \imath_1^\alpha}\int_{\imath_1}^{\imath_2}\varphi(x^\alpha)\, d_\alpha x - \frac{1}{2}\left[\frac{\varphi(\imath_1^\alpha)+\varphi(\imath_2^\alpha)}{2} + \varphi\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)\right] \end{aligned} \tag{3}$$

Proof. Integrating by parts

$$\begin{aligned} & \int_0^1 (1-2t^\alpha) D_\alpha(\varphi)\left(t^\alpha \imath_1^\alpha + (1-t^\alpha)\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right) d_\alpha t \\ & + \int_0^1 (1-2t^\alpha) D_\alpha(\varphi)\left(t^\alpha\frac{\imath_1^\alpha+\imath_2^\alpha}{2} + (1-t^\alpha)\imath_2^\alpha\right) d_\alpha t \\ = & (1-2t^\alpha)\varphi\left(t^\alpha \imath_1^\alpha + (1-t^\alpha)\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)\Big|_0^1 \\ & + 2\alpha\int_0^1 \varphi\left(t^\alpha \imath_1^\alpha + (1-t^\alpha)\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right) d_\alpha t \\ & + (1-2t^\alpha)\varphi\left(t^\alpha\frac{\imath_1^\alpha+\imath_2^\alpha}{2} + (1-t^\alpha)\imath_2^\alpha\right)\Big|_0^1 \\ & + 2\alpha\int_0^1 \varphi\left(t^\alpha\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\imath_1^\alpha + (1-t^\alpha)\imath_2^\alpha\right) d_\alpha t \end{aligned}$$

$$\begin{aligned}
&= -\varphi\left(\imath_1^{\alpha}\right)-\varphi\left(\frac{\imath_1^{\alpha}+\imath_2^{\alpha}}{2}\right)+\frac{4\alpha}{\imath_1^{\alpha}-\imath_2^{\alpha}}\int_{\left(\frac{\imath_1^{\alpha}+\imath_2^{\alpha}}{2}\right)^{\frac{1}{\alpha}}}^{\imath_1}\varphi\left(x^{\alpha}\right)d_{\alpha}x\\
&\quad -\varphi\left(\imath_2^{\alpha}\right)-\varphi\left(\frac{\imath_1^{\alpha}+\imath_2^{\alpha}}{2}\right)+\frac{4\alpha}{\imath_2^{\alpha}-\imath_1^{\alpha}}\int_{\left(\frac{\imath_1^{\alpha}+\imath_2^{\alpha}}{2}\right)^{\frac{1}{\alpha}}}^{\imath_2}\varphi\left(x^{\alpha}\right)d_{\alpha}x\\
&= -\left[\varphi\left(\imath_1^{\alpha}\right)+\varphi\left(\imath_2^{\alpha}\right)+2f\left(\frac{\imath_1^{\alpha}+\imath_2^{\alpha}}{2}\right)\right]+\frac{4\alpha}{\imath_2^{\alpha}-\imath_1^{\alpha}}\int_{\imath_1}^{\imath_2}\varphi\left(x^{\alpha}\right)d_{\alpha}x.
\end{aligned}$$

■

Remark 12 *If we choose $\alpha = 1$ in the Lemma 11, then we obtain the Lemma 1.*

Theorem 13 *Let $\alpha \in (0,1]$ and $\varphi : I \subset R^{+} \to R$ be an $\alpha-$fractional differentiable function on I° and $D_{\alpha}(\varphi)$ be an $\alpha-$fractional integrable function on I with $0 \leq \imath_1 < \imath_2$. If $|\varphi'|$ be a convex function on I, then the following inequality for conformable fractional integral holds:*

