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Research Article

The Bogomolov multiplier of Lie algebras

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Abstract

In this paper, we extend the notion of the Bogomolov multipliers and the CP-extensions to Lie algebras. Then, we compute the Bogomolov multipliers for Abelian, Heisenberg and nilpotent Lie algebras of class at most 6. Finally, we compute the Bogomolov multipliers of complex simple and semisimple Lie algebras.

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1. Introduction

During the study of continuous transformation groups in the end of 19th century, Sophus Lie found *Lie algebras* as a new algebraic structure. This new structure played an important role in 19th and 20th centuries mathematical physics. (See [20, 28], for more information). *Lie theory* is studying objects like Lie algebras, Lie groups, Root systems, Weyl groups, Linear algebraic groups, etc. and some researches show its emphasis on modern mathematics. (See [5, 20] for more information). Furthermore, it is shown that one can associate a Lie algebra to a continuous or Lie group. For example, Lazard introduced a correspondence between some groups and some Lie algebras. (See [19], for more information). So theories of groups and Lie algebras are structurally similar and many concepts related to groups are defined analogously to Lie algebras. In this paper we want to define the Bogomolov multipliers for Lie algebras. This concept is known for groups and it is a group-theoretical invariant introduced as an obstruction to a problem in algebraic geometry which is called the rationality problem. This problem can be stated in the following way. Let V be a faithful representation of a group G over a field K. Then G acts naturally on the field of rational functions $K(V)$. Now the rationality problem or Noether's problem) can be stated as "is the field of G-invariant functions $K(V)^G$ is rational (purely transcendental) over K?" A question related to the above mentioned is whether there exist independent variables $x_1, ..., x_r$ such that $K(V)^G(x_1, ..., x_r)$ becomes a pure transcendental extension of K? Saltman in [25] give some examples of groups of order p^9 for which the answer to the Noether's problem was negative, even when taking $K = \mathbb{C}$.

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He used the notion of the unramified cohomology group $H^2_{nr}(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$. Bogomolov in [4] proved that it is canonically isomorphic to

$$B_0(G) = \bigcap \ker\{res^A_G : H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(A, \mathbb{Q}/\mathbb{Z})\},$$

where A is an abelian subgroup of G. The group $B_0(G)$ is a subgroup of the Schur multiplier $\mathcal{M}(G) = H^2(G, \mathbb{Q}/\mathbb{Z})$ and Kunyavskii in [18] named it the *Bogomolov multiplier* of G. Thus non triviallity of the Bogomolov multiplier leads to counter-examples to Noether's problem. But it's not always easy to calculate Bogomolov multipliers of groups. Moravec in [22] introduced an equivalent definition of the Bogomolov multiplier. In this sense, he used a notion of the non abelian exterior square $G \wedge G$ of a group G to obtain a new description of the Bogomolov multiplier. He showed that if G is a finite group, then $B_0(G)$ is non-canonically isomorphic to $\mathrm{Hom}(\tilde{B}_0(G), \mathbb{Q}/\mathbb{Z})$, where the group $\tilde{B}_0(G)$ can be described as a section of the non abelian exterior square of the group G. Also, he proved that $\tilde{B}_0(G) \cong \mathcal{M}(G)/\mathcal{M}_0(G)$, such that the Schur multiplier $\mathcal{M}(G)$ or the same $H^2(G, \mathbb{Q}/\mathbb{Z})$ interpreted as the kernel of the commutator homomorphism $G \wedge G \to [G, G]$ given by $x \wedge y \to [x, y]$, and $\mathcal{M}_0(G)$ is the subgroup of $\mathcal{M}(G)$ defined as $\mathcal{M}_0(G) =< x \wedge y \mid [x, y] = 0,\ x, y \in G >$. Thus in the class of finite groups, $\tilde{B}_0(G)$ is non-canonically isomorphic to $B_0(G)$. With this definition and similar to the Schur multiplier, the Bogomolov multiplier can be explained as a measure of the extent to which relations among commutators in a group fail to be consequences of universal relation. Furthermore, Moravec's method relates the Bogomolov multiplier to the concept of commuting probability of a group and shows that the Bogomolov multiplier plays an important role in commutativity preserving central extensions of groups, that are famous cases in K-theory. Now, It is interesting that the analogous theory of commutativity preserving exterior product can be developed to the field of Lie theory. In this paper, we introduce a non abelian commutativity preserving exterior product, and the Bogomolov multiplier of Lie algebras. Then we investigate their properties. Moreover we compute the Bogomolov multiplier for Heisenberg Lie algebras, nilpotent Lie algebras of dimension at most 6 and complex simple and semisimple Lie algebras.

2. Some notations and preliminaries

Let L be a finite dimensional Lie algebra. The following standard notations will be used throughout the paper.

- $[.,.]$ the Lie bracket.
- $L^2 = [L, L]$ the commutator subalgebra of L.
- $H(m)$ the Heisenberg Lie algebra of dimension $2m + 1$.
- $A(n)$ the abelian Lie algebra of dimension n.
- $\mathcal{M}(L) \cong \dfrac{R \cap F^2}{[R, F]}$ the Schur multiplier of L, such that $L \cong \dfrac{F}{R}$.

2.1. Exterior product [8]

Let L be a Lie algebra and M and N be ideals of L. The exterior product $M \wedge N$ is a Lie algebra generated by all symbols $m \wedge n$, subject to the following relations

(i) $\lambda(m \wedge n) = \lambda m \wedge n = m \wedge \lambda n$,
(ii) $(m + m') \wedge n = m \wedge n + m' \wedge n$,
(iii) $m \wedge (n + n') = m \wedge n + m \wedge n'$,
(iv) $[m, m'] \wedge n = m \wedge [m', n] - m' \wedge [m, n]$,
(v) $m \wedge [n, n'] = [n', m] \wedge n - [n, m] \wedge n'$,
(vi) $[(m \wedge n), (m' \wedge n')] = -[n, m] \wedge [m', n']$,
(vii) If $m = n$, then $m \wedge n = 0$,

for all $\lambda \in F$, $m, m' \in M$ and $n, n' \in N$.

2.2. Exterior pairing [8]

Let L be a Lie algebra. A function $\phi : M \times N \to L$ is called an exterior pairing, if we have

(i) $h(\lambda m, n) = h(m, \lambda n) = \lambda h(m, n)$,
(ii) $h(m + m', n) = h(m, n) + h(m', n)$,
(iii) $h(m, n + n') = h(m, n) + h(m, n')$,
(iv) $h([m, m'], n) = h(m, [m', n]) + h(m', [n, m])$,
(v) $h(m, [n, n']) = h([n', m], n) + h([m, n], n')$,
(vi) $[h(m, n), h(m', n')] = h([m, n], [m', n'])$,
(vii) If $m = n$, then $h(m, n) = 0$,

for all $\lambda \in F$, $m, m' \in M$ and $n, n' \in N$.

Note that the function $M \times N \to M \wedge N$ given by $(m, n) \to m \wedge n$ is the universal exterior pairing.

3. The commutativity preserving non abelian exterior product of Lie algebras

In this section, we intend to extend the results of [4, 6, 13, 14, 16, 18, 22] to the theory of Lie algebras.

Definition 3.1. Let K be a Lie algebra and M and N be ideals of K. A bilinear function $h : M \times N \to K$, is called a Lie-$\tilde{B}_0$-pairing, if we have

(i) $h(\lambda m, n) = h(m, \lambda n) = \lambda h(m, n)$,
(ii) $h(m + m', n) = h(m, n) + h(m', n)$,
(iii) $h(m, n + n') = h(m, n) + h(m, n')$,
(iv) $h([m, m'], n) = h(m, [m', n]) - h(m', [m, n])$,
(v) $h(m, [n, n']) = h([n', m], n) - h([n, m], n')$,
(vi) $h([n, m], [m', n']) = -[h(m, n), h(m', n')]$,
(vii) If $[m, n'] = 0$, then $h(m, n') = 0$,

for all $\lambda \in F$, $m, m' \in M$ and $n, n' \in N$.

Definition 3.2. A Lie algebra Homomorphism is a linear map $H \in \mathrm{Hom}(L, M)$ between Lie algebras L and M, such that it is compatible with the Lie bracket, that is

$$H : L \to M \quad , \quad H([x, y]) = [H(x), H(y)].$$

For example any vector space can be made into a Lie algebra with the trivial bracket.

Definition 3.3. A Lie-$\tilde{B}_0$-pairing $h : M \times N \to L$ is called universal, if for any Lie-$\tilde{B}_0$-pairing $h' : M \times N \to L'$, there is a unique Lie homomorphism $\theta : L \to L'$ such that $\theta h = h'$.

The following definition extends the concept of CP exterior product in [22] to the theory of Lie algebras.

Definition 3.4. Let L be a Lie algebra and M and N be ideals of L. The CP exterior product $M \curlywedge N$ is the Lie algebra generated by all symbols $m \curlywedge n$ subject to the following relations

(i) $\lambda(m \curlywedge n) = \lambda m \curlywedge n = m \curlywedge \lambda n$,
(ii) $(m + m') \curlywedge n = m \curlywedge n + m' \curlywedge n$,
(iii) $m \curlywedge (n + n') = m \curlywedge n + m \curlywedge n'$,
(iv) $[m, m'] \curlywedge n = m \curlywedge [m', n] - m' \curlywedge [m, n]$,

(v) $m \curlywedge [n, n'] = [n', m] \curlywedge n - [n, m] \curlywedge n'$,
(vi) $[(m \curlywedge n), (m' \curlywedge n')] = -[n, m] \curlywedge [m', n']$,
(vii) If $[m, n] = 0$, then $m \curlywedge n = 0$,

for all $\lambda \in F$, $m, m' \in M$ and $n, n' \in N$.

In the case $M = N = L$, we call $L \curlywedge L$ the curly exterior product of L.

Proposition 3.5. *The function $h : M \times N \to M \curlywedge N$ given by $(m, n) \longmapsto m \curlywedge n$, is a universal Lie-$\tilde{B}_0$-pairing.*

Proof. By Definitions 3.1, 3.3 and 3.4, the proof is straightforward. □

Theorem 3.6. *Let L be a Lie algebra and M and N be ideals of L. Then we have*

$$M \curlywedge N \cong \frac{M \wedge N}{\mathcal{M}_0(M, N)},$$

where $\mathcal{M}_0(M, N) =< m \wedge n \mid m \in M,\ n \in N,\ [m, n] = 0 >$.

