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Research Article

Traveling wave solutions in a higher dimensional lattice competition-cooperation system with stage structure

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Abstract

In this paper, we consider the existence of traveling wave solutions in a higher dimensional lattice competition-cooperation system with stage structure. We first construct a pair of upper and lower solutions. The upper solutions are allowed to be larger than positive equilibrium point. Then we establish the existence of traveling wave solutions by means of cross iterative and Schauder's fixed point theorem.

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1. Introduction

Lattice differential equations are the discrete versions of reaction-diffusion equations. In past few years, many authors have paid their attention on the existence of traveling wave solutions for lattice differential equations, see [3, 4, 7, 8, 13, 16, 17, 21] for one or two dimensional lattices and [15, 20, 22] for higher dimensional lattices, and also see the results for reaction-diffusion equations with or without stage structure [1, 2, 5, 6, 9, 10, 14, 18, 19].

In this paper we are concerned with the existence of traveling wave solutions of a higher dimensional lattice competition-cooperation system with stage structure

$$\begin{cases} u'_{1\eta}(t) = D_1(\Delta_n u_1)_\eta + \alpha_1 e^{-\gamma_1\tau_1} u_{1\eta}(t-\tau_1) - a_1 u^2_{1\eta}(t) - b_1 u_{1\eta}(t) u_{2\eta}(t), \\ u'_{2\eta}(t) = D_2(\Delta_n u_2)_\eta + \alpha_2 e^{-\gamma_2\tau_2} u_{2\eta}(t-\tau_2) + b_2 u_{1\eta}(t) u_{2\eta}(t) - a_2 u^2_{2\eta}(t), \end{cases} \tag{1.1}$$

where all the parameters are positive constants, $t > 0, (\Delta_n w)_\eta = \sum_{|\xi-\eta|=1,\xi\in\mathbb{Z}^n} w_\xi - 2nw_\eta$, $\eta \in \mathbb{Z}^n, |\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n, n \in \mathbb{Z}^+$. System (1.1) has four equilibria

$$\mathbf{0} = (0,0), \quad \Big(\frac{\alpha_1 e^{-\gamma_1\tau_1}}{a_1}, 0\Big), \quad \Big(0, \frac{\alpha_2 e^{-\gamma_2\tau_2}}{a_2}\Big), \quad \mathbf{K} = (k_1, k_2),$$

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where

$$k_1 = \frac{a_2\alpha_1 e^{-\gamma_1\tau_1} - b_1\alpha_2 e^{-\gamma_2\tau_2}}{a_1a_2 + b_1b_2} > 0, \quad k_2 = \frac{a_1\alpha_2 e^{-\gamma_2\tau_2} + b_2\alpha_1 e^{-\gamma_1\tau_1}}{a_1a_2 + b_1b_2} > 0$$

provided that

$$a_2\alpha_1 e^{-\gamma_1\tau_1} > b_1\alpha_2 e^{-\gamma_2\tau_2}. \tag{1.2}$$

We are interested in the existence of traveling wave solutions of higher dimensional lattice (1.1) connecting $\mathbf{0}$ with $\mathbf{K}$. In this case, the reaction terms of (1.1) satisfy partial monotonicity. We notice that there are some existence results about traveling wave solutions for systems with partial monotonicity. For example, Huang et al. [8, 9] considered the existence of traveling wave solutions of continuous and discrete reaction-diffusion systems with partial monotonicity by using upper and lower solutions and Schauder's fixed point theorem; Li et al. [12] considered the existence and asymptotic behavior of traveling wave solutions of competition-cooperation system on 1D lattice. Recently, Li et al. [11] considered the existence of traveling wave solutions of diffusive and competition-cooperation system with stage structure. However, their results are not applied to system (1.1). Hence we need to extend the above methods to higher dimensional lattice system (1.1).

Motivated by techniques in [8, 9, 11, 12], we will establish the existence of traveling wave solutions of system (1.1) connecting $\mathbf{0}$ with $\mathbf{K}$ by Schauder's fixed point theorem and upper and lower solutions method. The upper solutions are allowed to be larger than positive equilibrium point, which is different from [8, 9].

This paper is organized as follows. In Section 2, we first construct a pair of upper and lower solutions and prove the continuity and compactness of operators, then we establish the existence of traveling wave solutions by means of the cross iterative and Schauder's fixed point theorem.

2. The existence of traveling wave solutions

In this paper, we use the usual notations for the standard ordering in $\mathbb{R}^2$. That is, for $u = (u_1, u_2)$ and $v = (v_1, v_2)$, we denote $u \le v$ if $u_i \le v_i, i = 1, 2$, and $u < v$ if $u \le v$ but $u \ne v$. In particular, we denote $u \ll v$ if $u \le v$ but $u_i \ne v_i, i = 1, 2$. If $u \le v$, we also denote $(u, v] = \{\omega \in \mathbb{R}^2, u < \omega \le v\}, [u, v) = \{\omega \in \mathbb{R}^2, u \le \omega < v\}$, $[u, v] = \{\omega \in \mathbb{R}^2, u \le \omega \le v\}$. In the following, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^2$ or $\mathbb{R}^n$ and $\|\cdot\|$ denotes the supremum norm in $C([-\tau, 0], \mathbb{R}^2)$.

