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Research Article

On K-pseudoframes for subspaces

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Abstract

In this paper, the concept of K-pseudoframes for subspaces of Hilbert spaces, as a generalization of both K-frames and pseudoframes, is introduced and some of their properties and their characterizations are investigated. Next, duals of K-pseudoframes are discussed. Finally, the concept of pseudoatomic system is introduced and its relations with K-pseudoframe are studied.

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1. Introduction

Frames in Hilbert spaces were first proposed by Duffin and Schaeffer to deal with non-harmonic Fourier series in 1952 [6], and were widely studied from 1986 since the great work by Daubechies et al. [4].

For special applications some types of frames were proposed, such as the fusion frames [2, 3] to deal with hierarchical data processing, g-frames [12] by Eldar, K-frames [7] by Găvruţa to study the atomic systems with respect to a bounded linear operator K in Hilbert spaces. From [7], we know that K-frames are more general than ordinary frames in the sense that the lower frame bound only holds for the elements in the range of K. Many properties for ordinary frames may not hold for K-frames, such as the corresponding synthesis operator for K-frames is not surjective, the frame operator for K-frames is not isomorphic for all $f \in \mathcal{H}$, the alternate dual reconstruction pair for K-frames is not interchangeable in general (see Example 3.2 in [13]). The concept of pseudoframe for subspaces was introduced by Li [11]. This sequences can go beyond a concerned subspace $\mathcal{X} \subset \mathcal{H}$.

In Section 2, we review some of the standard facts on pseudoframes, K-frames and atomic systems. Section 3 contains our main results on a generalization of both pseudoframes and K-frames, namely K-pseudoframes. In the last section, we introduce the concept of pseudoatomic system and we discuss some relations between K-pseudoframes and pseudoatomic systems.

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2. Preliminary

In this section, we recall some necessary concepts for our main results.

Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{X}$ be a closed subspace of $\mathcal{H}$. Also let $P_{\mathcal{X}}$ be the orthogonal projection on $\mathcal{X}$. We denote by $B(\mathcal{H},\mathcal{K})$ the set of all bounded linear operators from $\mathcal{H}$ into a Hilbert space $\mathcal{K}$ and we abbreviate $B(\mathcal{H},\mathcal{H})$ by $B(\mathcal{H})$. For $K \in B(\mathcal{H},\mathcal{K})$ let $R(K)$ denotes the range of K. Also we apply $K^\dagger$ for the pseudoinverse of K (if exists).

Let $\mathbb{J} \subseteq \mathbb{Z}$. A sequence $\{x_n\}_{n\in\mathbb{J}}$ is a Bessel sequence in $\mathcal{H}$ if there is a constant $M < \infty$ such that

$$\sum_{n\in\mathbb{J}} |\langle f, x_n\rangle|^2 \leq M\|f\|^2, \qquad (f \in \mathcal{H})$$

We shall say that $\{x_n\}_{n\in\mathbb{J}}$ is a Bessel sequence with respect to a closed subspace $\mathcal{X}$ of $\mathcal{H}$ if there is a constant $M < \infty$ such that

$$\sum_{n\in\mathbb{J}} |\langle f, x_n\rangle|^2 \leq M\|f\|^2, \qquad (f \in \mathcal{X}).$$

Definition 2.1. ([10]) Let $\{x_n\}_{n\in\mathbb{J}}$ and $\{x_n^*\}_{n\in\mathbb{J}}$ be two sequences in $\mathcal{H}$. We say $\{x_n\}_{n\in\mathbb{J}}$ is a pseudoframe for the subspace $\mathcal{X}$ with respect to $\{x_n^*\}_{n\in\mathbb{J}}$ if

$$f = \sum_{n\in\mathbb{J}} \langle f, x_n^*\rangle x_n, \qquad (f \in \mathcal{X}).$$

This definition is not symmetric (see [10]), i.e., there exists $f \in \mathcal{X}$ such that

$$\sum_{n\in\mathbb{J}} \langle f, x_n^*\rangle x_n \neq \sum_{n\in\mathbb{J}} \langle f, x_n\rangle x_n^*.$$

The sequence $\{x_n^*\}_{n\in\mathbb{J}}$ is called a dual pseudoframe of $\{x_n\}_{n\in\mathbb{J}}$.