Proof. By using Definition 2.2, the function $h : M \times N \to M \curlywedge N$ given by $(m, n) \longmapsto (m \curlywedge n)$ is an exterior pairing. So it induces a homomorphism $\tilde{h} : M \wedge N \to M \curlywedge N$, given by $(m \wedge n) \longmapsto m \curlywedge n$, for all $m \in M$ and $n \in N$. Clearly $\mathcal{M}_0(M, N) \subseteq \ker \tilde{h}$, so we have the homomorphism $h^* : (M \wedge N)/\mathcal{M}_0(M, N) \to M \curlywedge N$ given by $(m \wedge n) + \mathcal{M}_0(M, N) \longmapsto (m \curlywedge n)$. On the other hand, the map $l^* : M \curlywedge N \to (M \wedge N)/\mathcal{M}_0(M, N)$ given by $(m \curlywedge n) \longmapsto (m \wedge n) + \mathcal{M}_0(M, N)$ is induced by the Lie-$\tilde{B}_0$-pairing $l : M \times N \to (M \wedge N)/\mathcal{M}_0(M, N)$ given by $(m, n) \longmapsto (m \wedge n) + \mathcal{M}_0(M, N)$. Now it is easy to see that $h^* l^* = l^* h^* = 1$. Thus l^* is an isomorphism. □

It is known that $\kappa : M \times N \to [M, N]$ given by $(m, n) \longmapsto [m, n]$ is an exterior pairing. So for all $m \in M$ and $n \in N$, it induces a homomorphism $\tilde{\kappa} : M \wedge N \to [M, N]$, such that $\tilde{\kappa}(m \wedge n) = [m, n]$. Moreover, the kernel of $\tilde{\kappa}$ is denoted by $\mathcal{M}(M, N)$. It can easily seen that $\mathcal{M}_0(M, N) \leq \mathcal{M}(M, N)$, thus there is a homomorphism $\kappa^* : M \wedge N/\mathcal{M}_0(M, N) \to [M, N]$ given by $m \wedge n + \mathcal{M}_0(M, N) \longmapsto [m, n]$, with $\ker \kappa^* \cong \mathcal{M}(M, N)/\mathcal{M}_0(M, N)$. Similar to groups, we denote $\mathcal{M}(M, N)/\mathcal{M}_0(M, N)$ by $\tilde{B}_0(M, N)$, and we call it the Bogomolov multiplier of the pair of Lie algebras (M, N). Therefore, we have an exact sequence

$$0 \to \tilde{B}_0(M, N) \to M \curlywedge N \to [M, N] \to 0.$$

In the case $M = N = L$, $\mathcal{M}_0(L, L) =< l \wedge l' \mid l, l' \in L\ ,\ [l, l'] = 0 >$ and we denote it by $\mathcal{M}_0(L)$.

It is known that the kernel of $\tilde{\kappa} : L \wedge L \to L^2$ given by $l \wedge l' \longmapsto [l, l']$ is the Schur multiplier of L. On the other hand $\mathcal{M}_0(L) \leq \mathcal{M}(L) = \ker \tilde{\kappa}$. So there is a homomorphism $\kappa^* : L \wedge L/\mathcal{M}_0(L) \to L^2$ given by $l \wedge l' + \mathcal{M}_0(L) \longmapsto [l, l']$ and $\ker \kappa^* \cong \mathcal{M}(L)/\mathcal{M}_0(L)$. Similar to groups, we denote $\mathcal{M}(L)/\mathcal{M}_0(L)$ by $\tilde{B}_0(L)$, and we call it the Bogomolov multiplier of the Lie algebra L. So we have an exact sequence

$$0 \to \tilde{B}_0(L) \to L \curlywedge L \to L^2 \to 0.$$

Proposition 3.7. *Let L be a Lie algebra and M, N and K be ideals of L, such that $K \subseteq M \cap N$. Then there is an isomorphism*

$$M/K \curlywedge N/K \cong (M \curlywedge N)/T,$$

where $T =< m \curlywedge n \mid m \in M, n \in N, [m, n] \in K >$.

Proof. The function $\phi : M \times N \to M/K \curlywedge N/K$ given by $(m, n) \to (m + K) \curlywedge (n + K)$ is a well-defined Lie-$\tilde{B}_0$-pairing. Thus there is a homomorphism $\phi^* : M \curlywedge N \to M/K \curlywedge N/K$ with $m \curlywedge n \longmapsto (m + K) \curlywedge (n + K)$. Clearly $T \subseteq \ker \phi^*$, so we have the homomorphism $\psi : (M \curlywedge N)/T \to M/K \curlywedge N/K$ given by $m \curlywedge (n + T) \longmapsto (m + K) \curlywedge (n + K)$. On the other hand, the map $\varphi^* : M/K \curlywedge N/K \to (M \curlywedge N)/T$ given by $(m+K) \curlywedge (n+K) \longmapsto (m \curlywedge n) + T$ is induced by the Lie-$\tilde{B_0}$-pairing $\varphi : M/K \times N/K \to (M \curlywedge N)/T$ given by $(m+K, n+K) \longmapsto (m \curlywedge n) + T$. One can check that, $\varphi^*\psi = \psi\varphi^* = 1$. Thus, φ^* is an isomorphism, and the proof is complete. □

Now, we give the behaviour of the CP exterior product respect to a direct sum of Lie algebras.

Proposition 3.8. *Let L_1 and L_2 be ideals of a Lie algebra L. Then*

$$(L_1 \oplus L_2) \curlywedge (L_1 \oplus L_2) \cong L_1 \curlywedge L_1 \oplus L_2 \curlywedge L_2.$$

Proof. The result obtained by using a similar way to that of [8]. □

4. Hopf-type formula for the Bogomolov multiplier of Lie algebras

Let L be a Lie algebra with a free presentation $L \cong F/R$. By the well-known Hopf formula [8], we have an isomorphism $\mathcal{M}(L) \cong (R \cap F^2)/[R, F]$. Here we intend to give the similar formula for $\tilde{B_0}(L)$.

In the following $K(F)$ denotes $\{[x, y] \mid x, y \in F\}$.

Proposition 4.1. *Let L be a Lie algebra with the free presentation $L \cong F/R$, then*

$$\tilde{B_0}(L) \cong \frac{R \cap F^2}{< K(F) \cap R >}.$$

Proof. From [8], $L \wedge L \cong F^2/[R, F]$ and $L^2 \cong F^2/(R \cap F^2)$. Moreover $\ker \tilde{\kappa} = \mathcal{M}(L) \cong (R \cap F^2)/[R, F]$ and $\mathcal{M}_0(L)$ can be considered as the subalgebra of $F/[R, F]$ generated by all commutators in $F/[R, F]$ that belong to $\mathcal{M}(L)$. Thus we have the following isomorphism for $\mathcal{M}_0(L)$,

$$\mathcal{M}_0(L) \cong < K(\frac{F}{[R, F]}) \cap \frac{R}{[R, F]} > = \frac{< K(F) \cap R > + [R, F]}{[R, F]} = \frac{< K(F) \cap R >}{[R, F]}.$$

Therefore $\tilde{B_0}(L) = \mathcal{M}(L)/\mathcal{M}_0(L) \cong R \cap F^2/< K(F) \cap R >$ as required. □

Proposition 4.2. *Let L be a Lie algebra and M be an ideal of L. Then the sequence*

$$\tilde{B_0}(L) \to \tilde{B_0}(\frac{L}{M}) \to \frac{M}{< K(L) \cap M >} \to \frac{L}{L^2} \to \frac{L/M}{(L/M)^2} \to 0,$$

is exact.

Proof. Suppose $0 \to R \to F \xrightarrow{\pi} L \to 0$ be a free presentation of L and let $T = \ker(F \to L/M)$. We have $M \cong T/R$. The inclusion maps $R \cap F^2 \xrightarrow{f} T \cap F^2$, $T \cap F^2 \xrightarrow{g} T$, $T \xrightarrow{h} F$ and $F \xrightarrow{k} F$ induce the sequence of homomorphisms

$$\frac{R \cap F^2}{< K(F) \cap R >} \xrightarrow{f^*} \frac{T \cap F^2}{< K(F) \cap T >} \xrightarrow{g^*} \frac{T}{< K(F) \cap T > + R} \xrightarrow{h^*} \frac{F}{R + F^2} \xrightarrow{k^*}$$

$\frac{F}{T + F^2} \to 0$. Note that $\frac{T}{< K(F) \cap T > + R} \cong \frac{M}{< K(L) \cap M >}$, $\frac{F}{R + F^2} \cong \frac{L}{L^2}$ and $\frac{F}{T + F^2} \cong \frac{L/M}{(L/M)^2}$. Now by using Proposition 4.1, we have

$\tilde{B_0}(L) \cong \frac{R \cap F^2}{< K(F) \cap R >}$ and $\tilde{B_0}(\frac{L}{M}) \cong \frac{T \cap F^2}{< K(F) \cap T >}$. Moreover,

$\mathrm{im} f^* = \ker g^* = \frac{R \cap F^2}{< K(F) \cap T >}$, $\mathrm{im} g^* = \ker h^* = \frac{T \cap F^2}{< K(F) \cap T > + R}$,

$\mathrm{im} h^* = \ker k^* = \frac{T}{R + F^2}$, and K^* is an epimorphism. Therefore, the above sequence is exact. □

Proposition 4.3. *Let L be a Lie algebra with a free presentation $L \cong F/R$, and M be an ideal of L, such that $T = \ker(F \to L/M)$. Then the sequence*

$$0 \to \frac{R \cap < K(F) \cap T >}{< K(F) \cap R >} \to \tilde{B_0}(L) \to \tilde{B_0}(\frac{L}{M}) \to \frac{M \cap L^2}{< K(L) \cap M >} \to 0,$$

is exact.

Proof. Suppose $0 \to R \to F \xrightarrow{\pi} L \to 0$ be a free presentation of L and let $T = \ker(F \to L/M)$. We have $M \cong T/R$. The inclusion maps

$$R \cap < K(F) \ \cap \ T > \xrightarrow{f} R \ \cap F^2, \quad R \ \cap F^2 \xrightarrow{g} T \cap F^2$$

and the map $T \cap F^2 \xrightarrow{h} (T \cap F^2) + R$ induce the sequence of homomorphisms $0 \to \frac{R \cap < K(F) \cap T >}{< K(F) \cap R >} \xrightarrow{f^*} \frac{R \cap F^2}{< K(F) \cap R >} \xrightarrow{g^*} \frac{T \cap F^2}{< K(F) \cap T >} \xrightarrow{h^*}$ $\frac{(T \cap F^2) + R}{< K(F) \cap T > +R} \to 0$. It is straightforward to verify that $< K(L) \cap M >= \frac{< K(F) \cap T > +R}{R}$ and $M \cap L^2 = \frac{T}{R} \cap \frac{F^2 + R}{R} = \frac{(T \cap F^2) + R}{R}$. Therefore we have

$$\frac{M \cap L^2}{< K(L) \cap M >} = \frac{((T \cap R^2) + R)/R}{(< K(F) \cap T > +R)/R} \cong \frac{(T \cap F^2) + R}{< K(F) \cap T > +R}.$$

Now by using Proposition 4.1, we have $\tilde{B_0}(L) \cong \frac{R \cap F^2}{< K(F) \cap R >}$, $\tilde{B_0}(\frac{L}{M}) \cong \frac{T \cap F^2}{< K(F) \cap T >}$, and $\mathrm{im} f^* = \ker g^* = \frac{R \cap < K(F) \cap T >}{< K(F) \cap R >}$, $\mathrm{im} g^* = \ker h^* = \frac{R \cap F^2}{< K(F) \cap T >}$. Moreover h^* is an epimorphism. Thus, the above sequence is exact. □

For groups, the Schur multiplier is a universal object of central extensions. Recently, parallel to the classical theory of central extensions, Jezernik and Moravec in [13, 14] developed a version of extension that preserve commutativity. They showed that the Bogomolov multiplier is also the universal object parametrizing such extensions for a given group. Now we want to introduce a similar notion for Lie algebras.