$$\begin{aligned}
&\left|\frac{\alpha}{\imath_2^{\alpha}-\imath_1^{\alpha}}\int_{\imath_1}^{\imath_2}\varphi\left(x^{\alpha}\right)d_{\alpha}x-\frac{1}{2}\left[\frac{\varphi\left(\imath_1^{\alpha}\right)+\varphi\left(\imath_2^{\alpha}\right)}{2}+\varphi\left(\frac{\imath_1^{\alpha}+\imath_2^{\alpha}}{2}\right)\right]\right|\\
&\leq \frac{\left(\imath_2^{\alpha}-\imath_1^{\alpha}\right)}{32}\left[\left(\imath_1^{\alpha}\right)^{\alpha-1}\left|D_{\alpha}\left(\varphi\right)\left(\imath_1^{\alpha}\right)\right|\right.\\
&\quad \left.+2\left(\frac{\imath_1^{\alpha}+\imath_2^{\alpha}}{2}\right)^{\alpha-1}\left|D_{\alpha}\left(\varphi\right)\left(\frac{\imath_1^{\alpha}+\imath_2^{\alpha}}{2}\right)\right|+\left(\imath_2^{\alpha}\right)^{\alpha-1}\left|D_{\alpha}\left(\varphi\right)\left(\imath_2^{\alpha}\right)\right|\right].
\end{aligned}$$

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Proof. Since $|\varphi'|$ is a convex function on I, by the using the properties $D_{\alpha}\left(\varphi \circ \Lambda\right)(t) = \varphi'\left(\Lambda\left(t\right)\right)D_{\alpha}\left(\Lambda\left(t\right)\right)$ and $D_{\alpha}\left(\varphi\left(t\right)\right) = t^{1-\alpha}\varphi'\left(t\right)$ and by Lemma 11 and using the well known absolute value inequality, and we have

$$\begin{aligned}
&\left|\frac{\alpha}{\imath_2^{\alpha}-\imath_1^{\alpha}}\int_{\imath_1}^{\imath_2}\varphi\left(x^{\alpha}\right)d_{\alpha}x-\frac{1}{2}\left[\frac{\varphi\left(\imath_1^{\alpha}\right)+\varphi\left(\imath_2\right)}{2}+\varphi\left(\frac{\imath_1^{\alpha}+\imath_2^{\alpha}}{2}\right)\right]\right|\\
&\leq \frac{1}{4}\int_0^1\left|1-2t^{\alpha}\right|\left|D_{\alpha}\left(\varphi\right)\left(t^{\alpha}\imath_1^{\alpha}+\left(1-t^{\alpha}\right)\frac{\imath_1^{\alpha}+\imath_2^{\alpha}}{2}\right)\right|d_{\alpha}t\\
&\quad +\frac{1}{4}\int_0^1\left|1-2t^{\alpha}\right|\left|D_{\alpha}\left(\varphi\right)\left(t^{\alpha}\frac{\imath_1^{\alpha}+\imath_2^{\alpha}}{2}+\left(1-t^{\alpha}\right)\imath_2^{\alpha}\right)\right|d_{\alpha}t\\
&\leq \frac{\alpha\left(\imath_2^{\alpha}-\imath_1^{\alpha}\right)}{8}\left(\left(\imath_1^{\alpha}\right)^{\alpha-1}\left|D_{\alpha}\left(\varphi\right)\left(\imath_1\right)\right|\int_0^1\left|1-2t^{\alpha}\right|t^{\alpha}d_{\alpha}t\right)\\
&\quad +\frac{\alpha\left(\imath_2^{\alpha}-\imath_1^{\alpha}\right)}{8}\left(\left(\frac{\imath_1^{\alpha}+\imath_2^{\alpha}}{2}\right)^{\alpha-1}\left|D_{\alpha}\left(\varphi\right)\left(\frac{\imath_1^{\alpha}+\imath_2^{\alpha}}{2}\right)\right|\int_0^1\left|1-2t^{\alpha}\right|\left(1-t^{\alpha}\right)d_{\alpha}t\right)\\
&\quad +\frac{\alpha\left(\imath_2^{\alpha}-\imath_1^{\alpha}\right)}{8}\left(\left(\frac{\imath_1^{\alpha}+\imath_2^{\alpha}}{2}\right)^{\alpha-1}\left|D_{\alpha}\left(\varphi\right)\left(\frac{\imath_1^{\alpha}+\imath_2^{\alpha}}{2}\right)\right|\int_0^1\left|1-2t^{\alpha}\right|t^{\alpha}d_{\alpha}t\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha\left(\imath_2^\alpha-\imath_1^\alpha\right)}{8}\left(\left(\imath_2^\alpha\right)^{\alpha-1}\left|D_\alpha\left(\varphi\right)\left(\imath_2^\alpha\right)\right|\int_0^1\left|1-2t^\alpha\right|\left(1-t^\alpha\right)d_\alpha t\right)\\
=\ &\frac{\left(\imath_2^\alpha-\imath_1^\alpha\right)}{32}\Big[\left(\imath_1^\alpha\right)^{\alpha-1}\left|D_\alpha\left(\varphi\right)\left(\imath_1^\alpha\right)\right|\\
&+2\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)^{\alpha-1}\left|D_\alpha\left(\varphi\right)\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)\right|+\left(\imath_2^\alpha\right)^{\alpha-1}\left|D_\alpha\left(\varphi\right)\left(\imath_2^\alpha\right)\right|\Big].
\end{aligned}
$$