Let $x^* = \{x_n^*\}_{n\in\mathbb{J}}$ be a Bessel sequence with respect to $\mathcal{X}$ and $x = \{x_n\}_{n\in\mathbb{J}}$ be a Bessel sequence in $\mathcal{H}$. Define

$$U_{x^*} : \mathcal{X} \longrightarrow l^2(\mathbb{J}), Uf = \{\langle f, x_n^*\rangle\}_{n\in\mathbb{J}}, \qquad (f \in \mathcal{X}), \tag{2.1}$$

and

$$V_x : l^2(\mathbb{J}) \longrightarrow \mathcal{H}, V(\{c_n\}_{n\in\mathbb{J}}) = \sum_{n\in\mathbb{J}} c_n x_n, \qquad (\{c_n\}_{n\in\mathbb{J}} \in l^2(\mathbb{J})). \tag{2.2}$$

Then $\{x_n\}_{n\in\mathbb{J}}$ is a pseudoframe with respect to $\{x_n^*\}_n$ if and only if

$$V_x U_{x^*} P_{\mathcal{X}} = P_{\mathcal{X}}.$$

For more details see [11].

Now let us remind the concepts of K-frame, the atomic system of K, K-exact frame and K-minimal frame for $K \in B(\mathcal{H})$.

Definition 2.2. ([7]) A sequence $\{x_n\}_{n\in\mathbb{J}} \subseteq \mathcal{H}$ is called a K-frame for $\mathcal{X} \subseteq \mathcal{H}$, if there exist constants $A, B > 0$ such that

$$A\|K^*f\|^2 \leq \sum_{n\in\mathbb{J}} |\langle f, x_n\rangle|^2 \leq B\|f\|^2, \qquad (f \in \mathcal{H}).$$

We call A and B the lower and the upper frame bounds for the K-frame $\{x_n\}_{n\in\mathbb{J}}$, respectively. Obviously if $K = I$, then the K-frame is the ordinary frame [13].

Definition 2.3. Let $\{x_n\}_{n\in\mathbb{J}}$ be a K-frame. A Bessel sequence $\{x_n^*\}_{n\in\mathbb{J}} \subseteq \mathcal{H}$ is called a K-dual of $\{x_n\}_{n\in\mathbb{J}}$ if

$$Kf = \sum_{n\in\mathbb{J}} \langle f, x_n^*\rangle x_n, \qquad (f \in \mathcal{H}).$$

For more details see [1].

Definition 2.4. A sequence $\{x_n\}_{n\in\mathbb{J}}$ is called an atomic system for K, if the following conditions are satisfied

(i) The sequence $\{x_n\}_{n\in\mathbb{J}}$ is a Bessel sequence;
(ii) For any $x \in \mathcal{H}$, there exists $a_x = \{a_n\}_{n\in\mathbb{J}} \in l^2(\mathbb{J})$ such that $Kx = \sum_{n\in\mathbb{J}} a_n x_n$, where $\|a_x\|_{l^2(\mathbb{J})} \leq C\|x\|$, C is a positive constant independently of x.

In Theorem 3.1 of [13], it is shown that $\{x_n\}_{n\in\mathbb{J}}$ is an atomic system for K if and only if $\{x_n\}_{n\in\mathbb{J}}$ is a K-frame for $\mathcal{H}$.

Definition 2.5. A K-frame $\{x_n\}_{n\in\mathbb{J}}$ of $\mathcal{H}$ is called

(i) K-exact frame if for every j the sequence $\{x_n\}_{n\neq j}$ is not a K-frame for $\mathcal{H}$,
(ii) K-minimal frame whenever for each $\{c_n\}_{n\in\mathbb{J}} \in l^2(\mathbb{J})$ with $\sum_{n\in\mathbb{J}} c_n x_n = 0$ we get $c_n = 0$ for all n.