Definition 4.4. Let L, M and C be Lie algebras. An exact sequence of Lie algebras $0 \to M \xrightarrow{\chi} C \xrightarrow{\pi} L \to 0$ is called a commutativity preserving extension (CP extension) of M by L, if commuting pairs of elements of L have commuting lifts in C. A special type of CP extension with the central kernel is named a central CP extension.

Proposition 4.5. *Let $e : 0 \to M \xrightarrow{\chi} C \xrightarrow{\pi} L \to 0$ be a central extension. Then e is a CP extension if and only if $\chi(M) \cap K(C) = 0$.*

Proof. Suppose that e is a CP central extension. Let $[c_1, c_2] \in \chi(M) \cap K(C)$, then there is a commuting lift $(c_1', c_2') \in C \times C$ of the commuting pair $(\pi(c_1), \pi(c_2))$, such that $\pi(c_1') = \pi(c_1)$ and $\pi(c_2') = \pi(c_2)$. So for some $a, b \in \chi(M)$, we have $c_1' = c_1 + a$, $c_2' = c_2 + b$. Therefore $0 = [c_1', c_2'] = [c_1 + a, c_2 + b] = [c_1, c_2]$. Hence $\chi(M) \cap K(C) = 0$. Conversely suppose that $\chi(M) \cap K(C) = 0$. Choose $x, y \in L$ with $[x, y] = 0$, we have $x = \pi(c_1)$ and $y = \pi(c_2)$ for some $c_1, c_2 \in C$. Therefore $\pi([c_1, c_2]) = 0$. Hence $[c_1, c_2] \in \chi(M) \cap K(C) = 0$, so $[c_1, c_2] = 0$. Thus the central extension e is a CP extension. □

Definition 4.6. An abelian ideal M of a Lie algebra L is called a CP Lie subalgebra of L if the extension $0 \to M \to L \to \frac{L}{M} \to 0$ is a CP extension.

Moreover, by using Proposition 4.5 an abelian ideal M of a Lie algebra L is a CP Lie subalgebra of L if $M \cap K(L) = 0$.

Now we obtain an explicit formula for the Bogomolov multiplier of a direct product of two Lie algebras. The following lemma gives a free presentation for $L_1 \oplus L_2$, in terms of the given free presentation for L_1 and L_2. It will be used in the rest.

Lemma 4.7. ([24, Lemma 2.1]) *Let L_1 and L_2 be Lie algebras with free presentations F_1/R_1 and F_2/R_2, respectively. Let $F = F_1 * F_2$ be the free product of F_1 and F_2. Then $0 \to R \to F \to L_1 \oplus L_2 \to 0$ is a free presentation for $L_1 \oplus L_2$, where $R = R_1 + R_2 + [F_2, F_1]$.*

Proposition 4.8. *Let L_1 and L_2 be Lie algebras. Then*

$$\tilde{B_0}(L_1 \oplus L_2) \cong \tilde{B_0}(L_1) \oplus \tilde{B_0}(L_2).$$

Proof. By using Lemma 4.7, we have

$$\tilde{B_0}(L_1 \oplus L_2) = \frac{(R_1 + R_2 + [F_2, F_1]) \cap (F_1 * F_2)^2}{< K(F_1 * F_2) \cap (R_1 + R_2 + [F_2, F_1]) >}.$$

Let $F = F_1 * F_2$, then the epimorphism $F \to F_1 \times F_2$ induces the following epimorphism

$$\alpha : \frac{R \cap F^2}{< K(F) \cap R >} \to \frac{R_1 \cap {F_1}^2}{< K(F_1) \cap R_1 >} \oplus \frac{R_2 \cap {F_2}^2}{< K(F_2) \cap R_2 >}$$

$$x+ < K(F) \cap R > \longmapsto (x_1+ < K(F_1) \cap R_1 > \, , \; x_2+ < K(F_2) \cap R_2 >)$$

where $x = x_1 + x_2$, $x_1 \in R_1 \cap {F_1}^2$ and $x_2 \in R_2 \cap {F_2}^2$.
On the other hand

$$\beta : \frac{R_1 \cap {F_1}^2}{< K(F_1) \cap R_1 >} \oplus \frac{R_2 \cap {F_2}^2}{< K(F_2) \cap R_2 >} \to \frac{R \cap F^2}{< K(F) \cap R >}$$

given by

$$(x_1+ < K(F_1) \cap R_1 > \, , \; x_2+ < K(F_2) \cap R_2 >) \longmapsto x+ < K(F) \cap R >$$

is a well-defined homomorphism. It is easy to check that β is a right inverse to α, so α is an epimorphism. Now, we show that α is a monomorphism. Let $x+ < K(F) \cap R > \in \ker \alpha$, such that, $x = t_1 + t_2$. So we have $t_1 \in < K(F_1) \cap R_1 >$ and $t_2 \in < K(F_2) \cap R_2 >$. Since $t_1, t_2 \in < R \cap K(F) >$, then $x \in < K(F) \cap R >$. Thus α is a monomorphism. □

5. Computing the Bogomolov multiplier of Heisenberg Lie algebras

We use the symbol $H(m)$ for the Heisenberg Lie algebra of dimension $2m + 1$. The Heisenberg Lie algebra L is a Lie algebra such that $L^2 = Z(L)$ and $\dim L^2 = 1$. Such Lie algebras are odd dimensional with basis $v_1, \ldots, v_{2m}, v$ and the only non zero multiplications between basis elements are $[v_{2i-1}, v_{2i}] = -[v_{2i}, v_{2i-1}] = v$ for $i = 1, 2, \ldots, m$.

Theorem 5.1. *With the above notations and assumptions $\tilde{B_0}(H(1)) = 0$.*

Proof. Since $H(1) \wedge H(1) =< v_1 \wedge v_2, v_1 \wedge v, v_2 \wedge v >$, an element $w \in \mathcal{M}(H(1)) \leq H(1) \wedge H(1)$ can be written as $w = \alpha_1(v_1 \wedge v_2) + \alpha_2(v_1 \wedge v) + \alpha_3(v_2 \wedge v)$, for $\alpha_1, \alpha_2, \alpha_3 \in F$. Now, considering $\tilde{\kappa} : H(1) \wedge H(1) \to H(1)^2$ with $\ker \tilde{\kappa} = \mathcal{M}(H(1))$, we have $\tilde{\kappa}(w) = 0$, and hence $\alpha_1[v_1, v_2] + \alpha_2[v_1, v] + \alpha_3[v_2, v] = 0$. On the other hand, $[v_1, v] = [v_2, v] = 0$, $[\alpha_1 v_1, v_2] = \alpha_1[v_1, v_2] = \alpha_1 v = 0$. Hence $v_1 \wedge v, v_2 \wedge v, \alpha_1(v_1 \wedge v_2) \in \mathcal{M}_0(H(1))$. Thus $w \in \mathcal{M}_0(H(1))$, and so $\mathcal{M}(H(1)) \subseteq \mathcal{M}_0(H(1))$. Therefore $\tilde{B_0}(H(1)) = 0$. □

Theorem 5.2. *$\tilde{B}_0(H(m)) = 0$, for all $m \geq 2$.*

Proof. We know that

$$H(m) =< v_1, v_2, \dots v_{2m}, v \mid [v_{2i-1}, v_{2i}] = -[v_{2i}, v_{2i-1}] = v, i = 1 \dots m > .$$

We can see that

$$H(m) \wedge H(m) =< v_1 \wedge v_2, v_1 \wedge v_3, \dots, v_1 \wedge v_{2m}, v_2 \wedge v_3, v_2 \wedge v_4, \dots, v_2 \wedge v_{2m}, \dots v_{2m-1} \wedge v_{2m}, v_1 \wedge v, \dots, v_{2m} \wedge v >.$$

Also for all i, $1 \leq i \leq m$, $v_i \wedge v = v_i \wedge [v_{2i-1}, v_{2i}] = [v_{2i}, v_i] \wedge v_{2i-1} - [v_{2i-1}, v_i] \wedge v_{2i} = 0$. Thus $H(m) \wedge H(m) =< v_1 \wedge v_2, v_1 \wedge v_3, \dots, v_1 \wedge v_{2m}, v_2 \wedge v_3, \dots, v_2 \wedge v_{2m}, \dots, v_{2m-1} \wedge v_{2m} >$. Now, for all $w \in \mathcal{M}(H(m)) \leq H(m) \wedge H(m)$, there exist $\alpha_1, \dots \alpha_{2m^2-2m}, \beta_1, \dots, \beta_m \in F$, such that $w = \alpha_1(v_1 \wedge v_3) + \alpha_2(v_1 \wedge v_4) + \dots + \alpha_{2m^2-2m}(v_{2m-2} \wedge v_{2m}) + \beta_1(v_1 \wedge v_2)$ $+\beta_2(v_3 \wedge v_4) + \dots + \beta_m(v_{2m-1} \wedge v_{2m})$. Let $\tilde{\kappa} : H(m) \wedge H(m) \to H(m)^2$ be given by $x \wedge y \to [x, y]$. Since $\tilde{\kappa}(w) = 0$, we have $\alpha_1[v_1, v_3] + \alpha_2[v_1, v_4] + \dots + \alpha_{2m^2-2m}[v_{2m-2}, v_{2m}] + \beta_1[v_1, v_2] + \beta_2[v_3, v_4] + \dots + \beta_m[v_{2m-1}, v_{2m}] = 0$. So, $(\beta_1 + \beta_2 + \dots + \beta_m)v = 0$. Hence, $w = \alpha_1(v_1 \wedge v_3) + \alpha_2(v_1 \wedge v_4) + \dots + \alpha_{2m^2-2m}(v_{2m-2} \wedge v_{2m}) + \beta_1(v_1 \wedge v_2 - v_3 \wedge v_4) + \beta_2(v_3 \wedge v_4 - v_5 \wedge v_6) + \dots + \beta_{m-1}(v_{2m-3} \wedge v_{2m-2} - v_{2m-1} \wedge v_{2m})$. On the other hand, $[v_1, v_3] = [v_1, v_4] = \dots = [v_{2m-2}, v_{2m}] = 0$. Thus $v_1 \wedge v_3, v_1 \wedge v_4, \dots v_{2m-2} \wedge v_{2m} \in M_0(H(m))$. We can see that $[v_1 + v_4, v_2 + v_3] = 0$. So, $(v_1 + v_4) \wedge (v_2 + v_3) \in \mathcal{M}(H(m))$. Hence, $v_1 \wedge v_2 + v_1 \wedge v_3 + v_4 \wedge v_2 + v_4 \wedge v_3 \in \mathcal{M}_0(H(m))$. Thus $(v_1 \wedge v_2) - (v_3 \wedge v_4) \in \mathcal{M}_0(H(m))$. By a same way, we have

$$((v_3 \wedge v_4) - (v_5 \wedge v_6)), \dots, ((v_{2m-3} \wedge v_{2m-2}) - (v_{2m-1} \wedge v_{2m})) \in \mathcal{M}_0(H(m)).$$