Let

$$C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^2) = \{(\phi, \psi) \in C(\mathbb{R}, \mathbb{R}^2) : \mathbf{0} \le (\phi(s), \psi(s)) \le \mathbf{M}, s \in \mathbb{R}\},$$

where $\mathbf{K} \le \mathbf{M} := (M_1, M_2)$.

Denote

$$\begin{cases} f_1(u_\eta(s), v_\eta(s)) = \alpha_1 e^{-\gamma_1\tau_1} u_\eta(-\tau_1) - a_1 u_\eta^2(0) - b_1 u_\eta(0) v_\eta(0), \\ f_2(u_\eta(s), v_\eta(s)) = \alpha_2 e^{-\gamma_2\tau_2} v_\eta(-\tau_2) + b_2 u_\eta(0) v_\eta(0) - a_2 v_\eta^2(0). \end{cases}$$

Lemma 2.1. *For the functional $f(\phi, \psi) = (f_1(\phi, \psi), f_2(\phi, \psi))$,*

(A) *there exist $L_i > 0$ such that*

$$|f_i(u_{1\eta}, v_{1\eta}) - f_i(u_{2\eta}, v_{2\eta})| \le L_i \parallel U - V \parallel$$

for $U = (u_{1\eta}, v_{1\eta}), V = (u_{2\eta}, v_{2\eta}) \in C([-\tau, 0], \mathbb{R}^2)$ with $\boldsymbol{0} \le (u_{i\eta}(s), v_{i\eta}(s)) \le \boldsymbol{M}, s \in [-\tau, 0], i = 1, 2$,

and it satisfies partially quasimonotone condition:

(PQM) *there exist $\beta_1 > 0$ and $\beta_2 > 0$ such that*

$$f_1(u_{1\eta}(s), v_{1\eta}(s)) - f_1(u_{2\eta}(s), v_{1\eta}(s)) + \beta_1[u_{1\eta}(0) - u_{2\eta}(0)] \ge 2nD_1[u_{1\eta}(0) - u_{2\eta}(0)],$$
$$f_1(u_{1\eta}(s), v_{1\eta}(s)) - f_1(u_{1\eta}(s), v_{2\eta}(s)) \le 0,$$
$$f_2(u_{1\eta}(s), v_{1\eta}(s)) - f_2(u_{2\eta}(s), v_{2\eta}(s)) + \beta_2[v_{1\eta}(0) - v_{2\eta}(0)] \ge 2nD_2[v_{1\eta}(0) - v_{2\eta}(0)]$$

for $u_{i\eta}(s), v_{i\eta}(s) \in C([-\tau, 0], \mathbb{R}), i = 1, 2,$ *with*
$\mathbf{0} \le (u_{2\eta}(s), v_{2\eta}(s)) \le (u_{1\eta}(s), v_{1\eta}(s)) \le \mathbf{M}$ *for* $s \in [-\tau, 0]$.

Proof. (A) is obvious. Next we check (PQM). For any $(u_{1\eta}, v_{1\eta}), (u_{2\eta}, v_{2\eta}) \in C([-\tau, 0], \mathbb{R}^2)$ with $\mathbf{0} \le (u_{2\eta}(s), v_{2\eta}(s)) \le (u_{1\eta}(s), v_{1\eta}(s)) \le \mathbf{M}$, let $\beta_1 > 0$ and $\beta_2 > 0$ satisfying $\beta_1 > 2a_1M_1 + b_1M_2 + 2nD_1$ and $\beta_2 > 2a_2M_2 + 2nD_2$, we have

$$
\begin{aligned}
f_1(u_{1\eta}, v_{1\eta}) &- f_1(u_{2\eta}, v_{1\eta}) - 2nD_1[u_{1\eta}(0) - u_{2\eta}(0)] \\
&\ge \{-a_1[u_{1\eta}(0) + u_{2\eta}(0)] - b_1 v_{1\eta}(0)\}[u_{1\eta}(0) - u_{2\eta}(0)] - 2nD_1[u_{1\eta}(0) - u_{2\eta}(0)] \\
&\ge -(2a_1M_1 + b_1M_2 + 2nD_1)[u_{1\eta}(0) - u_{2\eta}(0)] \\
&\ge -\beta_1[u_{1\eta}(0) - u_{2\eta}(0)],
\end{aligned}
$$

$$
f_1(u_{1\eta}, v_{1\eta}) - f_1(u_{1\eta}, v_{2\eta}) = -b_1 u_{1\eta}(0)[v_{1\eta}(0) - v_{2\eta}(0)] \le 0,
$$

$$
\begin{aligned}
f_2(u_{1\eta}, v_{1\eta}) &- f_2(u_{2\eta}, v_{2\eta}) - 2nD_2[u_{1\eta}(0) - u_{2\eta}(0)] \\
&\ge -a_2[v_{1\eta}(0) + v_{2\eta}(0)][v_{1\eta}(0) - v_{2\eta}(0)] - 2nD_2[v_{1\eta}(0) - v_{2\eta}(0)] \\
&\ge -(2a_2M_2 + 2nD_2)[v_{1\eta}(0) - v_{2\eta}(0)] \\
&\ge -\beta_2[v_{1\eta}(0) - v_{2\eta}(0)].
\end{aligned}
$$

□

We give the definition of traveling wave solutions of (1.1) .