Note that every K-exact frame is a K-minimal frame [1].
We need the following theorem for our next section.

Theorem 2.6. (Douglas Theorem) [5] *Let $\mathcal{H}$, $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. For any bounded linear operators $L_1 \in B(\mathcal{H}_1, \mathcal{H})$ and $L_2 \in B(\mathcal{H}_2, \mathcal{H})$, the following statements are equivalent*

(i) *$R(L_1) \subseteq R(L_2)$;*
(ii) *$L_1 L_1^* \leq \lambda^2 L_2 L_2^*$ for some $\lambda \geq 0$ and*
(iii) *there exists a bounded operator $M \in B(\mathcal{H}_1, \mathcal{H}_2)$ so that $L_1 = L_2 M$.*

For more results on K-frames, see [8, 9].

3. K-pseudoframe for subspaces

In this section, we define the concept of K-pseudoframes and after making an operator type equivalent condition, we give some properties of K-pseudoframes for subspaces. Also a characterization of K-dual pseudoframe is presented. Next a complete sequence in $\mathcal{H}$ with respect to $\mathcal{X}$ is introduced and its relations with K-pseudoframe and K-dual pseudoframe are studied.

Definition 3.1. Let $\mathcal{X}$ be a closed subspace of $\mathcal{H}$ and $K \in B(\mathcal{H})$. Let $\{x_n\}_{n\in\mathbb{J}}$ and $\{x_n^*\}_{n\in\mathbb{J}}$ be sequences in $\mathcal{H}$. We say $\{x_n\}_{n\in\mathbb{J}}$ is a K-pseudoframe for the subspace $\mathcal{X}$ with respect to $\{x_n^*\}_{n\in\mathbb{J}}$ if

$$Kf = \sum_{n\in\mathbb{J}} \langle f, x_n^* \rangle x_n, \qquad (f \in \mathcal{X}). \tag{3.1}$$

In general, for a K-frame $\{f_n\}_{n\in\mathbb{J}}$ we know that if $Kf = \sum_{n\in\mathbb{J}} \langle f, g_n \rangle f_n$, then $K^* f = \sum_{n\in\mathbb{J}} \langle f, f_n \rangle g_n$ for all $f \in \mathcal{H}$ (see [1]). Also for a pseudoframe $\{x_n\}_{n\in\mathbb{J}}$ with respect to $\{x_n^*\}_{n\in\mathbb{J}}$ it is well known that $f = \sum_{n\in\mathbb{J}} \langle f, x_n^* \rangle x_n$ dose not imply that $f = \sum_{n\in\mathbb{J}} \langle f, x_n \rangle x_n^*$, for any $f \in \mathcal{X}$.

Definition 3.2. Let $\{x_n\}_{n\in\mathbb{J}}$ is a K-pseudoframe for $\mathcal{X}$ with respect to $\{x_n^*\}_{n\in\mathbb{J}}$. We say that $\{x_n\}_{n\in\mathbb{J}}$ is interchangeable with $\{x_n^*\}_{n\in\mathbb{J}}$ for K if

$$K^* f = \sum_{n\in\mathbb{J}} \langle f, x_n \rangle x_n^*, \qquad (f \in \mathcal{X}).$$

Remark 3.3. Let $\{x_n\}_{n\in\mathbb{J}}$ be an interchangeable K-pseudoframe with respect to $\{x_n^*\}_{n\in\mathbb{J}}$. If $\mathcal{X} = \mathcal{H}$, then $\{x_n\}_{n\in\mathbb{J}}$ and $\{x_n^*\}_{n\in\mathbb{J}}$ are two atomic systems [7], so they are K-frames.

One can easily see that $\{x_n\}_{n\in\mathbb{J}}$ is a K-pseudoframe for $\mathcal{X}$ with respect to $\{x_n^*\}_{n\in\mathbb{J}}$ if and only if $V_x U_{x^*} P_{\mathcal{X}} = K P_{\mathcal{X}}$, where U_{x^*} and V_x are defined as (2.1) and (2.2).
In the following theorem we construct some K-pseudoframe for a Bessel sequence.