Therefore $w \in \mathcal{M}_0(H(m))$ and so $\mathcal{M}(H(m)) \subseteq \mathcal{M}_0(H(m))$. Hence $\tilde{B}_0(H(m)) = 0$ as required. □

Theorem 5.3. *Let L be an n-dimensional Lie algebra with $\dim L^2 = 1$. Then $\tilde{B}_0(L) = 0$.*

Proof. By Lemma 3.3 in [23], $L \cong H(m) \oplus A(n-2m-1)$ for some m. Now using Theorem 5.2 and Proposition 4.8, we have

$$\tilde{B}_0(L) \cong \tilde{B}_0(H(m) \oplus A(n-2m-1)) \cong \tilde{B}_0(H(m)) \oplus \tilde{B}_0(A(n-2m-1)).$$

Since $\tilde{B}_0(H(m)) = \tilde{B}_0(A(n-2m-1)) = 0$, the result follows. □

6. Computing the Bogomolov multiplier of nilpotent Lie algebras of dimension at most 6

This section is devoted to obtain the Bogomolov multiplier for the nilpotent Lie algebras of dimension at most 6. We need the classification of these Lie algebras in [7, 9]. The following results are obtained by using notations and terminology used in [1, 6, 14, 15].

Theorem 6.1. *Let L be a nilpotent Lie algebra of dimension at most 2. Then $\tilde{B}_0(L) = 0$.*

Proof. Since L is abelian, its Bogomolov multiplier is trivial. □

From [9], there are two nilpotent Lie algebras of dimension 3, the abelian one, which we denote by $L_{3,1}$ and $L_{3,2} \cong H(1)$ with basis v, v_1, v_2 and the only non-zero Lie bracket $[v_1, v_2] = v$.

Theorem 6.2. *Let L be a nilpotent Lie algebra of dimension 3. Then $\tilde{B}_0(L) = 0$.*

Proof. $L_{3,1}$ is abelian so $\tilde{B}_0(L_{3,1}) = 0$. Now since $L_{3,2} \cong H(1)$, the result is obtained by using Theorem 5.1. □

From [9], there are three nilpotent Lie algebras of dimension 4, which are isomorphic to $L_{4,1}, L_{4,2}, L_{4,3}$ and $L_{4,k} \cong L_{3,k} \oplus I, k = 1, 2$ (where I is a 1-dimensional abelian ideal). $L_{4,3}$ has the basis x_1, x_2, x_3, x_4, with the only non-zero brackets $[x_1, x_2] = x_3$, $[x_1, x_3] = x_4$.

Theorem 6.3. *Let L be a nilpotent Lie algebra of dimension 4. Then $\tilde{B}_0(L) = 0$.*

Proof. Using Proposition 4.8 and Theorem 6.2, we have

$$\tilde{B}_0(L_{4,k}) \cong \tilde{B}_0(L_{3,k}) \oplus \tilde{B}_0(I) = 0, \text{ for } k = 1, 2.$$

Now let $L \cong L_{4,3} =< x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4 >$, we have $x_2 \wedge x_4 = x_3 \wedge x_4 = 0$. So, $L_{4,3} \wedge L_{4,3} =< x_1 \wedge x_2, x_1 \wedge x_3, x_1 \wedge x_4, x_2 \wedge x_3 >$. Hence, for all $w \in \mathcal{M}(L_{4,3}) \leq L_{4,3} \wedge L_{4,3}$, there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in F$, such that $w = \alpha_1(x_1 \wedge x_2) + \alpha_2(x_1 \wedge x_3) + \alpha_3(x_1 \wedge x_4) + \alpha_4(x_2 \wedge x_3)$. Now, considering $\tilde{\kappa} : L_{4,3} \wedge L_{4,3} \to L^2_{4,3}$ given by $x \wedge y \to [x, y]$. Since $\tilde{\kappa}(w) = 0$, we have $\alpha_1[x_1, x_2] + \alpha_2[x_1, x_3] + \alpha_3[x_1, x_4] + \alpha_4[x_2, x_3] = 0$. So $\alpha_1 x_3 + \alpha_2 x_4 = 0$. On the other hand, $[x_1, x_4] = [x_2, x_3] = [x_2, x_4] = [x_3, x_4] = 0$, $[\alpha_1 x_1, x_2] = \alpha_1[x_1, x_2] = \alpha_1 x_3 = 0$ and $[\alpha_2 x_1, x_3] = \alpha_2[x_1, x_3] = \alpha_2 x_4 = 0$. Hence $(x_1 \wedge x_4), (x_2 \wedge x_3), \alpha_1(x_1 \wedge x_2), \alpha_2(x_1 \wedge x_3) \in \mathcal{M}_0(L_{4,3})$. So $\mathcal{M}(L_{4,3}) \subseteq \mathcal{M}_0(L_{4,3})$. Thus $\tilde{B}_0(L_{4,3}) = 0$. □

From [9], the 5-dimensional Lie algebras are $L_{5,k} \cong L_{4,k} \oplus I$, for $k = 1, 2, 3$. Where I is a 1-dimensional abelian ideal and

- $L_{5,4} \cong< x_1, ..., x_5 \mid [x_1, x_2] = [x_3, x_4] = x_5 >$,
- $L_{5,5} \cong< x_1, ..., x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_5 >$,
- $L_{5,6} \cong< x_1, ..., x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = [x_2, x_3] = x_5 >$,
- $L_{5,7} \cong< x_1, ..., x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5 >$,
- $L_{5,8} \cong< x_1, ..., x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5 >$,
- $L_{5,9} \cong< x_1, ..., x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5 >$.

Theorem 6.4. *Let L be a nilpotent Lie algebra of dimension 5. Then $\tilde{B}_0(L) \neq 0$ if and only if $L \cong L_{5,6}$.*

Proof. Using Theorem 6.3 and Proposition 4.8, $\tilde{B}_0(L_{5,1}) = \tilde{B}_0(L_{5,2}) = \tilde{B}_0(L_{5,3}) = 0$. Let $L \cong L_{5,4}$, one can see that

$$L_{5,4} \wedge L_{5,4} =< x_1 \wedge x_2, x_1 \wedge x_3, x_1 \wedge x_4, x_2 \wedge x_3, x_2 \wedge x_4, x_3 \wedge x_4 > .$$

Hence for all $w \in \mathcal{M}(L_{5,4}) \leq L_{5,4} \wedge L_{5,4}$, there exist $\alpha_1, \alpha_2, \ldots, \alpha_6 \in F$, such that $w = \alpha_1(x_1 \wedge x_2) + \alpha_2(x_1 \wedge x_3) + \alpha_3(x_1 \wedge x_4) + \alpha_4(x_2 \wedge x_3) + \alpha_5(x_2 \wedge x_4) + \alpha_6(x_3 \wedge x_4)$. Since $\tilde{\kappa}(w) = 0$, we have $\alpha_1[x_1, x_2] + \alpha_2[x_1, x_3] + \alpha_3[x_1, x_4] + \alpha_4[x_2, x_3] + \alpha_5[x_2, x_4] + \alpha_6[x_3, x_4] = 0$. Thus $(\alpha_1 + \alpha_6)x_5 = 0$ and $\alpha_1 + \alpha_6 = 0$. Hence $w = \alpha_1((x_1 \wedge x_2) - (x_3 \wedge x_4)) + \alpha_2(x_1 \wedge x_3) + \alpha_3(x_1 \wedge x_4) + \alpha_4(x_2 \wedge x_3) + \alpha_5(x_2 \wedge x_4)$. We can see that $[x_1 + x_4, x_2 + x_3] = 0$, so $(x_1 + x_4) \wedge (x_2 + x_3) \in \mathcal{M}_0(L_{5,4})$. Hence $(x_1 \wedge x_2) + (x_1 \wedge x_3) + (x_4 \wedge x_2) + (x_4 \wedge x_3) \in \mathcal{M}_0(L_{5,4})$, so $(x_1 \wedge x_2) - (x_3 \wedge x_4) \in \mathcal{M}_0(L_{5,4})$. Also we know $(x_1 \wedge x_3), (x_1 \wedge x_4), (x_2 \wedge x_3)$ and $(x_2 \wedge x_4) \in \mathcal{M}_0(L_{5,4})$. Therefore $w \in \mathcal{M}_0(L_{5,4})$ and $\mathcal{M}(L_{5,4}) \subseteq \mathcal{M}_0(L_{5,4})$. Thus $\tilde{B}_0(L_{5,4}) = 0$.

Let $L \cong L_{5,5}$, it can be shown that

$$L_{5,5} \wedge L_{5,5} =< x_1 \wedge x_2, x_1 \wedge x_3, x_1 \wedge x_4, x_2 \wedge x_3, x_2 \wedge x_4, x_3 \wedge x_4 > .$$

Hence for all $w \in \mathcal{M}(L_{5,5}) \leq L_{5,5} \wedge L_{5,5}$, there exist $\alpha_1, \alpha_2, \ldots, \alpha_6 \in F$, such that $w = \alpha_1(x_1 \wedge x_2) + \alpha_2(x_1 \wedge x_3) + \alpha_3(x_1 \wedge x_4) + \alpha_4(x_2 \wedge x_3) + \alpha_5(x_2 \wedge x_4) + \alpha_6(x_3 \wedge x_4)$. Since $\tilde{\kappa}(w) = 0$, we have $\alpha_1[x_1, x_2] + \alpha_2[x_1, x_3] + \alpha_3[x_1, x_4] + \alpha_4[x_2, x_3] + \alpha_5[x_2, x_4] + \alpha_6[x_3, x_4] = 0$. Thus $\alpha_1 x_3 + (\alpha_2 + \alpha_5)x_5 = 0$, and $\alpha_1 = \alpha_2 + \alpha_5 = 0$. Hence $w = \alpha_1(x_1 \wedge x_2) + \alpha_2((x_1 \wedge x_3) - (x_2 \wedge x_4)) + \alpha_3(x_1 \wedge x_4) + \alpha_4(x_2 \wedge x_3)$. On the other hand $[x_1, x_4] = [x_2, x_3] = [\alpha_1 x_1, x_2] = \alpha_1[x_1, x_2] = \alpha_1 x_3 = 0$. So $(x_1 \wedge x_4), (x_2 \wedge x_3), \alpha_1(x_1 \wedge x_2) \in \mathcal{M}_0(L_{5,5})$. We can see that $[x_1 + x_2 + x_3, x_1 + x_2 + x_4] = 0$, so $(x_1 + x_2 + x_3) \wedge (x_1 + x_2 + x_4) \in \mathcal{M}_0(L_{5,5})$. Hence $(x_1 \wedge x_4) + (x_2 \wedge x_4) + (x_3 \wedge x_1) + (x_3 \wedge x_2) + (x_3 \wedge x_4) \in \mathcal{M}_0(L_{5,5})$ and

$(x_1 \wedge x_3) - (x_2 \wedge x_4) \in \mathcal{M}_0(L_{5,5})$. Therefore $\mathcal{M}(L_{5,5}) \subseteq \mathcal{M}_0(L_{5,5})$, and hence $\tilde{B}_0(L_{5,5}) = 0$. Similarly, $\tilde{B}_0(L_{5,7}) = \tilde{B}_0(L_{5,8}) = \tilde{B}_0(L_{5,9}) = 0$.