Definition 2.2. A traveling wave solution of (1.1) has special form $u_{1\eta}(t) = \phi(\sigma \cdot \eta + ct), u_{2\eta}(t) = \psi(\sigma \cdot \eta + ct)$, where $\phi(\pm\infty)$ and $\psi(\pm\infty)$ both exist, $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n) \in \mathbb{R}^n$ is a unit vector, $c > 0$ is the wave speed, (ϕ, ψ) is the wave profile.

Denoting $\sigma \cdot \eta + ct$ by t, we search for the solutions of system

$$
\begin{cases}
c\phi'(t) = D_1 \sum_{k=1}^{n} [\phi(t + \sigma_k) - 2\phi(t) + \phi(t - \sigma_k)] + f_1^c(\phi_t, \psi_t), \\
c\psi'(t) = D_2 \sum_{k=1}^{n} [\psi(t + \sigma_k) - 2\psi(t) + \psi(t - \sigma_k)] + f_2^c(\phi_t, \psi_t)
\end{cases}
\tag{2.1}
$$

satisfying

$$
\lim_{t \to -\infty} (\phi(t), \psi(t)) = \mathbf{0}, \quad \lim_{t \to +\infty} (\phi(t), \psi(t)) = \mathbf{K}, \tag{2.2}
$$

where $\phi_t(s) = \phi(t + s), \psi_t(s) = \psi(t + s), s \in [-\tau, 0], \tau = \max\{\tau_1, \tau_2\}, f_i^c(\phi, \psi), i = 1, 2,$ is defined by

$$
\begin{cases}
f_1^c(\phi_t, \psi_t) = \alpha_1 e^{-\gamma_1\tau_1} \phi(t - c\tau_1) - a_1\phi^2(t) - b_1\phi(t)\psi(t), \\
f_2^c(\phi_t, \psi_t) = \alpha_2 e^{-\gamma_2\tau_2} \psi(t - c\tau_2) + b_2\phi(t)\psi(t) - a_2\psi^2(t).
\end{cases}
$$

Define $F = (F_1, F_2) : C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^2) \to C(\mathbb{R}, \mathbb{R}^2)$ by

$$
F_i(\phi, \psi)(t) = \frac{1}{c} e^{-\frac{\beta_i}{c}t} \int_{-\infty}^{t} e^{\frac{\beta_i}{c}s} H_i(\phi, \psi)(s) ds, \quad i = 1, 2,
$$

where $H = (H_1, H_2) : C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^2) \to C(\mathbb{R}, \mathbb{R}^2)$ is defined by

$$
\begin{cases}
H_1(\phi, \psi)(t) = f_1^c(\phi_t, \psi_t) + \beta_1\phi(t) + D_1 \sum_{k=1}^{n} [\phi(t + \sigma_k) - 2\phi(t) + \phi(t - \sigma_k)], \\
H_2(\phi, \psi)(t) = f_2^c(\phi_t, \psi_t) + \beta_2\psi(t) + D_2 \sum_{k=1}^{n} [\psi(t + \sigma_k) - 2\psi(t) + \psi(t - \sigma_k)].
\end{cases}
$$

Then F is well defined and for any $(\phi, \psi) \in C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^2)$, we have

$$
c[F_i(\phi, \psi)]'(t) = -\beta_i F_i(\phi, \psi)(t) + H_i(\phi, \psi)(t), \quad i = 1, 2. \tag{2.3}
$$

We only need to find a fixed point of (2.3) satisfying (2.2).

Let $\mu \in (0, \min\{\beta_1/c, \beta_2/c\})$ and equip $C(\mathbb{R}, \mathbb{R}^2)$ with the norm $|\cdot|_\mu$ defined by

$$|\Phi|_\mu = \sup_{t\in\mathbb{R}} |\Phi(t)|e^{-\mu|t|} \quad \text{and} \quad B_\mu(\mathbb{R}, \mathbb{R}^2) = \{\Phi \in C(\mathbb{R}, \mathbb{R}^2) : \sup_{t\in\mathbb{R}} |\Phi(t)|e^{-\mu|t|} < \infty\}.$$

Then it is easy to check that $(B_\mu(\mathbb{R}, \mathbb{R}^2), |\cdot|_\mu)$ is a Banach space.