Theorem 3.4. *Let $x = \{x_n\}_{n\in\mathbb{J}}$ be a Bessel sequence in $\mathcal{H}$, $K \in B(\mathcal{H})$, $\mathcal{X}$ be a closed subspace of $\mathcal{H}$ and $K(\mathcal{X}) \subseteq \mathcal{X}$. If $\mathcal{X} \subseteq \overline{span}\{x_n : n \in \mathbb{J}\}$ and $R(V_x)$ is closed, then the set of all linear operators $U : \mathcal{X} \longrightarrow l^2(\mathbb{J})$ satisfying $V_xUP_{\mathcal{X}} = KP_{\mathcal{X}}$ is given by*

$$U = V_x^{\dagger}KP_{\mathcal{X}} + W - V_x^{\dagger}V_xWP_{\mathcal{X}}, \tag{3.2}$$

where $V_x^{\dagger}$ is the pseudoinverse of V_x, and $W : l^2(\mathbb{J}) \longrightarrow \mathcal{H}$ is a bounded linear operator. Moreover, let U be given by (3.2), then $\{x_n^* = U^*e_n\}_{n\in\mathbb{J}}$ is a dual K-pseudoframe for $\mathcal{X}$ with respect to $\{x_n\}_{n\in\mathbb{J}}$, where $\{e_n\}_{n\in\mathbb{J}}$ is the standard orthonormal basis for $l^2(\mathbb{J})$.

Proof. Since $R(V_x)$ is closed, the pseudoframe $V_x^{\dagger}$ of V_x exists and $V_xV_x^{\dagger} = P_{R(V_x)}$, where $P_{R(V_x)}$ stands for the orthogonal projection onto $R(V_x)$. It follows that, with U as in (3.2),

$$\begin{aligned} V_xUP_{\mathcal{X}} &= V_x(V_x^{\dagger}KP_{\mathcal{X}} + W - V_x^{\dagger}V_xWP_{\mathcal{X}})P_{\mathcal{X}} \\ &= V_xV_x^{\dagger}KP_{\mathcal{X}}^2 + V_xWP_{\mathcal{X}} - V_xV_x^{\dagger}V_xWP_{\mathcal{X}}^2 \\ &= P_{R(V_x)}KP_{\mathcal{X}} + V_xWP_{\mathcal{X}} - V_xWP_{\mathcal{X}} \\ &= P_{R(V_x)}P_{\mathcal{X}}K = P_{\mathcal{X}}K = KP_{\mathcal{X}}. \end{aligned}$$

Now let $U : \mathcal{X} \longrightarrow l^2(\mathbb{J})$ satisfies $V_xUP_{\mathcal{X}} = KP_{\mathcal{X}}$. Letting $W = U$ we get

$$\begin{aligned} V_x^{\dagger}KP_{\mathcal{X}} + W - V_x^{\dagger}V_xWP_{\mathcal{X}} &= V_x^{\dagger}KP_{\mathcal{X}} + U - V_x^{\dagger}V_xUP_{\mathcal{X}} \\ &= V_x^{\dagger}KP_{\mathcal{X}} + U - V_x^{\dagger}KP_{\mathcal{X}} \\ &= U. \end{aligned}$$

For the last part of theorem, let $x_n^* := U^*e_n$ then for all $f \in \mathcal{H}$ we have

$$\begin{aligned} \sum_{n\in\mathbb{J}}\langle P_{\mathcal{X}}f, x_n^*\rangle x_n &= \sum_{n\in\mathbb{J}}\langle P_{\mathcal{X}}f, U^*e_n\rangle x_n \\ &= \sum_{n\in\mathbb{J}}\langle UP_{\mathcal{X}}f, e_n\rangle x_n \\ &= \sum_{n\in\mathbb{J}}(UP_{\mathcal{X}}f)(n)x_n \\ &= V_xUP_{\mathcal{X}}f \\ &= KP_{\mathcal{X}}. \end{aligned}$$

□

In Theorem 3.4, we characterized all operators U satisfying $V_xUP_{\mathcal{X}} = KP_{\mathcal{X}}$. Now for a given $\{x_n^*\}_{n\in\mathbb{J}}$ we are going to characterize all operators V which satisfies $VU_{x^*}P_{\mathcal{X}} = KP_{\mathcal{X}}$.