where

$$
\begin{aligned}
&\int_0^1\left|1-2t^\alpha\right|t^\alpha d_\alpha t\\
=\ &\int_0^1\left|1-2t^\alpha\right|t^\alpha t^{\alpha-1}dt\\
=\ &\int_0^{\frac{1}{2^{1/\alpha}}}\left(1-2t^\alpha\right)t^\alpha t^{\alpha-1}dt+\int_{\frac{1}{2^{1/\alpha}}}^1\left(2t^\alpha-1\right)t^\alpha t^{\alpha-1}dt\\
=\ &\frac{1}{4\alpha}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1\left|1-2t^\alpha\right|\left(1-t^\alpha\right)d_\alpha t\\
=\ &\int_0^1\left|1-2t^\alpha\right|\left(1-t^\alpha\right)t^{\alpha-1}dt\\
=\ &\int_0^{\frac{1}{2^{1/\alpha}}}\left(1-2t^\alpha\right)\left(1-t^\alpha\right)t^{\alpha-1}dt+\int_{\frac{1}{2^{1/\alpha}}}^1\left(2t^\alpha-1\right)\left(1-t^\alpha\right)t^{\alpha-1}dt\\
=\ &\frac{1}{4\alpha}.
\end{aligned}
$$

■

Remark 14 *If we choose $\alpha=1$ in Theorem (13), then we obtain the following inequality:*

$$
\begin{aligned}
&\left|\frac{1}{\imath_2-\imath_1}\int_{\imath_1}^{\imath_2}\varphi\left(x\right)dx-\frac{1}{2}\left[\frac{\varphi\left(\imath_1\right)+\varphi\left(\imath_2\right)}{2}+\varphi\left(\frac{\imath_1+\imath_2}{2}\right)\right]\right|\\
\leq\ &\frac{\imath_2-\imath_1}{32}\left[\left|\varphi'\left(\imath_1\right)\right|+2\left|\varphi'\left(\frac{\imath_1+\imath_2}{2}\right)\right|+\left|\varphi'\left(\imath_2\right)\right|\right].
\end{aligned}
$$

Theorem 15 *Let $\alpha\in(0,1]$and $\varphi:I\subset R^+\to R$ be an $\alpha-$fractional differentiable function on $I^\circ$and $D_\alpha(\varphi)$ be an $\alpha-$fractional integrable function on I with $0\leq\imath_1<\imath_2$ and $p,q>1$, $1/p+1/q=1$. If $\left|\varphi'\right|^q$ be a convex function on I, then the following inequality for conformable fractional integral holds:*

$$
\left|\frac{\alpha}{\imath_2^\alpha-\imath_1^\alpha}\int_{\imath_1}^{\imath_2}\varphi\left(x^\alpha\right)d_\alpha x-\frac{1}{2}\left[\frac{\varphi\left(\imath_1^\alpha\right)+\varphi\left(\imath_2^\alpha\right)}{2}+\varphi\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)\right]\right|
$$

$$\leq \frac{\alpha(\imath_2^\alpha-\imath_1^\alpha)}{8}\left(\frac{1}{\alpha(1+p)}\right)^{\frac{1}{p}}\left(\frac{1}{2\alpha}\right)^{\frac{1}{q}}\times$$
$$\left[\left((\imath_1^{q\alpha})^{\alpha-1}(D_\alpha(\varphi)(\imath_1^\alpha))^q+\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)^{q(\alpha-1)}\left(D_\alpha(\varphi)\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)\right)^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)^{q(\alpha-1)}\left(D_\alpha(\varphi)\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)\right)^q+(\imath_2^{q\alpha})^{\alpha-1}(D_\alpha(\varphi)(\imath_2^\alpha))^q)^{\frac{1}{q}}\right].$$