Definition 2.3. A pair of functions $\bar{\Phi} = (\bar{\phi}, \bar{\psi}), \underline{\Phi} = (\underline{\phi}, \underline{\psi}) \in C(\mathbb{R}, \mathbb{R}^2)$ is called a weak upper solution and a weak lower solution of (2.1), respectively, if there exist finitely many constants $T_i, i = 1, \cdots, p$ such that $\bar{\Phi}$ and $\underline{\Phi}$ are differentiable in $\mathbb{R} \setminus T$, $T := \{T_i : i = 1, \cdots, p\}$ and satisfy

$$\begin{cases} c\bar{\phi}'(t) \geq D_1 \sum_{k=1}^{n} [\bar{\phi}(t+\sigma_k) - 2\bar{\phi}(t) + \bar{\phi}(t-\sigma_k)] + f_1^c(\bar{\phi}_t, \underline{\psi}_t) \quad \text{for } t \in \mathbb{R} \setminus T, \\ c\bar{\psi}'(t) \geq D_2 \sum_{k=1}^{n} [\bar{\psi}(t+\sigma_k) - 2\bar{\psi}(t) + \bar{\psi}(t-\sigma_k)] + f_2^c(\bar{\phi}_t, \bar{\psi}_t) \quad \text{for } t \in \mathbb{R} \setminus T, \end{cases} \tag{2.4}$$

and

$$\begin{cases} c\underline{\phi}'(t) \leq D_1 \sum_{k=1}^{n} [\underline{\phi}(t+\sigma_k) - 2\underline{\phi}(t) + \underline{\phi}(t-\sigma_k)] + f_1^c(\underline{\phi}_t, \bar{\psi}_t) \quad \text{for } t \in \mathbb{R} \setminus T, \\ c\underline{\psi}'(t) \leq D_2 \sum_{k=1}^{n} [\underline{\psi}(t+\sigma_k) - 2\underline{\psi}(t) + \underline{\psi}(t-\sigma_k)] + f_2^c(\underline{\phi}_t, \underline{\psi}_t) \quad \text{for } t \in \mathbb{R} \setminus T. \end{cases} \tag{2.5}$$

Next we will construct a pair of upper and lower solutions of (2.1) satisfying

(A1) $\mathbf{0} \leq (\underline{\phi}(t), \underline{\psi}(t)) \leq (\bar{\phi}(t), \bar{\psi}(t)) \leq \mathbf{M}, t \in \mathbb{R}$;

(A2) $\lim_{t\to-\infty} (\bar{\phi}(t), \bar{\psi}(t)) = \mathbf{0}, \quad \lim_{t\to+\infty} (\underline{\phi}(t), \underline{\psi}(t)) = \lim_{t\to+\infty} (\bar{\phi}(t), \bar{\psi}(t)) = \mathbf{K}$.

To do this, we need to assume

$$a_1 k_1 > b_1 k_2, \tag{2.6}$$

which implies (1.2) holds.

Similar to [11, 12], it is easy to prove the following lemma.

Lemma 2.4. *Let*

$$\Delta_i(\lambda, c) := D_i \sum_{k=1}^{n} (e^{\lambda\sigma_k} + e^{-\lambda\sigma_k} - 2) - c\lambda + \alpha_i e^{-\gamma_i\tau_i} e^{-\lambda c\tau_i}, \quad i = 1, 2.$$

Then, there exist $c_1^* > 0$ and $c_2^* > 0$ such that $\Delta_1(\lambda, c) = 0$ and $\Delta_2(\lambda, c) = 0$, respectively, have only two positive roots $0 < \lambda_1 < \lambda_3$ and $0 < \lambda_2 < \lambda_4$, and

$$\Delta_1(\lambda, c) \begin{cases} > 0, & \lambda < \lambda_1 \text{ or } \lambda > \lambda_3, \\ < 0, & \lambda_1 < \lambda < \lambda_3, \end{cases} \quad \text{and} \quad \Delta_2(\lambda, c) \begin{cases} > 0, & \lambda < \lambda_2 \text{ or } \lambda > \lambda_4, \\ < 0, & \lambda_2 < \lambda < \lambda_4, \end{cases}$$

for $0 < c < c_i^*$, $\Delta_i(\lambda, c) = 0$ has no real roots on $\mathbb{R}, i = 1, 2$.

Now we construct a pair of upper and lower solutions for $c > c^* := \max\{c_1^*, c_2^*\}$.

For fixed

$$\nu \in \Big(1, \min\Big\{2, \frac{\lambda_3}{\lambda_1}, \frac{\lambda_4}{\lambda_2}, \frac{\lambda_1 + \lambda_2}{\lambda_1}\Big\}\Big),$$

consider $h_i(t) = e^{\lambda_i t} - qe^{\nu\lambda_i t}$, i=1,2, where $q > 1$ is large enough. By direct calculation, it shows that $h_i(t)$ has a unique global maximum $\varrho_i = \varrho_i(q) > 0$ at $t_i^* = t_i^*(q) = -\frac{1}{(\nu-1)\lambda_i} \ln q\nu < 0$, and

$$\lim_{q\to\infty} \varrho_i(q) = \lim_{q\to\infty} e^{\lambda_i t_i^*(q)} = \lim_{q\to\infty} qe^{\nu\lambda_i t_i^*(q)} = 0, \quad i = 1, 2.$$

We also have $h_i(t)$ is increasing in $t \leq t_i^*$ and decreasing in $t \geq t_i^*$. So $h_i(t) = A_i$ has only two different roots t_{i*} and $t_i(t_{i*} < t_i^* < t_i)$ and $t_i - t_{i*} > \tilde{t}$, where $A_i = e^{\lambda_i(t_i^*-\tilde{t})} - qe^{\nu\lambda_i(t_i^*-\tilde{t})} > 0, \tilde{t} = \max\{1, c\tau_1, c\tau_2\}, i = 1, 2$.