Let $L \cong L_{5,6}$, we obtain that

$$L_{5,6} \wedge L_{5,6} = < x_1 \wedge x_2, x_1 \wedge x_3, x_1 \wedge x_4, x_1 \wedge x_5, x_2 \wedge x_3, x_2 \wedge x_5 > .$$

Hence for all $w \in \mathcal{M}(L_{5,6}) \leq L_{5,6} \wedge L_{5,6}$ there exist $\alpha_1, \alpha_2, \dots, \alpha_6 \in F$, such that $w = \alpha_1(x_1 \wedge x_2) + \alpha_2(x_1 \wedge x_3) + \alpha_3(x_1 \wedge x_4) + \alpha_4(x_1 \wedge x_5) + \alpha_5(x_2 \wedge x_3) + \alpha_6(x_2 \wedge x_5)$. Since $\tilde{\kappa}(w) = 0$, we have $\alpha_1[x_1, x_2] + \alpha_2[x_1, x_3] + \alpha_3[x_1, x_4] + \alpha_4[x_1, x_5] + \alpha_5[x_2, x_3] + \alpha_6[x_2, x_5] = 0$. Thus $\alpha_1 x_3 + \alpha_2 x_4 + (\alpha_3 + \alpha_5)x_5 = 0$. Therefore $\alpha_1 x_3 = \alpha_2 x_4 = (\alpha_3 + \alpha_5)x_5 = 0$. On the other hand, $[\alpha_1 x_1, x_2] = \alpha_1[x_1, x_2] = \alpha_1 x_3 = 0$ and $[\alpha_2 x_1, x_3] = \alpha_2[x_1, x_3] = \alpha_2 x_4 = 0$. So $\alpha_1(x_1 \wedge x_2), \alpha_2(x_1 \wedge x_3), (x_1 \wedge x_5), (x_2 \wedge x_5) \in \mathcal{M}_0(L_{5,6})$. Thus $w = \alpha_3(x_1 \wedge x_4 - x_2 \wedge x_3) + \tilde{w}$, where $\tilde{w} \in \mathcal{M}_0(L_{5,6})$. Let B be a generating set for $\mathcal{M}_0(L_{5,6})$, then $\mathcal{M}(L_{5,6}) = < B,\ x_1 \wedge x_4 - x_2 \wedge x_3 >$. Hence $\dim \tilde{B}_0(L_{5,6}) = 1$. So $\tilde{B}_0(L_{5,6}) \cong A(1)$. $\square$

From [9], the 6-dimensional Lie algebras are $L_{6,k} \cong L_{5,k} \oplus I$, for $k = 1, \dots, 9$ where I is a 1-dimensional abelian ideal and

- $L_{6,10} \cong < x_1, ..., x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_6, [x_4, x_5] = x_6 >$,
- $L_{6,11} \cong < x_1, ..., x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = [x_2, x_3] = [x_2, x_5] = x_6 >$,
- $L_{6,12} \cong < x_1, ..., x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = [x_2, x_5] = x_6 >$,
- $L_{6,13} \cong < x_1, ..., x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_5, [x_1, x_5] = [x_3, x_4] = x_6 >$,
- $L_{6,14} \cong < x_1, ..., x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = [x_2, x_3] = x_5, [x_2, x_5] = x_6, [x_3, x_4] = -x_6 >$,
- $L_{6,15} \cong < x_1, ..., x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = [x_2, x_3] = x_5, [x_1, x_5] = [x_2, x_4] = x_6 >$,
- $L_{6,16} \cong < x_1, ..., x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_2, x_5] = x_6, [x_3, x_4] = -x_6 >$,
- $L_{6,17} \cong < x_1, ..., x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_1, x_5] = [x_2, x_3] = x_6 >$,
- $L_{6,18} \cong < x_1, ..., x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_1, x_5] = x_6 >$,
- $L_{6,19}(\epsilon) \cong < x_1, ..., x_6 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, [x_2, x_4] = x_6, [x_3, x_5] = \epsilon x_6 >$,
- $L_{6,20} \cong < x_1, ..., x_6 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, [x_1, x_5] = [x_2, x_4] = x_6 >$,
- $L_{6,21}(\epsilon) \cong < x_1, ..., x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, [x_1, x_4] = x_6, [x_2, x_5] = \epsilon x_6 >$,
- $L_{6,22}(\epsilon) \cong < x_1, ..., x_6 \mid [x_1, x_2] = x_5, [x_1, x_3] = x_6, [x_2, x_4] = \epsilon x_6, [x_3, x_4] = x_5 >$,
- $L_{6,23} \cong < x_1, ..., x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_5, [x_1, x_4] = x_6 >$,
- $L_{6,24}(\epsilon) \cong < x_1, ..., x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_5, [x_1, x_4] = \epsilon x_6, [x_2, x_3] = x_6 >$,
- $L_{6,25} \cong < x_1, ..., x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_1, x_4] = x_6 >$,
- $L_{6,26} \cong < x_1, ..., x_6 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, [x_2, x_3] = x_6 >$.

Theorem 6.5. *Let L be a nilpotent Lie algebra of dimension 6. Then $\tilde{B}_0(L) \neq 0$ if and only if L is isomorphic to one of the Lie algebras $L_{6,6}$, $L_{6,13}$, $L_{6,14}$, $L_{6,15}$, $L_{6,22}(\epsilon)$ where $\epsilon \geq 1$, $L_{6,23}$ and $L_{6,24}(\epsilon)$ where $\epsilon \neq 1$.*

Proof. Using Proposition 4.8 and Theorem 6.4, $\tilde{B}_0(L_{6,1}) = \tilde{B}_0(L_{6,2}) = \tilde{B}_0(L_{6,3}) = \tilde{B}_0(L_{6,4}) = \tilde{B}_0(L_{6,5}) = \tilde{B}_0(L_{6,7}) = \tilde{B}_0(L_{6,8}) = \tilde{B}_0(L_{6,9}) = 0$ and $\tilde{B}_0(L_{6,6}) \neq 0$. Also similar to the calculations in Theorems 6.3 and 6.4, we have $\tilde{B}_0(L_{6,13}), \tilde{B}_0(L_{6,14}), \tilde{B}_0(L_{6,15}), \tilde{B}_0(L_{6,23}) \neq 0$ and $\tilde{B}_0(L_{6,22}(\epsilon))(\epsilon = 0), \tilde{B}_0(L_{6,24}(\epsilon))(\epsilon = 1), \tilde{B}_0(L_{6,10}), \tilde{B}_0(L_{6,12}),$ $\tilde{B}_0(L_{6,16}), \tilde{B}_0(L_{6,17}), \tilde{B}_0(L_{6,18}), \tilde{B}_0(L_{6,20}), \tilde{B}_0(L_{6,25}), \tilde{B}_0(L_{6,26}) = 0$.

Let $L \cong L_{6,11}$, one can compute that

$$L_{6,11} \wedge L_{6,11} = < x_1 \wedge x_2, x_1 \wedge x_3, x_1 \wedge x_4, x_1 \wedge x_5, x_2 \wedge x_3, x_2 \wedge x_4, x_2 \wedge x_5, x_3 \wedge x_4, x_3 \wedge x_5, x_4 \wedge x_5 > .$$

Hence for all $w \in \mathcal{M}(L_{6,11}) \leq L_{6,11} \wedge L_{6,11}$, there exist $\alpha_1, \alpha_2, \ldots, \alpha_{10} \in F$, such that $w = \alpha_1(x_1 \wedge x_2) + \alpha_2(x_1 \wedge x_3) + \alpha_3(x_1 \wedge x_4) + \alpha_4(x_1 \wedge x_5) + \alpha_5(x_2 \wedge x_3) + \alpha_6(x_2 \wedge x_4) + \alpha_7(x_2 \wedge x_5) + \alpha_8(x_3 \wedge x_4) + \alpha_9(x_3 \wedge x_5) + \alpha_{10}(x_4 \wedge x_5)$. Since $\tilde{\kappa}(w) = 0$, we have $\alpha_1[x_1, x_2] + \alpha_2[x_1, x_3] + \alpha_3[x_1, x_4] + \alpha_4[x_1, x_5] + \alpha_5[x_2, x_3] + \alpha_6[x_2, x_4] + \alpha_7[x_2, x_5] + \alpha_8[x_3, x_4] + \alpha_9[x_3, x_5] + \alpha_{10}[x_4, x_5] = 0$. Thus $\alpha_1 x_3 + \alpha_2 x_4 + (\alpha_3 + \alpha_5 + \alpha_7)x_6 = 0$, and $\alpha_1 = \alpha_2 = \alpha_3 + \alpha_5 + \alpha_7 = 0$. Hence $\alpha_7 = -\alpha_3 - \alpha_5$, so $w = \alpha_3(x_1 \wedge x_4 - x_2 \wedge x_5) + \alpha_4(x_1 \wedge x_5) + \alpha_5(x_2 \wedge x_3 - x_2 \wedge x_5) + \alpha_6(x_2 \wedge x_4) + \alpha_8(x_3 \wedge x_4) + \alpha_9(x_3 \wedge x_5) + \alpha_{10}(x_4 \wedge x_5)$. On the other hand $[x_1, x_5] = [x_2, x_4] = [x_3, x_4] = [x_3, x_5] = [x_4, x_5] = 0$. So $(x_1 \wedge x_5), (x_2 \wedge x_4), (x_3 \wedge x_4), (x_3 \wedge x_5)$ and $(x_4 \wedge x_5) \in \mathcal{M}_0(L_{6,11})$. We can see that $[x_1 + x_2 + x_5, x_1 + x_2 + x_4] = 0$, so $(x_1 + x_2 + x_5) \wedge (x_1 + x_2 + x_4) \in \mathcal{M}_0(L_{6,11})$. Hence $(x_1 \wedge x_4 - x_2 \wedge x_5) \in \mathcal{M}_0(L_{6,11})$. Also $[x_2 + x_5, x_2 + x_3] = 0$, so $(x_2 \wedge x_3 - x_2 \wedge x_5) \in \mathcal{M}_0(L_{6,11})$. Therefore $\mathcal{M}(L_{6,11}) \subseteq \mathcal{M}_0(L_{6,11})$, and $\tilde{B_0}(L_{6,11}) = 0$.