Theorem 3.5. *Let $x^* = \{x_n^*\}_{n\in\mathbb{J}}$ be a Bessel sequence with respect to $\mathcal{X}$ such that $P_{\mathcal{X}}(\overline{span}\{x_n^* : n \in \mathbb{J}\}) = \mathcal{X}$. If $R(U_{x^*}P_{\mathcal{X}})$ is closed and K is a bounded operator such that $K(\mathcal{X}) \subseteq \mathcal{X}$, then the class of all operators satisfying $VU_{x^*}P_{\mathcal{X}} = KP_{\mathcal{X}}$ is given by*

$$V = K(U_{x^*}P_{\mathcal{X}})^{\dagger} + W(I - U_{x^*}P_{\mathcal{X}}(U_{x^*}P_{\mathcal{X}})^{\dagger}). \tag{3.3}$$

Also $\{x_n\}_{n\in\mathbb{J}} := \{Ve_n\}_{n\in\mathbb{J}}$ is a K-dual pseudoframe for $\mathcal{X}$ with respect to $\{x_n^*\}_{n\in\mathbb{J}}$.

Proof. Since $R(U_{x^*}P_{\mathcal{X}})$ is closed the pseudoinverse $(U_{x^*}P_{\mathcal{X}})^{\dagger}$ exists.
Thus

$$\begin{aligned} VU_{x^*}P_{\mathcal{X}} &= (K(U_{x^*}P_{\mathcal{X}})^{\dagger} + W(I - U_{x^*}P_{\mathcal{X}}(U_{x^*}P_{\mathcal{X}})^{\dagger}))(U_{x^*}P_{\mathcal{X}}) \\ &= K(U_{x^*}P_{\mathcal{X}})^{\dagger}U_{x^*}P_{\mathcal{X}} + W(I - U_{x^*}P_{\mathcal{X}}(U_{x^*}P_{\mathcal{X}})^{\dagger})U_{x^*}P_{\mathcal{X}} \\ &= K(U_{x^*}P_{\mathcal{X}})^{\dagger}(U_{x^*}P_{\mathcal{X}}) + WU_{x^*}P_{\mathcal{X}} - WU_{x^*}P_{\mathcal{X}} \\ &= K(U_{x^*}P_{\mathcal{X}})^{\dagger}(U_{x^*}P_{\mathcal{X}}) = KP_{\mathcal{X}}. \end{aligned}$$

If $x_n := Ve_n$, then similar to the proof of Theorem 3.4, we obtain $\sum_{n\in\mathbb{J}}\langle P_{\mathcal{X}}f, x_n^*\rangle x_n = VU_{x^*}P_{\mathcal{X}}f$. □

Proposition 3.6. *Let $\{x_n\}_{n\in\mathbb{J}}$ be a pseudoframe for $\mathcal{X}$ with respect to $\{x_n^*\}_{n\in\mathbb{J}}$ and $K \in B(\mathcal{H})$.*

(i) *If $K(\mathcal{X}) \subseteq \mathcal{X}$, then $\{x_n\}_{n\in\mathbb{J}}$ is a K-pseudoframe for $\mathcal{X}$ with respect to $\{K^*x_n^*\}_{n\in\mathbb{J}}$.*
(ii) *If $R(K^*)$ is closed and $\{x_n^*\}_{n\in\mathbb{J}} \subseteq R(K^*)$, then $\{x_n\}_{n\in\mathbb{J}}$ is a pseudoframe for $K(\mathcal{X})$ with respect to $\{K^{*\dagger}x_n^*\}_{n\in\mathbb{J}}$, where $K^{*\dagger}$ is the pseudoinverse of K^*.*