For any given $\lambda > 0$, there exists a unique $\varepsilon_i = \varepsilon_i(\lambda) > 0$ such that

$$k_i - k_i(1-\varepsilon_i)e^{-\lambda t_i} = A_i, \quad i = 1, 2.$$

Then $\lim\limits_{\lambda\to 0} k_i\varepsilon_i(\lambda) = A_i$ and

$$e^{\lambda_i t} - qe^{\nu\lambda_i t} \geq k_i - k_i(1-\varepsilon_i)e^{-\lambda t}, \quad t_{i*} \leq t \leq t_i, \quad i = 1, 2. \tag{2.7}$$

We can see that for sufficiently small λ and large enough $q > 1$,

$$0 < \max\{\varepsilon_1, \varepsilon_2\} \ll \min\Big\{1, \frac{1}{a_2k_2}, \frac{a_1k_1 - b_1k_2}{a_1k_1}\Big\}.$$

Define the continuous functions as follows:

$$\bar{\phi}(t) = \begin{cases} e^{\lambda_1 t}, & t \leq t_3, \\ k_1 + k_1e^{-\lambda t}, & t > t_3, \end{cases} \qquad \bar{\psi}(t) = \begin{cases} e^{\lambda_2 t} + qe^{\nu\lambda_2 t}, & t \leq t_4, \\ k_2 + k_2e^{-\lambda t}, & t > t_4, \end{cases}$$

and

$$\underline{\phi}(t) = \begin{cases} e^{\lambda_1 t} - qe^{\nu\lambda_1 t}, & t \leq t_1, \\ k_1 - k_1(1-\varepsilon_1)e^{-\lambda t}, & t > t_1, \end{cases} \qquad \underline{\psi}(t) = \begin{cases} e^{\lambda_2 t} - qe^{\nu\lambda_2 t}, & t \leq t_2, \\ k_2 - k_2(1-\varepsilon_2)e^{-\lambda t}, & t > t_2, \end{cases}$$

where $\lambda > 0$ is small enough and $q > 1$ is large enough, which will be determined later. Then

$$M_1 := \sup_{t\in\mathbb{R}} \bar{\phi}(t) > k_1, \quad M_2 := \sup_{t\in\mathbb{R}} \bar{\psi}(t) > k_2,$$

$\bar{\phi}(t), \bar{\psi}(t), \underline{\phi}(t)$ and $\underline{\psi}(t)$ satisfy (A1) and (A2) and

$$\max\{t_1 + c\tau_1, t_2 + c\tau_2\} \ll t_4 \ll \min\{0, t_3\}$$

for small enough $\lambda > 0$ and large enough $q > 1$. By the choice of ν, we have $\Delta_i(\nu\lambda_i, c) < 0, i = 1, 2$.

Lemma 2.5. *Assume that (2.6) holds. Then $(\bar{\phi}(t), \bar{\psi}(t))$ is a weak upper solution and $(\underline{\phi}(t), \underline{\psi}(t))$ is a weak lower solution of (2.1), respectively.*

Proof. Without loss of generality, assume $\sigma_k > 0$, otherwise, we only need to distinguish them from positive, negative or zero. Define

$$\begin{aligned} P(\phi,\psi)(t) \;&:=\; D_1\sum_{k=1}^{n}[\phi(t+\sigma_k) - 2\phi(t) + \phi(t-\sigma_k)] - c\phi'(t) \\ &\quad +\alpha_1e^{-\gamma_1\tau_1}\phi(t-c\tau_1) - a_1\phi^2(t) - b_1\phi(t)\psi(t), \\ Q(\phi,\psi)(t) \;&:=\; D_2\sum_{k=1}^{n}[\psi(t+\sigma_k) - 2\psi(t) + \psi(t-\sigma_k)] - c\psi'(t) \\ &\quad +\alpha_2e^{-\gamma_2\tau_2}\psi(t-c\tau_2) + b_2\phi(t)\psi(t) - a_2\psi^2(t). \end{aligned}$$

We have two cases to verify for $\bar{\phi}(t)$.