Proof. Since $|\varphi'|^q$ is a convex function on I, by the using the properties $D_\alpha(\varphi\circ\Lambda)(t)=\varphi'(\Lambda(t))D_\alpha(\Lambda(t))$ and $D_\alpha(\varphi(t))=t^{1-\alpha}\varphi'(t)$ and by Lemma 11 and using the well known Hölder inequality, we have

$$\left|\frac{\alpha}{\imath_2-\imath_1}\int_{\imath_1}^{\imath_2}\varphi(x^\alpha)d_\alpha x-\frac{1}{2}\left[\frac{\varphi(\imath_1)+\varphi(\imath_2)}{2}+\varphi\left(\frac{\imath_1+\imath_2}{2}\right)\right]\right|$$
$$\leq \frac{1}{4}\left(\int_0^1(1-2t^\alpha)^p d_\alpha t\right)^{\frac{1}{p}}\left(\int_0^1\left(D_\alpha(\varphi)\left(t^\alpha\imath_1^\alpha+(1-t^\alpha)\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)\right)^q d_\alpha t\right)^{\frac{1}{q}}$$
$$+\frac{1}{4}\left(\int_0^1(1-2t^\alpha)^p d_\alpha t\right)^{\frac{1}{p}}\left(\int_0^1\left(D_\alpha(\varphi)\left(t^\alpha\frac{\imath_1^\alpha+\imath_2^\alpha}{2}+(1-t^\alpha)\imath_2^\alpha\right)\right)^q d_\alpha t\right)^{\frac{1}{q}}$$
$$\leq \frac{\alpha(\imath_2^\alpha-\imath_1^\alpha)}{8}\left(\frac{1}{\alpha(1+p)}\right)^{\frac{1}{p}}\left((\imath_1^{q\alpha})^{\alpha-1}(D_\alpha(\varphi)(\imath_1^\alpha))^q\int_0^1 t^\alpha d_\alpha t\right.$$
$$\left.+\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)^{q(\alpha-1)}\left(D_\alpha(\varphi)\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)\right)^q\int_0^1(1-t^\alpha)d_\alpha t\right)^{\frac{1}{q}}$$
$$+\frac{\alpha(\imath_2^\alpha-\imath_1^\alpha)}{8}\left(\frac{1}{\alpha(1+p)}\right)^{\frac{1}{p}}\left(\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)^{q(\alpha-1)}\left(D_\alpha(\varphi)\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)\right)^q\int_0^1 t^\alpha d_\alpha t\right.$$
$$\left.+(\imath_2^{q\alpha})^{\alpha-1}(D_\alpha(\varphi)(\imath_2^\alpha))^q\int_0^1(1-t^\alpha)d_\alpha t\right)^{\frac{1}{q}}$$
$$= \frac{\alpha(\imath_2^\alpha-\imath_1^\alpha)}{8}\left(\frac{1}{\alpha(1+p)}\right)^{\frac{1}{p}}\left(\frac{1}{2\alpha}\right)^{\frac{1}{q}}\times$$
$$\left[\left((\imath_1^{q\alpha})^{\alpha-1}(D_\alpha(\varphi)(\imath_1^\alpha))^q+\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)^{q(\alpha-1)}\left(D_\alpha(\varphi)\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)\right)^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)^{q(\alpha-1)}\left(D_\alpha(\varphi)\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)\right)^q+(\imath_2^{q\alpha})^{\alpha-1}(D_\alpha(\varphi)(\imath_2^\alpha))^q)^{\frac{1}{q}}\right].$$