Let $L \cong L_{6,19}(\epsilon)$ where $\epsilon \geq 0$, we can write

$$L_{6,19}(\epsilon) \wedge L_{6,19}(\epsilon) = < x_1 \wedge x_2, x_1 \wedge x_3, x_1 \wedge x_4, x_1 \wedge x_5, x_2 \wedge x_3, x_2 \wedge x_4, x_3 \wedge x_4, x_3 \wedge x_5 > .$$

For all $w \in \mathcal{M}(L_{6,19}(\epsilon)) \leq L_{6,19}(\epsilon) \wedge L_{6,19}(\epsilon)$, there exist $\alpha_1, \alpha_2, \ldots, \alpha_8 \in F$, such that $w = \alpha_1(x_1 \wedge x_2) + \alpha_2(x_1 \wedge x_3) + \alpha_3(x_1 \wedge x_4) + \alpha_4(x_1 \wedge x_5) + \alpha_5(x_2 \wedge x_3) + \alpha_6(x_2 \wedge x_4) + \alpha_7(x_3 \wedge x_4) + \alpha_8(x_3 \wedge x_5)$. Since $\tilde{\kappa}(w) = 0$, we have $\alpha_1[x_1, x_2] + \alpha_2[x_1, x_3] + \alpha_3[x_1, x_4] + \alpha_4[x_1, x_5] + \alpha_5[x_2, x_3] + \alpha_6[x_2, x_4] + \alpha_7[x_3, x_4] + \alpha_8[x_3, x_5] = 0$. Thus $\alpha_1 x_4 + \alpha_2 x_5 + (\alpha_6 + \epsilon\alpha_8)x_6 = 0$, and $\alpha_1 = \alpha_2 = \alpha_6 + \epsilon\alpha_8 = 0$. Hence $w = \alpha_3(x_1 \wedge x_4) + \alpha_4(x_1 \wedge x_5) + \alpha_5(x_2 \wedge x_3) + \alpha_7(x_3 \wedge x_4) + \alpha_8((x_3 \wedge x_5) - \epsilon(x_2 \wedge x_4))$. Also, $[x_1, x_4] = [x_1, x_5] = [x_2, x_3] = [x_3, x_4] = 0$. So $(x_1 \wedge x_4), (x_1 \wedge x_5), (x_2 \wedge x_3), (x_3 \wedge x_4) \in \mathcal{M}_0(L_{6,19}(\epsilon))$. We can see that $[x_3 + \epsilon x_2 + x_4, x_3 + \epsilon x_2 + x_5] = 0$, so $(x_3 + \epsilon x_2 + x_4) \wedge (x_3 + \epsilon x_2 + x_5) \in \mathcal{M}_0(L_{6,19}(\epsilon))$. Hence $(x_3 \wedge x_5) + \epsilon(x_2 \wedge x_5) + (x_4 \wedge x_3) + \epsilon(x_4 \wedge x_2) + (x_4 \wedge x_5) \in \mathcal{M}_0(L_{6,19}(\epsilon))$. Thus $(x_3 \wedge x_5) - \epsilon(x_2 \wedge x_4) \in \mathcal{M}_0(L_{6,19}(\epsilon))$. Therefore $\mathcal{M}(L_{6,19}(\epsilon)) \subseteq \mathcal{M}_0(L_{6,19}(\epsilon))$, and $\tilde{B_0}(L_{6,19}(\epsilon)) = 0$, where $\epsilon \geq 0$. Similarly $\tilde{B_0}(L_{6,21}(\epsilon)) = 0$, where $\epsilon \geq 0$.

For $L \cong L_{6,22}(\epsilon)$ where $\epsilon \geq 1$, we have
$L_{6,22}(\epsilon) \wedge L_{6,22}(\epsilon) = < x_1 \wedge x_2, x_1 \wedge x_3, x_1 \wedge x_4, x_1 \wedge x_6, x_2 \wedge x_3, x_2 \wedge x_4, x_3 \wedge x_4, x_3 \wedge x_5, x_3 \wedge x_6, x_4 \wedge x_6 >$.
Now for all $w \in \mathcal{M}(L_{6,22}(\epsilon)) \leq L_{6,22}(\epsilon) \wedge L_{6,22}(\epsilon)$, there exist $\alpha_1, \alpha_2, \ldots, \alpha_{10} \in F$, such that $w = \alpha_1(x_1 \wedge x_2) + \alpha_2(x_1 \wedge x_3) + \alpha_3(x_1 \wedge x_4) + \alpha_4(x_1 \wedge x_6) + \alpha_5(x_2 \wedge x_3) + \alpha_6(x_2 \wedge x_4) + \alpha_7(x_3 \wedge x_4) + \alpha_8(x_3 \wedge x_5) + \alpha_9(x_3 \wedge x_6) + \alpha_{10}(x_4 \wedge x_6)$. Since $\tilde{\kappa}(w) = 0$, we have $\alpha_1[x_1, x_2] + \alpha_2[x_1, x_3] + \alpha_3[x_1, x_4] + \alpha_4[x_1, x_6] + \alpha_5[x_2, x_3] + \alpha_6[x_2, x_4] + \alpha_7[x_3, x_4] + \alpha_8[x_3, x_5] + \alpha_9[x_3, x_6] + \alpha_{10}[x_4, x_6] = 0$. Thus $(\alpha_1 + \alpha_7)x_5 + (\alpha_2 + \epsilon\alpha_6)x_6 = 0$, so $\alpha_7 = -\alpha_1$ and $\alpha_2 = -\epsilon\alpha_6$. Hence $w = \alpha_1(x_1 \wedge x_2 - x_3 \wedge x_4) + \alpha_3(x_1 \wedge x_4) + \alpha_4(x_1 \wedge x_6) + \alpha_5(x_2 \wedge x_3) + \alpha_6(x_2 \wedge x_4 - \epsilon(x_1 \wedge x_3)) + \alpha_8(x_3 \wedge x_5) + \alpha_9(x_3 \wedge x_6) + \alpha_{10}(x_4 \wedge x_6)$. On the other hand $[x_1, x_4] = [x_1, x_6] = [x_2, x_3] = [x_3, x_5] = [x_3, x_6] = [x_4, x_6] = 0$. So $(x_1 \wedge x_4), (x_1 \wedge x_6), (x_2 \wedge x_3), (x_3 \wedge x_5), (x_3 \wedge x_6)$ and $(x_4 \wedge x_6) \in \mathcal{M}_0(L_{6,22}(\epsilon))$. Thus $w = \alpha_1(x_1 \wedge x_2 - x_3 \wedge x_4) + \alpha_6(x_2 \wedge x_4 - \epsilon(x_1 \wedge x_3)) + \tilde{w}$, where $\tilde{w} \in \mathcal{M}_0(L_{6,22}(\epsilon))$. Let B be a generating set for $\mathcal{M}_0(L_{6,22}(\epsilon))$, then $\mathcal{M}(L_{6,22}(\epsilon)) = < B,\ x_1 \wedge x_2 - x_3 \wedge x_4, x_2 \wedge x_4 - \epsilon(x_1 \wedge x_3) >$. Hence $\dim \tilde{B_0}(L_{6,22}(\epsilon)) \geq 2$. So $\tilde{B_0}(L_{6,22}(\epsilon)) \neq 0$.

Let $L \cong L_{6,24}(\epsilon)$ where $\epsilon \neq 1$, we can compute
$L_{6,24}(\epsilon) \wedge L_{6,24}(\epsilon) = < x_1 \wedge x_2, x_1 \wedge x_3, x_1 \wedge x_4, x_2 \wedge x_3, x_2 \wedge x_4, x_2 \wedge x_5, x_2 \wedge x_6, x_3 \wedge x_4, x_4 \wedge x_5, x_4 \wedge x_6 >$.
So, for all $w \in \mathcal{M}(L_{6,24}(\epsilon)) \leq L_{6,24}(\epsilon) \wedge L_{6,24}(\epsilon)$, there exist $\alpha_1, \alpha_2, \ldots, \alpha_{10} \in F$, such

that $w = \alpha_1(x_1 \wedge x_2) + \alpha_2(x_1 \wedge x_3) + \alpha_3(x_1 \wedge x_4) + \alpha_4(x_2 \wedge x_3) + \alpha_5(x_2 \wedge x_4) + \alpha_6(x_2 \wedge x_5) + \alpha_7(x_2 \wedge x_6) + \alpha_8(x_3 \wedge x_4) + \alpha_9(x_4 \wedge x_5) + \alpha_{10}(x_4 \wedge x_6)$. Since $\tilde{\kappa}(w) = 0$, we have $\alpha_1[x_1, x_2]+\alpha_2[x_1, x_3]+\alpha_3[x_1, x_4]+\alpha_4[x_2, x_3]+\alpha_5[x_2, x_4]+\alpha_6[x_2, x_5]+\alpha_7[x_2, x_6]+\alpha_8[x_3, x_4]+\alpha_9[x_4, x_5]+\alpha_{10}[x_4, x_6] = 0$. Thus $\alpha_1 x_3+(\alpha_2+\alpha_5)x_5+(\alpha_4+\epsilon\alpha_3)x_6 = 0$, so $\alpha_1 = 0, \alpha_5 = -\alpha_2$ and $\alpha_4 = -\epsilon\alpha_3$. Hence $w = \alpha_2(x_1 \wedge x_3 - x_2 \wedge x_4) + \alpha_3(x_1 \wedge x_4 - \epsilon(x_2 \wedge x_3)) + \alpha_6(x_2 \wedge x_5) + \alpha_7(x_2 \wedge x_6) + \alpha_8(x_3 \wedge x_4) + \alpha_9(x_4 \wedge x_5) + \alpha_{10}(x_4 \wedge x_6)$. Also we have, $[x_2, x_5] = [x_2, x_6] = [x_3, x_4] = [x_4, x_5] = [x_4, x_6] = 0$. So $(x_2 \wedge x_5), (x_2 \wedge x_6), (x_3 \wedge x_4), (x_4 \wedge x_5)$ and $(x_4 \wedge x_6) \in \mathcal{M}_0(L_{6,24}(\epsilon))$. Thus $w = \alpha_2(x_1 \wedge x_3 - x_2 \wedge x_4) + \alpha_3(x_1 \wedge x_4 - \epsilon(x_2 \wedge x_3)) + \tilde{w}$, where $\tilde{w} \in \mathcal{M}_0(L_{6,24}(\epsilon))$. Let B be a generating set for $\mathcal{M}_0(L_{6,24}(\epsilon))$, then $\mathcal{M}(L_{6,24}(\epsilon)) =< B,\ x_1\wedge x_3-x_2\wedge x_4, x_1\wedge x_4-\epsilon(x_2\wedge x_3) >$. Hence $\dim \tilde{B}_0(L_{6,24}(\epsilon)) \geq 2$. So $\tilde{B}_0(L_{6,24}(\epsilon)) \neq 0$.