(i) For $t < t_3$, since $\bar{\phi}(t\pm\sigma_k) \leq e^{\lambda_1(t\pm\sigma_k)}$ and $\bar{\phi}(t-c\tau_1) \leq e^{\lambda_1(t-c\tau_1)}$, it follows that

$$\begin{aligned} P(\bar{\phi}, \underline{\psi})(t) \;&\leq\; D_1\sum_{k=1}^{n}[\bar{\phi}(t+\sigma_k) - 2\bar{\phi}(t) + \bar{\phi}(t-\sigma_k)] - c\bar{\phi}'(t) + \alpha_1e^{-\gamma_1\tau_1}\bar{\phi}(t-c\tau_1) \\ &\leq\; e^{\lambda_1 t}\Delta_1(\lambda_1, c) = 0. \end{aligned}$$

(ii) For $t > t_3$, since $t_3 \gg t_2 + c\tau_1, \bar{\phi}(t \pm \sigma_k) \leq k_1 + k_1 e^{-\lambda(t\pm\sigma_k)}$ and $\bar{\phi}(t - c\tau_1) \leq k_1 + k_1 e^{-\lambda(t-c\tau_1)}$, we have

$$\begin{aligned} P(\bar{\phi}, \underline{\psi})(t) &\leq e^{-\lambda t}\Big\{k_1\Big[D_1 \sum_{k=1}^{n}(e^{\lambda\sigma_k} - 2 + e^{-\lambda\sigma_k}) + c\lambda\Big] \\ &\quad + k_1\Big[\alpha_1 e^{-\gamma_1\tau_1} e^{\lambda c\tau_1} - a_1 k_1(2 + e^{-\lambda t}) - b_1 k_2(\varepsilon_2 - (1-\varepsilon_2)e^{-\lambda t})\Big]\Big\} \\ &:= e^{-\lambda t} I_1(\lambda). \end{aligned}$$

$I_1(\lambda) < 0$ for sufficiently small λ since $I_1(0) = 2k_1(b_1k_2 - a_1k_1 - b_1k_2\varepsilon_2) < 0$ by (2.6).

For $\bar{\psi}(t)$, we also have two cases to verify.

(i) For $t < t_4$, because of $t_4 \to -\infty$ as $q \to \infty$, we have

$$\Pi_1(q) := \frac{1}{q} e^{(\frac{\lambda_1+\lambda_2}{\lambda_1} - \nu)\lambda_1 t} + e^{\lambda_1 t} \to 0 \text{ as } q \to \infty.$$

Since $\bar{\psi}(t\pm\sigma_k) \leq e^{\lambda_2(t\pm\sigma_k)} + q e^{\nu\lambda_2(t\pm\sigma_k)}, \bar{\psi}(t-c\tau_2) \leq e^{\lambda_2(t-c\tau_2)} + q e^{\nu\lambda_2(t-c\tau_2)}$ and $\bar{\phi}(t) \leq e^{\lambda_1 t}$, it follows that for sufficiently large $q > 1$,

$$\begin{aligned} Q(\bar{\phi}, \bar{\psi})(t) &\leq D_2 \sum_{k=1}^{n}[\bar{\psi}(t+\sigma_k) - 2\bar{\psi}(t) + \bar{\psi}(t-\sigma_k)] - c\bar{\psi}'(t) + \alpha_2 e^{-\gamma_2\tau_2}\bar{\psi}(t - c\tau_2) + b_2\bar{\phi}(t)\bar{\psi}(t) \\ &\leq q e^{\nu\lambda_2 t}[\Delta_2(\nu\lambda_2, c)] + b_2\Pi_1(q)] \leq 0. \end{aligned}$$

(ii) For $t > t_4$, since $\bar{\phi}(t) \leq k_1 + k_1 e^{-\lambda t}$ and $\bar{\psi}(t \pm \sigma_k) \leq k_2 + k_2 e^{-\lambda(t\pm\sigma_k)}, \bar{\psi}(t - c\tau_2) \leq k_2 + k_2 e^{-\lambda(t-c\tau_2)}$, we have

$$\begin{aligned} Q(\bar{\phi}, \bar{\psi})(t) &\leq k_2 e^{-\lambda t}\Big[\sum_{k=1}^{n} D_2(e^{\lambda\sigma_k} - 2 + e^{-\lambda\sigma_k}) + c\lambda \\ &\quad + \alpha_2 e^{-\gamma_2\tau_2} e^{\lambda c\tau_2} + (b_2k_1 - a_2k_2)(2 + e^{-\lambda t})\Big] \\ &:= e^{-\lambda t} I_2(\lambda). \end{aligned}$$

$I_2(\lambda) < 0$ for sufficiently small λ since $I_2(0) = 2k_2(b_2k_1 - a_2k_2) = -2k_2\alpha_2 e^{-\gamma_2\tau_2} < 0$.

We have two cases to verify for $\underline{\phi}(t)$.