where

$$\int_0^1(1-2t^\alpha)^p d_\alpha t = \int_0^1(1-2t^\alpha)^p t^{\alpha-1}dt$$

$$
\begin{aligned}
&= \int_0^{\frac{1}{2^{1/\alpha}}} (1-2t^\alpha)^p t^{\alpha-1} dt + \int_{\frac{1}{2^{1/\alpha}}}^1 (2t^\alpha - 1)^p t^{\alpha-1} dt \\
&= \frac{1}{\alpha(1+p)}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^1 t^\alpha d_\alpha t &= \int_0^1 t^\alpha t^{\alpha-1} dt = \frac{1}{2\alpha} \\
\int_0^1 1 - t^\alpha d_\alpha t &= \int_0^1 (1-t^\alpha) t^{\alpha-1} dt = \frac{1}{2\alpha}
\end{aligned}
$$

■

Corollary 16 *If we choose $\alpha = 1$ in Theorem 15, then we obtain the following inequality.*

$$
\begin{aligned}
&\left| \frac{1}{\imath_2 - \imath_1} \int_{\imath_1}^{\imath_2} \varphi(x)\, dx - \frac{1}{2} \left[\frac{\varphi(\imath_1) + \varphi(\imath_2)}{2} + \varphi\left(\frac{\imath_1 + \imath_2}{2} \right) \right] \right| \\
\leq\; & \frac{\imath_2 - \imath_1}{8} \left(\frac{1}{1+p} \right)^{\frac{1}{p}} \left(\frac{1}{2} \right)^{\frac{1}{q}} \times \\
& \left[\left((\varphi'(\imath_1))^q + \left(\varphi'\left(\frac{\imath_1 + \imath_2}{2} \right) \right)^q \right)^{\frac{1}{q}} + \left(\left(\varphi'\left(\frac{\imath_1 + \imath_2}{2} \right) \right)^q + (\varphi'(\imath_2))^q \right)^{\frac{1}{q}} \right].
\end{aligned}
$$

Remark 17 *If we choose $p = q = 2$ in Corollary 16, then we obtain the following inequality.*

$$
\begin{aligned}
&\left| \frac{1}{\imath_2 - \imath_1} \int_{\imath_1}^{\imath_2} \varphi(x)\, dx - \frac{1}{2} \left[\frac{\varphi(\imath_1) + \varphi(\imath_2)}{2} + \varphi\left(\frac{\imath_1 + \imath_2}{2} \right) \right] \right| \\
\leq\; & \frac{\imath_2 - \imath_1}{8} \frac{1}{6^{1/2}} \times \left[\left((\varphi'(\imath_1))^2 + \left(\varphi'\left(\frac{\imath_1 + \imath_2}{2} \right) \right)^2 \right)^{\frac{1}{2}} \right. \\
& \left. + \left(\left(\varphi'\left(\frac{\imath_1 + \imath_2}{2} \right) \right)^2 + (\varphi'(\imath_2))^2 \right)^{\frac{1}{2}} \right].
\end{aligned}
$$

Theorem 18 *Let $\alpha \in (0,1]$ and $\varphi : I \subset R^+ \to R$ be an $\alpha-$fractional differentiable function on I° and $D_\alpha(\varphi)$ be an $\alpha-$fractional integrable function on I with $0 \leq \imath_1 < \imath_2$ and $q \geq 1$. If $|\varphi'|^q$ be a convex function on I, then the following inequality for conformable fractional integral holds:*

$$
\begin{aligned}
&\left| \frac{\alpha}{\imath_2^\alpha - \imath_1^\alpha} \int_{\imath_1}^{\imath_2} \varphi(x^\alpha)\, d_\alpha x - \frac{1}{2} \left[\frac{\varphi(\imath_1^\alpha) + \varphi(\imath_2^\alpha)}{2} + \varphi\left(\frac{\imath_1^\alpha + \imath_2^\alpha}{2} \right) \right] \right| \\
\leq\; & \frac{\alpha(\imath_2^\alpha - \imath_1^\alpha)}{8} \left(\frac{1}{2\alpha} \right)^{1-\frac{1}{q}} \left(\frac{1}{4\alpha} \right)^{\frac{1}{q}} \times
\end{aligned}
$$

$$\left(\left(\imath_1^{q\alpha}\right)^{\alpha-1}\left(D_\alpha(\varphi)(\imath_1^\alpha)\right)^q+\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)^{q(\alpha-1)}\left(D_\alpha(\varphi)\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)\right)^q\right)^{\frac{1}{q}}$$
$$+\left(\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)^{q(\alpha-1)}\left(D_\alpha(\varphi)\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)\right)^q+\left(\imath_2^{q\alpha}\right)^{\alpha-1}\left(D_\alpha(\varphi)(\imath_2^\alpha)\right)^q\right)^{\frac{1}{q}}.$$