Proof. (i) For all $f \in \mathcal{X}$ we have $f = \sum_{n\in\mathbb{J}}\langle f, x_n^*\rangle x_n$. Also $K(\mathcal{X}) \subseteq \mathcal{X}$ implies that

$$Kf = \sum_{n\in\mathbb{J}}\langle Kf, x_n^*\rangle x_n = \sum_{n\in\mathbb{J}}\langle f, K^*x_n^*\rangle x_n, \qquad (f \in \mathcal{X}).$$

Trivially $\{K^*x_n^*\}_{n\in\mathbb{J}}$ is a Bessel sequence with respect to $\mathcal{X}$. Indeed

$$\sum_{n\in\mathbb{J}}|\langle f, K^*x_n^*\rangle|^2 = \sum_{n\in\mathbb{J}}|\langle Kf, x_n^*\rangle|^2 \leq B\|K\|^2\|f\|^2 \leq M\|f\|^2, \qquad (f \in \mathcal{X}).$$

(ii) Since $R(K^*)$ is closed, the pseudoinverse of K^* exists. For any $f \in \mathcal{X}$ we have

$$Kf = \sum_{n\in\mathbb{J}}\langle f, x_n^*\rangle x_n = \sum_{n\in\mathbb{J}}\langle f, K^*K^{*\dagger}x_n^*\rangle x_n = \sum_{n\in\mathbb{J}}\langle Kf, K^{*\dagger}x_n^*\rangle x_n. \tag{3.4}$$

Also $\{K^{*\dagger}x_n^*\}_{n\in\mathbb{J}}$ is a Bessel sequence with respect to $K(\mathcal{X})$, since for any $f \in K(\mathcal{X})$

$$\sum_{n\in\mathbb{J}}|\langle f, K^{*\dagger}x_n^*\rangle|^2 = \sum_{n\in\mathbb{J}}|\langle (K^{*\dagger})^*f, x_n^*\rangle|^2 \leq B\|(K^{*\dagger})^*\|^2\|f\|^2 \leq M\|f\|^2.$$

□

As an application of Proposition 3.6, we get the following example.

An example of K-pseudoframe on $L^2(\mathbb{R})$

We know that an integral transform is any transform T on $L^2(\mathbb{R})$ of the following form

$$(Tf)(u) = \int_{\mathbb{R}} \kappa(t,u)f(t)dt,$$

where $\kappa \in L^2(\mathbb{R}^2)$. Also $\|T\| = \|\kappa\|$, so the fact that $\kappa \in L^2(\mathbb{R}^2)$ implies that T is bounded and $(T^*f)(u) = \int_{\mathbb{R}} \overline{\kappa(t,u)}f(t)dt$.

Let ϕ be defined by its Fourier transform as follows

$$\hat{\phi}(\gamma) = \begin{cases} 1 & a.e. -\frac{1}{4} \leq \gamma < \frac{1}{4} \\ 2-4|\gamma| & a.e. \frac{1}{4} \leq |\gamma| < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Choose $\Omega = \{\gamma \in \mathbb{R} : |\hat{\phi}(\gamma) \geq 1|\} = [-\frac{1}{4}, \frac{1}{4})$ and $\mathcal{X} = PW_\Omega = \{f \in L^2(\mathbb{R}) : Supp\hat{f} \subseteq \Omega\}$. As in Example 1 of [10], select ϕ^* such that

$$\hat{\phi^*}(\gamma) = \begin{cases} 1 & a.e. -\frac{1}{4} \leq \gamma < \frac{1}{4} \\ 3-8|\gamma| & a.e. \frac{1}{4} \leq |\gamma| < \frac{3}{8} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{\tau_n\phi\}_{n\in\mathbb{J}}$ and $\{\tau_n\phi^*\}_{n\in\mathbb{J}}$ form a pair of pseudoframe for $\mathcal{X}$, where $(\tau_n f)(x) = f(x-n)$. Now for any $\kappa(x,y) = r(x)s(y)$ such that $r \in \mathcal{X}, \kappa \in L^2(\mathbb{R}^2)$ we have

$$(Kf)(x) = \int_{\mathbb{R}} \kappa(x,y)f(y)dy = r(x)\int_{\mathbb{R}} s(y)f(y)dy$$