(i) For $t < t_1 < 0$, because of $t_1 \to -\infty$ as $q \to \infty$, we have

$$\Pi_2(q) := \frac{a_1}{q} e^{(2-\nu)\lambda_1 t} + b_1\Big(\frac{1}{q} e^{(\frac{\lambda_1+\lambda_2}{\lambda_1} - \nu)\lambda_1 t} + e^{(\lambda_1 + \nu(\lambda_2 - \lambda_1))t}\Big) \to 0 \text{ as } q \to \infty.$$

Since $\underline{\phi}(t \pm \sigma_k) \geq e^{\lambda_1(t\pm\sigma_k)} - q e^{\nu\lambda_1(t\pm\sigma_k)}, \bar{\phi}(t - c\tau_1) \geq e^{\lambda_1(t-c\tau_1)} - q e^{\nu\lambda_1(t-c\tau_1)}$ and $\bar{\psi}(t) \leq e^{\lambda_2 t} + q e^{\nu\lambda_2 t}$, it follows that for sufficiently large $q > 1$,

$$\begin{aligned} P(\underline{\phi}, \bar{\psi})(t) &\geq -q e^{\nu\lambda_1 t}\Delta_1(\nu\lambda_1, c) + (e^{\lambda_1 t} - q e^{\nu\lambda_1 t})[-a_1(e^{\lambda_1 t} - q e^{\nu\lambda_1 t}) - b_1(e^{\lambda_2 t} + q e^{\nu\lambda_2 t})] \\ &\geq -q e^{\nu\lambda_1 t}[\Delta_1(\nu\lambda_1, c) + r_1\Pi_2(q)] \geq 0. \end{aligned}$$

(ii) For $t > t_1$, since $\bar{\psi}(t) \leq k_2 + k_2 e^{-\lambda t}$ and $\underline{\phi}(t \pm \sigma_k) \geq k_1 - k_1(1-\varepsilon_1)e^{-\lambda(t\pm\sigma_k)}, \underline{\phi}(t - c\tau_1) \geq k_1 - k_1(1-\varepsilon_1)e^{-\lambda(t-c\tau_1)}$ by (2.7) and $t_1 - t_{1*} > \tilde{t}$, we have

$$\begin{aligned} P(\underline{\phi}, \bar{\psi})(t) &\geq e^{-\lambda t}\Big\{-k_1(1-\varepsilon_1)\Big[D_1 \sum_{k=1}^{n}(e^{\lambda\sigma_k} - 2 + e^{-\lambda\sigma_k}) + c\lambda\Big] - k_1(1-\varepsilon_1)\alpha_1 e^{-\gamma_1\tau_1} e^{\lambda c\tau_1} \\ &\quad - a_1k_1^2[-2(1-\varepsilon_1) + (1-\varepsilon_1)^2 e^{-\lambda t}] - b_1k_1k_2[\varepsilon_1 - (1-\varepsilon_1)e^{-\lambda t}]\Big\} \\ &:= e^{-\lambda t} I_3(\lambda). \end{aligned}$$

$I_3(\lambda) > 0$ for sufficiently small λ since $I_3(0) = \varepsilon_1 k_1(a_1k_1 - b_1k_2 - a_1k_1\varepsilon_1) > 0$ by (2.6) and $\varepsilon_1 < \frac{a_1k_1 - b_1k_2}{a_1k_1}$.

For $\underline{\psi}(t)$, we only have two cases to verify.

(i) For $t < t_2 < 0$, since $\underline{\psi}(t \pm \sigma_k) \geq e^{\lambda_2(t\pm\sigma_k)} - qe^{\nu\lambda_2(t\pm\sigma_k)}, \underline{\psi}(t - c\tau_2) \geq e^{\lambda_2(t-c\tau_2)} - qe^{\nu\lambda_2(t-c\tau_2)}$ and $\underline{\psi}(t) \leq e^{\lambda_2 t}$, for sufficiently large $q > 1$, we have

$$\begin{aligned}Q(\underline{\phi}, \underline{\psi})(t) &\geq D_2 \sum_{k=1}^{n} [\underline{\psi}(t + \sigma_k) - \underline{\psi}(t) + \underline{\psi}(t - \sigma_k)] - c\underline{\psi}'(t) + \alpha_2 e^{-\gamma_2\tau_2}\underline{\psi}(t - c\tau_2) - a_2\underline{\psi}^2(t)\\ &\geq -qe^{\nu\lambda_2 t}\Big[\Delta_2(\nu\lambda_2, c) + \frac{a_2}{q}e^{(2-\nu)\lambda_2 t}\Big] \geq 0.\end{aligned}$$

(ii) For $t > t_2$, we can divide this case into two subcases: (a) $t_2 < t \leq \max\{t_1, t_2\}$, (b) $t > \max\{t_1, t_2\}$. In fact, if $t_2 \geq t_1$, it only has case (b).