Proof. Since $|\varphi'|^q$ is a convex function on I, by the using the properties $D_\alpha(\varphi\circ\Lambda)(t)=\varphi'(\Lambda(t))D_\alpha(\Lambda(t))$ and $D_\alpha(\varphi(t))=t^{1-\alpha}\varphi'(t)$ and assume that $q\geq 1$, by Lemma 11 and using the well known power-mean inequality, we have

$$\begin{aligned}
&\left|\frac{\alpha}{\imath_2^\alpha-\imath_1^\alpha}\int_{\imath_1}^{\imath_2}\varphi(x^\alpha)\,d_\alpha x-\frac{1}{2}\left[\frac{\varphi(\imath_1^\alpha)+\varphi(\imath_2^\alpha)}{2}+\varphi\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)\right]\right| \\
\leq\ & \frac{\alpha(\imath_2^\alpha-\imath_1^\alpha)}{8}\left(\int_0^1(1-2t^\alpha)\,d_\alpha t\right)^{1-\frac{1}{q}}\times \\
&\left(\left(\imath_1^{q\alpha}\right)^{\alpha-1}\left(D_\alpha(\varphi)(\imath_1^\alpha)\right)^q\int_0^1(1-2t^\alpha)t^\alpha d_\alpha t\right. \\
&\left.+\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)^{q(\alpha-1)}\left(D_\alpha(\varphi)\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)\right)^q\int_0^1(1-2t^\alpha)(1-t^\alpha)\,d_\alpha t\right)^{\frac{1}{q}} \\
&+\frac{\alpha(\imath_2^\alpha-\imath_1^\alpha)}{8}\left(\int_0^1(1-2t^\alpha)\,d_\alpha t\right)^{1-\frac{1}{q}}\times \\
&\left(\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)^{q(\alpha-1)}\left(D_\alpha(\varphi)\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)\right)^q\int_0^1(1-2t^\alpha)t^\alpha d_\alpha t\right. \\
&\left.+\left(\imath_2^{q\alpha}\right)^{\alpha-1}\left(D_\alpha(\varphi)(\imath_2)\right)^q\int_0^1(1-2t^\alpha)(1-t^\alpha)\,d_\alpha t\right)^{\frac{1}{q}} \\
=\ & \frac{\alpha(\imath_2^\alpha-\imath_1^\alpha)}{8}\left(\frac{1}{2\alpha}\right)^{1-\frac{1}{q}}\left(\frac{1}{4\alpha}\right)^{\frac{1}{q}}\times \\
&\left(\left(\imath_1^{q\alpha}\right)^{\alpha-1}\left(D_\alpha(\varphi)(\imath_1^\alpha)\right)^q+\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)^{q(\alpha-1)}\left(D_\alpha(\varphi)\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)\right)^q\right)^{\frac{1}{q}} \\
&+\left(\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)^{q(\alpha-1)}\left(D_\alpha(\varphi)\left(\frac{\imath_1^\alpha+\imath_2^\alpha}{2}\right)\right)^q+\left(\imath_2^{q\alpha}\right)^{\alpha-1}\left(D_\alpha(\varphi)(\imath_2^\alpha\imath_2)\right)^q\right)^{\frac{1}{q}}.
\end{aligned}$$

where

$$\int_0^1(1-2t^\alpha)\,d_\alpha t=\int_0^{\frac{1}{2^{1/\alpha}}}(1-2t^\alpha)t^{\alpha-1}dt+\int_{\frac{1}{2^{1/\alpha}}}^1(2t^\alpha-1)t^{\alpha-1}dt=\frac{1}{2\alpha}$$