If $\epsilon = 1$, we have $\alpha_1 x_3 + (\alpha_2 + \alpha_5)x_5 + (\alpha_4 + \alpha_3)x_6 = 0$, so $\alpha_1 = 0, \alpha_5 = -\alpha_2$ and $\alpha_4 = -\alpha_3$. Hence $w = \alpha_2(x_1 \wedge x_3 - x_2 \wedge x_4) + \alpha_3(x_1 \wedge x_4 - x_2 \wedge x_3) + \alpha_6(x_2 \wedge x_5) + \alpha_7(x_2 \wedge x_6) + \alpha_8(x_3 \wedge x_4) + \alpha_9(x_4 \wedge x_5) + \alpha_{10}(x_4 \wedge x_6)$. On the other hand $[x_1 + x_2 + x_4, x_1 + x_2 + x_3], [x_1+x_2+x_3, x_1+x_2+x_4] = 0$. So $(x_1\wedge x_3-x_2\wedge x_4), (x_1\wedge x_4-x_2\wedge x_3) \in \mathcal{M}_0(L_{6,24}(\epsilon))$. Therefore $\mathcal{M}(L_{6,24}(\epsilon)) \subseteq \mathcal{M}_0(L_{6,24}(\epsilon))$, and $\tilde{B}_0(L_{6,24}(\epsilon)) = 0$, where $\epsilon = 1$. □

One of the important results on the Schur multiplier of Lie algebras was presented by Moneyhun in [21]. He showed that for a Lie algebra L of dimension n, $\dim \mathcal{M}(L) = n(n-1)/2 - t(L)$, for some $t(L) \geq 0$. His results suggest the interesting question, ''can we classify Lie algebras of dimension n by $t(L)$?" The answer to this question can be found for $t(L) = 1, \ldots, 8$ in [3, 11, 12, 21]. On the other hand, from [23], we have an upper bound for the dimension of the Schur multiplier of a non abelian nilpotent Lie algebra as $\dim \mathcal{M}(L) = n(n-1)(n-2)/2 + 1 - s(L)$, for some $s(L) \geq 0$. Hence by the same motivation, we have the analogous question for classification of L according to the values of $s(L)$. It seems that classifying nilpotent Lie algebras by $s(L)$ helps to the classification of Lie algebras in term of $t(L)$. (See for instance [23]). Now, according to this classification, we will investigate the Bogomolov multiplier for some Lie algebras.

Theorem 6.6. *Let L be an n-dimensional nilpotent Lie algebra with $s(L) = 1$. Then, $\tilde{B}_0(L) = 0$.*

Proof. Since $s(L) = 1$, by Theorem 3.9 in [23], $L \cong L_{5,4}$. So $\tilde{B}_0(L) = 0$. □

Theorem 6.7. *Let L be a n-dimension nilpotent Lie algebra and $t(L) \leq 6$. Then, $\tilde{B}_0(L) = 0$.*

Proof. By Theorem 3.10 in [23] and Proposition 4.8, $\tilde{B}_0(L) = 0$. □

Theorem 6.8. *Let L be an n-dimensional nilpotent Lie algebra with $s(L) = 2$ and $dimL^2 = 2$. Then $\tilde{B}_0(L) = 0$.*

Proof. By Theorem 4.3 in [23], $L \cong L_{4,3}$ or $L \cong L_{5,4} \oplus A(1)$. Thus $\tilde{B}_0(L) = 0$. □

7. The Bogomolov multipliers of complex simple and semisimple Lie algebras

A simple group is a group with no non-trivial proper normal subgroup. The classification of finite simple groups is a major milestone in the history of mathematics. On the other hand with the help of the Jordan-Holder Theorem, a finite group can be written as a certain combination of simple groups. Also, in contrast to the classification of finite simple groups, the classification of simple Lie groups is simplified by using the manifold structure. In particular every Lie group has an dependent Lie algebra and in this regard, some authors have also gained some results. For example, Bosshardt showed that a Lie group is simple if and only if its Lie algebra is simple. (see [5, 20, 26] for more information). Theories of groups and Lie algebras are structurally similar, and many concepts related to groups, there are analogously defined concepts for Lie algebras. Eventually, this subject reduces the problem of finding simple Lie groups to classifying simple Lie algebras. In this section we obtain the Bogomolov multipliers of complex semisimple Lie algebras and complex simple Lie algebras.

Definition 7.1 ([10]). A Lie algebra L is called semisimple if the only commutative ideal of L is 0.

For example a 0-dimension Lie algebra, the special linear Lie algebra, the odd-dimension special orthogonal Lie algebra, the symplectic Lie algebra and the even-dimension special orthogonal Lie algebra for $(n > 1)$ are semisimple.

Definition 7.2 ([10]). A Lie algebra L is simple if it has no ideals other than 0 and L, and it is not abelian.

Theorem 7.3 ([27]). *A Lie algebra L is semisimple if and only if $L = \oplus_i L_i$, for simple Lie algebras L_i.*

We can therefore view a semisimple Lie algebra L as a direct sum of simple Lie algebras L_i, which have only 0 and L_i for their ideals. In particular, every simple Lie algebra is also semisimple.

Batten in [2] (Example 2 of Chapter 2), showed that any semisimple Lie algebra over a field of characteristic 0 has a trivial Schur multiplier. Also we know the Bogomolov multiplier is subalgebra of the Schur multiplier, so these semisimple Lie algebras have trivial Bogomolov multipliers. Therefore all simple Lie algebras over a field of characteristic 0 have trivial Bogomolov multipliers.

In this section, considering the classification of simple complex Lie algebras Cartan [5], we will show that the Bogomolov multipliers of an arbitrary complex semisimple Lie algebra is trivial. Using our computational method as used in Theorems 5.1, 5.2, 6.3, 6.4 and 6.5. Note that this result is a special case of results of Batten [2] (the Example 2 of Chapter 2), but it is worthy to state it here because we prove it by using computations on the Bogomolov multiplier regardless of its relation with the Schur multiplier.

In the following, E_{ij} denotes the matrix with 1 at the intersection of the i-th row and the j-th column and 0 everywhere else. The Lie bracket of E_{ij} and E_{kl} is given by $[E_{ij}, E_{kl}] = E_{ij}E_{kl} - E_{kl}E_{ij} = \delta_{jk}E_{il} - \delta_{il}E_{kj}$.

Theorem 7.4. *Every following semisimple Lie algebra over $\mathbb{C}$, has trivial Bogomolov multiplier.*

(i) $Sl(n+1,\mathbb{C})$,
(ii) $So(2n+1,\mathbb{C})$,
(iii) $Sp(n,\mathbb{C})$,
(iv) $So(2n,\mathbb{C})$, $n \geq 2$.

Proof. Let L be a special linear Lie algebra $Sl(n+1,\mathbb{C})$. From [17], $Sl(n+1,\mathbb{C})$ has the basis D_{ii+1}, E_{ij} such that $D_{ij} = E_{ii} - E_{jj}$. So for $j \neq i = 1 \dots n$, we have $Sl(n+1,\mathbb{C}) =< D_{ii+1}, E_{ij} \mid [D_{ii+1}, D_{i+1i+2}] = D_{ii+2}, [D_{ii+1}, E_{ij}] = 2E_{ij}\ ;\ j = i+1\ ,\ [D_{ii+1}, E_{ij}] = E_{ij}\ ;\ j \neq i+1 >$ mod $\mathcal{M}_0(Sl(n+1,\mathbb{C}))$. We can see that

$$Sl(n+1,\mathbb{C}) \wedge Sl(n+1,\mathbb{C}) =< D_{ii+1} \wedge D_{i+1i+2}, D_{ii+1} \wedge E_{ij} > \quad \text{mod } \mathcal{M}_0(Sl(n+1,\mathbb{C})).$$

Now for all $w \in \mathcal{M}(Sl(n+1,\mathbb{C})) \leq Sl(n+1,\mathbb{C}) \wedge Sl(n+1,\mathbb{C})$, there exist $\alpha_i, \beta_{ij} \in \mathbb{C}$, $i,j = 1 \ldots n+1$ and $\tilde{w} \in \mathcal{M}_0(Sl(n+1,\mathbb{C}))$, such that

$$w = \sum_{i=1}^{n} \alpha_i (D_{ii+1} \wedge D_{i+1i+2}) + \sum_{i,j=1}^{n} \beta_{ij} (D_{ii+1} \wedge E_{ij}) + \tilde{w}.$$

Since $\tilde{\kappa}(w) = 0$, we have $\sum_{i=1}^{n} \alpha_i [D_{ii+1}, D_{i+1i+2}] + \sum_{i,j=1}^{n} \beta_{ij} [D_{ii+1}, E_{ij}] = 0$. Now if $j = i+1$, then $\sum_{i=1}^{n} \alpha_i D_{ii+2} + \sum_{i=1}^{n} 2\beta_i E_{ii+1} = \sum_{i=1}^{n} \alpha_i (E_{ii} - E_{i+2i+2}) + 2\sum_{i=1}^{n} \beta_i E_{ii+1} = 0$. So for all $i = 1...n$, $\alpha_i = \beta_i = 0$. If $j \neq i+1$, then $\sum_{i=1}^{n} \alpha_i D_{ii+2} + \sum_{i,j=1,i<j}^{n} \beta_{i,j} E_{ij} = 0$. So for all i, j, $\alpha_i = \beta_{ij} = 0$. Hence $w \in \mathcal{M}_0(Sl(n+1,\mathbb{C}))$ and $\mathcal{M}(Sl(n+1,\mathbb{C})) \subseteq \mathcal{M}_0(Sl(n+1,\mathbb{C}))$. Therefore $\tilde{B}_0(Sl(n+1,\mathbb{C})) = 0$.