(a) In view of $t_4 \to 0$ as $q \to \infty$, $\underline{\psi}(t \pm \sigma_k) \geq k_2 - k_2(1 - \varepsilon_2)e^{-\lambda(t\pm\sigma_k)}, \underline{\psi}(t - c\tau_2) \geq k_2 - k_2(1 - \varepsilon_2)e^{-\lambda(t-c\tau_2)}$ by (2.7) and $t_2 - t_{2*} > \tilde{t}$, it follows that

$$\begin{aligned}Q(\underline{\phi}, \underline{\psi})(t) \geq &- k_2(1 - \varepsilon_2)[D_2 \sum_{k=1}^{n} (e^{\lambda\sigma_k} - 2 + e^{-\lambda\sigma_k}) + c\lambda]e^{-\lambda t}\\ &+ \alpha_2 e^{-\gamma_2\tau_2}[k_2 - k_2(1 - \varepsilon_2)e^{-\lambda(t-c\tau_2)}] - a_2[k_2 - k_2(1 - \varepsilon_2)e^{-\lambda t}]^2\\ &\to \alpha_2 e^{-\gamma_2\tau_2}A_2 - a_2A_2^2 \text{ as } \lambda \to 0.\end{aligned}$$

Then $Q(\underline{\phi}, \underline{\psi})(t) \geq 0$ for sufficiently small $\lambda > 0$ since λ is independent of q and $A_2 \to 0$ as $q \to \infty$.

(b) we can get

$$\begin{aligned}Q(\underline{\phi}, \underline{\psi})(t) \geq &e^{-\lambda t}\Big\{-k_2(1 - \varepsilon_2)\Big[D_2 \sum_{k=1}^{n} (e^{\lambda\sigma_k} - 2 + e^{-\lambda\sigma_k}) + c\lambda\Big]\\ &- k_2(1 - \varepsilon_2)\alpha_2 e^{-\gamma_2\tau_2}e^{\lambda c\tau_2} + b_1k_1k_2[-(1 - \varepsilon_1) - (1 - \varepsilon_2) + (1 - \varepsilon_1)(1 - \varepsilon_2)e^{-\lambda t}]\\ &- a_2k_2^2[-2(1 - \varepsilon_2) + (1 - \varepsilon_2)^2e^{-\lambda t}]\Big\}\\ :=&e^{-\lambda t}I_4(\lambda).\end{aligned}$$

$I_4(\lambda) > 0$ for sufficiently small λ since $I_4(0) = \varepsilon_2k_2(1 - a_2k_2\varepsilon_2 + b_2k_1\varepsilon_1) > 0$ by $a_2k_2 - b_2k_1 = \alpha_2 e^{-\gamma_2\tau_2}$ and $\varepsilon_2 < \frac{1}{a_2k_2}$.

□

Define

$$\Gamma([\underline{\phi}, \underline{\psi}], [\bar{\phi}, \bar{\psi}]) = \Big\{(\phi, \psi) \in C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^2) | (\underline{\phi}(t), \underline{\psi}(t)) \leq (\phi(t), \psi(t)) \leq (\bar{\phi}(t), \bar{\psi}(t)), \quad t \in \mathbb{R}\Big\}.$$

Obviously, $\Gamma([\underline{\phi}, \underline{\psi}], [\bar{\phi}, \bar{\psi}])$ is nonempty.

Similar to the proofs in [8, 9, 11, 12], we have the following lemma.

Lemma 2.6. *F has the following properties:*

(i) $F_1(\phi_2, \psi_1)(t) \leq F_1(\phi_1, \psi_2)(t), \quad F_2(\phi_1, \psi_2)(t) \leq F_2(\phi_2, \psi_1)(t)$ *for* $t \in \mathbb{R}$ *if* $(\phi_i, \psi_i) \in C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^2)$ *satisfy* $\mathbf{0} \leq (\phi_2(t), \psi_2(t)) \leq (\phi_1(t), \psi_1(t)) \leq \mathbf{M}$ *for* $t \in \mathbb{R}, i = 1, 2$;

(ii) $F = (F_1, F_2) : C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^2) \to C(\mathbb{R}, \mathbb{R}^2)$ *is continuous with respect to the norm* $|\cdot|_\mu$ *in* $B_\mu(\mathbb{R}, \mathbb{R}^2)$;

(iii) $F(\Gamma([\underline{\phi}, \underline{\psi}], [\bar{\phi}, \bar{\psi}])) \subset \Gamma([\underline{\phi}, \underline{\psi}], [\bar{\phi}, \bar{\psi}])$;

(iv) $F : \Gamma([\underline{\phi}, \underline{\psi}], [\bar{\phi}, \bar{\psi}]) \to \Gamma([\underline{\phi}, \underline{\psi}], [\bar{\phi}, \bar{\psi}])$ *is compact with respect to the norm* $|\cdot|_\mu$.

By Schauder's fixed point theorem, we have the following existence result.

Theorem 2.7. *Assume that (2.6) holds. Then, for every $c > c^*$, (1.1) has a traveling wave solution $(\phi(\sigma \cdot \eta + ct), \psi(\sigma \cdot \eta + ct))$ with the wave speed c which connects $\mathbf{0}$ with $\mathbf{K}$. Moreover, $\lim_{\xi\to-\infty} (\phi(\xi)e^{-\lambda_1\xi}, \psi(\xi)e^{-\lambda_2\xi}) = (1, 1)$, where $\xi = \sigma \cdot \eta + ct$.*

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