and

$$\int_0^1(1-2t^\alpha)t^\alpha d_\alpha t$$

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$$= \int_0^{\frac{1}{2^{1/\alpha}}} (1-2t^\alpha)\, t^\alpha t^{\alpha-1} dt + \int_{\frac{1}{2^{1/\alpha}}}^1 (2t^\alpha - 1)\, t^\alpha t^{\alpha-1} dt = \frac{1}{4\alpha},$$

$$\int_0^1 (1-2t^\alpha)(1-t^\alpha)\, d_\alpha t$$

$$= \int_0^{\frac{1}{2^{1/\alpha}}} (1-2t^\alpha)(1-t^\alpha)\, t^{\alpha-1} dt + \int_{\frac{1}{2^{1/\alpha}}}^1 (2t^\alpha - 1)(1-t^\alpha)\, t^{\alpha-1} dt = \frac{1}{4\alpha}.$$

■

Corollary 19 *If we choose $\alpha = 1$ in Theorem 18, then we obtain the following inequality.*

$$\left| \frac{1}{\imath_2 - \imath_1} \int_{\imath_1}^{\imath_2} \varphi(x)\, dx - \frac{1}{2} \left[\frac{\varphi(\imath_1) + \varphi(\imath_2)}{2} + \varphi\left(\frac{\imath_1 + \imath_2}{2}\right) \right] \right|$$
$$\leq \frac{\imath_2 - \imath_1}{8} \left(\frac{1}{2}\right)^{1+\frac{1}{q}} \times \left[\left((\varphi'(\imath_1))^q + \left(\varphi'\left(\frac{\imath_1+\imath_2}{2}\right)\right)^q \right)^{\frac{1}{q}} \right.$$
$$\left. + \left(\left(\varphi'\left(\frac{\imath_1+\imath_2}{2}\right)\right)^q + (\varphi'(\imath_2))^q \right)^{\frac{1}{q}} \right].$$

Remark 20 *If we choose $q = 1$ in Corollary 19, then we obtain Remark 14.*

4 Applications

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Let

$$A(\imath_1, \imath_2) = \frac{\imath_1 + \imath_2}{2},$$
$$L_p(\imath_1, \imath_2) = \left(\frac{\imath_2^{p+1} - \imath_1^{p+1}}{(p+1)(\imath_2 - \imath_1)} \right)^{1/p}, \imath_1 \neq \imath_2, p \in \mathbb{R}, p \neq -1, 0$$

be the arithmetic mean, generalized logarithmic mean for $\imath_1, \imath_2 > 0$ respectively.

Proposition 21 *Let $s \in (0, 1]$, $\imath_1, \imath_2 > 0$, then*

$$\left| L_n^n(\imath_1^\alpha, \imath_2^\alpha) - \frac{1}{2} \left[A(\imath_1^{n\alpha}, \imath_2^{n\alpha}) + A^n(\imath_1^\alpha, \imath_2^\alpha) \right] \right|$$
$$\leq \frac{n(\imath_2^\alpha - \imath_1^\alpha)}{16} \left[A\left((\imath_1^\alpha)^{n-1}, (\imath_2^\alpha)^{n-1} \right) + A^{n-1}(\imath_1^\alpha, \imath_2^\alpha) \right].$$

Proof. The claim follows from Theorem 13 applied to convex function $\varphi(\varkappa) = \varkappa^n$ where $n \in \mathbb{N}$. ■

Example 22 *If we take* $\alpha = 1, n = 2$ *in Proposition 21, then we can obtain inequality following*

$$\begin{aligned} & \left| L_2^2(\imath_1, \imath_2) - A\left[A(\imath_1, \imath_2), A^2(\imath_1, \imath_2)\right]\right| \\ \leq \; & \frac{(\imath_2 - \imath_1)}{4} A(\imath_1, \imath_2). \end{aligned}$$