Let L be one of the odd-dimension orthogonal Lie algebra $So(2n+1,\mathbb{C})$. From [17], $So(2n+1,\mathbb{C})$ has the basis H_i, ${K_i}^{\pm}$, ${L_{ij}}^{\pm}$, ${M_{ij}}^{\pm}$ such that $D_{ij} = E_{ij} - E_{ji}$ $(1 \leq i \neq j \leq 2n+1)$, $H_i := \sqrt{-1}D_{2i-12i}$ $(i = 1, ..., n)$, ${K_i}^{\pm} := D_{2i-12n+1} \pm \sqrt{-1}D_{2i2n+1}$ $(i = 1, ..., n)$, ${L_{ij}}^{\pm} := (D_{2i-12j-1} - D_{2i2j}) \pm \sqrt{-1}(D_{2i-12j} + D_{2i2j-1})$ $(1 \leq i < j \leq n)$, ${M_{ij}}^{\pm} := (D_{2i-12j} - D_{2i2j-1}) \pm \sqrt{-1}(D_{2i-12j-1} + D_{2i2j})$ $(1 \leq i < j \leq n)$. Also we have $[H_i, {K_i}^{\pm}] = \sqrt{-1}D_{2n+12i} \pm D_{2i-12n+1}$, $[H_i, {L_{ij}}^{\pm}] = -\sqrt{-1}D_{2i-12j} \pm D_{2i-12j-1}$, $[H_i, {M_{ij}}^{\pm}] = -\sqrt{-1}D_{2i-12j-1} \pm D_{2i-12j}$, $[{K_i}^{\pm}, {L_{ij}}^{\pm}] = [{K_i}^{\pm}, {M_{ij}}^{\pm}] = [{L_{ij}}^{\pm}, {M_{ij}}^{\pm}] = 0$. So in mod $\mathcal{M}_0(So(2n+1,\mathbb{C}))$, can be see that

$$So(2n+1,\mathbb{C}) =< H_i, {K_i}^{\pm}, {L_{ij}}^{\pm}, {M_{ij}}^{\pm} \mid [H_i, {K_i}^{\pm}], [H_i, {L_{ij}}^{\pm}], [H_i, {M_{ij}}^{\pm}] >$$

and in mod $\mathcal{M}_0(So(2n+1,\mathbb{C}))$, $So(2n+1,\mathbb{C}) \wedge So(2n+1,\mathbb{C}) =$

$$< H_i \wedge {K_i}^{\pm}, H_i \wedge {L_{ij}}^{\pm}, H_i \wedge {M_{ij}}^{\pm} >=< D_{2i-12i} \wedge D_{2i2n+1}, D_{2i-12i} \wedge D_{2i2j} > .$$

Now for all $w \in \mathcal{M}(So(2n+1,\mathbb{C})) \leq So(2n+1,\mathbb{C}) \wedge So(2n+1,\mathbb{C})$, there exist $\alpha_1, \alpha_2 \in \mathbb{C}$ and $\tilde{w} \in \mathcal{M}_0(So(2n+1,\mathbb{C}))$, such that $w = \alpha_1(D_{2i-12i} \wedge D_{2i2n+1}) + \alpha_2(D_{2i-12i} \wedge D_{2i2j}) + \tilde{w}$. Since $\tilde{\kappa}(w) = 0$, we have $\alpha_1[D_{2i-12i}, D_{2i2n+1}] + \alpha_2[D_{2i-12i}, D_{2i2j}] = \alpha_1 D_{2i-12n+1} + \alpha_2 D_{2i-12j}$ $= \alpha_1(E_{2i-12n+1} - E_{2n+12i-1}) + \alpha_2(E_{2i-12j} - E_{2j2i-1}) = 0$. Thus $\alpha_1 = \alpha_2 = 0$. Hence $\mathcal{M}(So(2n+1,\mathbb{C})) \subseteq \mathcal{M}_0(So(2n+1,\mathbb{C}))$, so $\tilde{B}_0(So(2n+1,\mathbb{C})) = 0$.

Let L be an even-dimension orthogonal Lie algebra $So(2n,\mathbb{C})$. From [17], $So(2n,\mathbb{C})$ has the basis H_i, ${L_{ij}}^{\pm}$, ${M_{ij}}^{\pm}$, such that $D_{ij} = E_{ij} - E_{ji}$ $(1 \leq i \neq j \leq 2n+1)$, $H_i := \sqrt{-1}D_{2i-12i}$ $(i = 1, ..., n)$, ${L_{ij}}^{\pm} := (D_{2i-12j-1} - D_{2i2j}) \pm \sqrt{-1}(D_{2i-12j} + D_{2i2j-1})$ $(1 \leq i < j \leq n)$, ${M_{ij}}^{\pm} := (D_{2i-12j} - D_{2i2j-1}) \pm \sqrt{-1}(D_{2i-12j-1} + D_{2i2j})$ $(1 \leq i < j \leq n)$. Also we have $[H_i, L_{ij}^{\pm}] = -\sqrt{-1}D_{2i-12j} \pm D_{2i-12j-1}$, $[H_i, M_{ij}^{\pm}] = -\sqrt{-1}D_{2i-12j-1} \pm D_{2i-12j}$, $[L_{ij}^{\pm}, M_{ij}^{\pm}] = 0$. Thus

$$So(2n,\mathbb{C}) =< H_i, {L_{ij}}^{\pm}, {M_{ij}}^{\pm} \mid [H_i, L_{ij}], [H_i, M_{ij}] > \quad \text{mod } \mathcal{M}_0(So(2n,\mathbb{C})).$$

We can write

$$So(2n,\mathbb{C}) \wedge So(2n,\mathbb{C}) =< H_i \wedge {L_{ij}}^{\pm}, H_i \wedge {M_{ij}}^{\pm} > \quad \text{mod } \mathcal{M}_0(So(2n,\mathbb{C})).$$

Now for all $w \in \mathcal{M}(So(2n,\mathbb{C})) \leq So(2n,\mathbb{C}) \wedge So(2n,\mathbb{C})$, there exist $\alpha_1, \alpha_2 \in \mathbb{C}$ and $\tilde{w} \in \mathcal{M}_0(So(2n,\mathbb{C}))$, such that $w = \alpha_1(H_i \wedge {L_{ij}}^{\pm}) + \alpha_2(H_i \wedge {M_{ij}}^{\pm}) + \tilde{w}$. Since $\tilde{\kappa}(w) = 0$, we have $\alpha_1[H_i, {L_{ij}}^{\pm}] + \alpha_2[H_i, {M_{ij}}^{\pm}] = \alpha_1(-\sqrt{-1}D_{2i-12j} \pm D_{2i-12j-1}) +$ $\alpha_2(-\sqrt{-1}D_{2i-12j-1} \pm D_{2i-12j}) = 0$. Thus $\alpha_1 = \alpha_2 = 0$. Hence $\mathcal{M}(So(2n,\mathbb{C})) \subseteq \mathcal{M}_0(So(2n,\mathbb{C}))$, so $\tilde{B}_0(So(2n,\mathbb{C})) = 0$.

Let L be a symplectic Lie algebra $Sp(n,\mathbb{C})$. From [17], $Sp(n,\mathbb{C})$ has the basis H_i, X_{ij}, Y_{ij}, Z_{ij}, U_i, V_i, such that $H_i = E_{ii} - E_{n+in+i}$ $(1 \leq i \leq n)$, $X_{ij} := E_{ij} - E_{n+jn+i}$ $(1 \leq i \neq j \leq n)$, $Y_{ij} := E_{in+j} + E_{jn+i}$ $(1 \leq i < j \leq n)$, $Z_{ij} := E_{n+ij} + E_{n+ji}$ $(1 \leq i < j \leq n)$, $U_i := E_{in+i}$ $(1 \leq i \leq n)$, and $V_i := E_{n+ii}$ $(1 \leq i \leq n)$. Since $[X_{ij}, Y_{ij}] = 2E_{in+i}$, $[X_{ij}, Z_{ij}] = -2E_{n+jj}$, $[X_{ij}, V_i] = -E_{n+ij} - E_{n+ji}$, $[Y_{ij}, Z_{ij}] = E_{ii} + E_{jj}$, $[Y_{ij}, V_i] = -E_{n+in+j} + E_{ji}$, $[Z_{ij}, U_i] = -E_{ij} + E_{n+jn+i}$, $[U_i, V_i] = E_{ii}$, $[X_{ij}, U_i] = [Y_{ij}, U_i] = [Z_{ij}, V_i] = 0$, $[H_i, X_{ij}] = -E_{n+jn+i}$, $[H_i, Y_{ij}] = E_{in+j} - E_{jn+i}$, $[H_i, Z_{ij}] = -E_{n+ij} - E_{n+ji}$, $[H_i, U_i] = 2E_{in+i}$ and $[H_i, V_i] = -2E_{n+ii}$, we have
$Sp(n,\mathbb{C}) =< H_i, X_{ij}, Y_{ij}, Z_{ij}, U_i, V_i \mid [X_{ij}, Y_{ij}], [X_{ij}, Z_{ij}], [X_{ij}, V_i], [Y_{ij}, Z_{ij}], [Y_{ij}, V_i], [Z_{ij}, U_i]$ $, [U_i, V_i], [H_i, X_{ij}], [H_i, Y_{ij}], [H_i, Z_{ij}], [H_i, U_i], [H_i, V_i] >$ mod $\mathcal{M}_0(Sp(n,\mathbb{C}))$.

We can write

$$Sp(n,\mathbb{C}) \wedge Sp(n,\mathbb{C}) =< X_{ij} \wedge Y_{ij}, X_{ij} \wedge Z_{ij}, X_{ij} \wedge V_i, Y_{ij} \wedge Z_{ij}, Y_{ij} \wedge V_i, Z_{ij} \wedge U_i,$$
$$U_i \wedge V_i, H_i \wedge X_{ij}, H_i \wedge Y_{ij}, H_i \wedge Z_{ij}, H_i \wedge U_i, H_i \wedge V_i > \mod \mathcal{M}_0(Sp(n,\mathbb{C})).$$

By using a similar method, the result follows. □

Since $So(2,\mathbb{C})$ is commutative, its Bogomolov multiplier is trivial. Therefore, we have

Corollary 7.5. *Following complex Lie algebras, have trivial Bogomolov multipliers.*

(i) $Sl(n+1,\mathbb{C})$,
(ii) $So(2n+1,\mathbb{C})$,
(iii) $Sp(n,\mathbb{C})$,
(iv) $So(2n,\mathbb{C})$.

Complex simple Lie algebras have been completely classified by Cartan [5]. They are classified into four infinite classes with five exceptional Lie algebras.

Theorem 7.6 ([10])**.** *Every simple Lie algebra over $\mathbb{C}$ is isomorphic to precisely one of the following Lie algebras*

(i) $Sl(n+1,\mathbb{C})$, $n \geq 1$,
(ii) $So(2n+1,\mathbb{C})$, $n \geq 2$,
(iii) $Sp(n,\mathbb{C})$, $n \geq 3$,
(iv) $So(2n,\mathbb{C})$, $n \geq 4$,
(v) *The exceptional Lie algebras* G_2 , F_4 , E_6 , E_7 *and* E_8.

Knapp in [17] showed that the five exceptional Lie algebras G_2, F_4, E_6, E_7, E_8 have dimensions $14, 52, 78, 133$ and 248, respectively.

Theorem 7.7. *Every complex simple Lie algebra has a trivial Bogomolov multiplier.*

Proof. By using Theorems 7.4 and 7.6, all complex simple Lie algebras in (i), (ii), (iii) and (iv) have trivial Bogomolov multipliers. Also we know any simple Lie algebra is semisimple. Thus the Example 2 of Chapter 2 in [2] showed that the exceptional Lie algebras G_2 , F_4 , E_6 , E_7 and E_8 have trivial Bogomolov multipliers. □

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