, so K is a bounded linear operator on $L^2(\mathbb{R})$ and $K(\mathcal{X}) \subseteq \mathcal{X}$. As an example of such a κ, let

$$r(x) = \frac{8sin(\frac{\pi}{4}x)}{\pi x}, s(y) = \frac{10sin(\frac{\pi}{5}y)}{\pi y}.$$

Obviously,

$$\hat{r}(\gamma) = \chi_{[-\frac{1}{8},\frac{1}{8})}(\gamma), \hat{s}(\gamma) = \chi_{[-\frac{1}{10},\frac{1}{10})}(\gamma) \in \mathcal{X},$$

and so $\hat{r}, \hat{s} \in \mathcal{X}$. Also

$$(Kf)(x) = r(x)\int_{\mathbb{R}} s(y)f(y)dy \in \mathcal{X}, \qquad (f \in \mathcal{X}).$$

Thus $K(\mathcal{X}) \subseteq \mathcal{X}$. Clearly K is self adjoint, which means $(K^*f)(x) = r(x)\int_{\mathbb{R}} s(y)f(y)dy$. Now by part (i) of Proposition 3.6, we have $\{\tau_n\phi\}_{n\in\mathbb{J}}$ is a K-pseudoframe for $\mathcal{X}$ with respect to
$\{K^*\tau_n\phi^*\}_{n\in\mathbb{J}} = \{\frac{8sin(\frac{\pi}{4}x)}{\pi x}\int_{\mathbb{R}}\frac{10sin(\frac{\pi}{5}y)}{\pi y}\tau_n\phi^*(y)dy\}_{n\in\mathbb{J}}$.

Proposition 3.7. *Let $\{x_n\}_{n\in\mathbb{J}}$ and $\{x_n^*\}_{n\in\mathbb{J}}$ be two sequences in $\mathcal{H}$, the operators U_{x^*}, V_x are defined as* (2.1), (2.2) *and $K \in B(\mathcal{H})$ with $K(\mathcal{X}) \subseteq \mathcal{X}$. Then $\{x_n^*\}_{n\in\mathbb{J}}$ is K^*-pseudoframe for $\mathcal{X}$ with respect to $\{x_n\}_{n\in\mathbb{J}}$ if and only if $KP_{\mathcal{X}} = P_{\mathcal{X}}V_xU_{x^*}$.*

Proof. For all $f, g \in \mathcal{H}$ we have

$$\begin{aligned}\langle P_{\mathcal{X}}f, V_xU_{x^*}g\rangle &= \overline{\langle V_xU_{x^*}g, P_{\mathcal{X}}f\rangle} = \overline{\langle\sum_{n\in\mathbb{J}}\langle g, x_n^*\rangle x_n, P_{\mathcal{X}}f\rangle}\\ &= \sum_{n\in\mathbb{J}}\langle P_{\mathcal{X}}f, x_n\rangle\langle x_n^*, g\rangle = \langle\sum_{n\in\mathbb{J}}\langle P_{\mathcal{X}}f, x_n\rangle x_n^*, g\rangle\\ &= \langle K^*P_{\mathcal{X}}f, g\rangle = \langle P_{\mathcal{X}}f, Kg\rangle.\end{aligned}$$

Hence

$$P_{\mathcal{X}}V_xU_{x^*} = P_{\mathcal{X}}K = KP_{\mathcal{X}}.$$

Conversely, if $P_{\mathcal{X}}V_xU_{x^*} = P_{\mathcal{X}}K = KP_{\mathcal{X}}$, then for any $f, g \in \mathcal{H}$

$$\begin{aligned}\langle P_{\mathcal{X}}f, V_xU_{x^*}g\rangle &= \langle\sum_{n\in\mathbb{J}}\langle P_{\mathcal{X}}f, x_n\rangle x_n^*, g\rangle\\ &= \langle P_{\mathcal{X}}f, Kg\rangle = \langle K^*P_{\mathcal{X}}f, g\rangle.\end{aligned}$$