Proposition 23 *Let* $\alpha \in (0,1]$, $\imath_1, \imath_2 > 0$, $p, q > 1$, *then*

$$\begin{aligned} & \left| L_n^n(\imath_1^\alpha, \imath_2^\alpha) - \frac{1}{2}\left[A(\imath_1^{n\alpha}, \imath_2^{n\alpha}) + A^n(\imath_1^\alpha, \imath_2^\alpha)\right]\right| \\ \leq \; & \frac{n\alpha(\imath_2^\alpha - \imath_1^\alpha)}{8}\left(\frac{1}{\alpha(1+p)}\right)^{\frac{1}{p}}\left(\frac{1}{2\alpha}\right)^{\frac{1}{q}} \times \\ & \left[\left(\left((\imath_1^\alpha)^{n-1}\right)^q + \left(\left(\frac{\imath_1^\alpha + \imath_2^\alpha}{2}\right)^{n-1}\right)^q\right)^{\frac{1}{q}}\right. \\ & \left. + \left(\left(\left(\frac{\imath_1^\alpha + \imath_2^\alpha}{2}\right)^{n-1}\right)^q + \left((\imath_2^\alpha)^{n-1}\right)^q\right)^{\frac{1}{q}}\right]. \end{aligned}$$

Proof. The claim follows from Theorem 15 applied to convex $\varphi(\varkappa) = \varkappa^n$ where $n \in \mathbb{N}$. ■

Example 24 *If we take* $\alpha = 1$, $p = q = n = 2$ *in Proposition 23, then we can obtain the following inequality*

$$\begin{aligned} & \left| L_2^2(\imath_1, \imath_2) - \frac{1}{2}\left[A\left(\imath_1^2, \imath_2^2\right) + A^2(\imath_1, \imath_2)\right]\right| \\ \leq \; & \frac{\imath_2 - \imath_1}{4} \times \left[\left(\imath_1^2 + \left(\frac{\imath_1 + \imath_2}{2}\right)^2\right)^{\frac{1}{2}} + \left(\left(\frac{\imath_1 + \imath_2}{2}\right)^2 + \imath_2^2\right)\right]^{\frac{1}{2}}. \end{aligned}$$

Proposition 25 *Let* $\alpha \in (0,1]$, $\imath_1, \imath_2 > 0$, $q \geq 1$, *then*

$$\begin{aligned} & \left| L_n^n(\imath_1^\alpha, \imath_2^\alpha) - \frac{1}{2}\left[A(\imath_1^{n\alpha}, \imath_2^{n\alpha}) + A^n(\imath_1^\alpha, \imath_2^\alpha)\right]\right| \\ \leq \; & \frac{\alpha(\imath_2^\alpha - \imath_1^\alpha)}{8}\left(\frac{1}{2\alpha}\right)^{1-\frac{1}{q}}\left(\frac{1}{4\alpha}\right)^{\frac{1}{q}} \times \\ & \left[\left(\left((\imath_1^\alpha)^{n-1}\right)^q + \left(\left(\frac{\imath_1^\alpha + \imath_2^\alpha}{2}\right)^{n-1}\right)^q\right)^{\frac{1}{q}}\right. \\ & \left. + \left(\left(\left(\frac{\imath_1^\alpha + \imath_2^\alpha}{2}\right)^{n-1}\right)^q + \left((\imath_2^\alpha)^{n-1}\right)^q\right)^{\frac{1}{q}}\right]. \end{aligned}$$

Proof. The claim follows from Theorem 18 applied to convex $\varphi(\varkappa) = \varkappa^n$ where $n \in \mathbb{N}$. ■

Example 26 *If we take $\alpha = 1, q = 2,\ n = 2$ in Proposition 25, then we can obtain inequality following*

$$
\begin{aligned}
&\left| L_2^2(\imath_1, \imath_2) - \frac{1}{2}\left[A\left(\imath_1^2, \imath_2^2\right) + A^2(\imath_1, \imath_2)\right]\right| \\
\leq\ & \frac{(\imath_2 - \imath_1)}{8}\left(\frac{1}{8}\right)^{\frac{1}{2}} \times \\
& \left(\imath_1^2 + \left(\frac{\imath_1 + \imath_2}{2}\right)^2\right)^{\frac{1}{2}} + \left(\left(\frac{\imath_1 + \imath_2}{2}\right)^2 + \imath_2^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

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