Thus $K^*P_{\mathcal{X}}f = \sum_{n\in\mathbb{J}}\langle P_{\mathcal{X}}f, x_n\rangle x_n^*$. □

Remark 3.8. By Proposition 3.7, $\{x_n\}_{n\in\mathbb{J}}$ interchanges by $\{x_n^*\}_{n\in\mathbb{J}}$ if and only if $P_{\mathcal{X}}V_xU_{x^*} = KP_{\mathcal{X}} = V_xU_{x^*}P_{\mathcal{X}}$.

The following theorem is a characterization of K-dual pseudoframes for a closed subspace $\mathcal{X}$ of $\mathcal{H}$.

Theorem 3.9. *Let $K \in B(\mathcal{H})$ and $\{x_n\}_{n\in\mathbb{J}}$ be a K-pseudoframe for $\mathcal{X}$ with respect to $\{x_n^*\}_{n\in\mathbb{J}}$. If $\{y_n^*\}_{n\in\mathbb{J}} = \{x_n^* + \phi^*e_n\}_{n\in\mathbb{J}}$ for a bounded linear operator $\phi : \mathcal{X} \longrightarrow l^2(\mathbb{J})$, then $\{x_n\}_{n\in\mathbb{J}}$ is K-pseudoframe for $\mathcal{X}$ with respect to $\{y_n^*\}_{n\in\mathbb{J}}$ if and only if $V_x\phi = 0$.*

Proof. For all $f \in \mathcal{X}$ we have

$$(\sum_{n\in\mathbb{J}}|\langle f, y_n^*\rangle|^2)^{\frac{1}{2}} \leq (\sum_{n\in\mathbb{J}}|\langle f, x_n^*\rangle|^2)^{\frac{1}{2}} + (\sum_{n\in\mathbb{J}}\langle f, \phi^*e_n\rangle|^2)^{\frac{1}{2}} \leq C\|f\| + \|\phi\|\|f\|.$$

So $\{y_n^*\}_{n\in\mathbb{J}}$ is a Bessel sequence with respect to $\mathcal{X}$. Also we have

$$\begin{aligned}\sum_{n\in\mathbb{J}}\langle f, y_n^*\rangle x_n &= \sum_{n\in\mathbb{J}}\langle f, x_n^*\rangle x_n + \langle f, \phi^*e_n\rangle\\ &= Kf + \sum_{n\in\mathbb{J}}\langle\phi f, e_n\rangle x_n = Kf + \sum_{n\in\mathbb{J}}(\phi f)(n)x_n\\ &= Kf + V_x\phi f = Kf.\end{aligned}$$

□

Another characterization of K-dual pseudoframes for $\{x_n\}_{n\in\mathbb{J}}$ is obtained in the following theorem.

Theorem 3.10. *Let $K \in B(\mathcal{H})$ and $\{x_n\}_{n\in\mathbb{J}}$ be K-pseudoframe for $\mathcal{X}$ with respect to $\{x_n{}^*\}_{n\in\mathbb{J}}$, U_{x^*}, V_x are defined by (2.1), (2.2) and $R(V_xU_{x^*})$ be closed. If $\{y^*{}_n\}_{n\in\mathbb{J}}$ be a K-dual pseudoframe for $\{x_n\}_{n\in\mathbb{J}}$ then there exists some bounded linear operator $\phi : \mathcal{X} \longrightarrow l^2(\mathbb{J})$ such that $K^*(V_xU_{x^*})^{\dagger^*}x_n{}^* + \phi^*e_n = y_n^*$ and $V_x\phi = 0$.*

Proof. For any $f \in \mathcal{X}$

$$\begin{aligned}\sum_{n\in\mathbb{J}}&\langle f, K^*(V_xU_{x^*})^{\dagger^*}P_{R(V_xU_{x^*})}x_n^*\rangle x_n\\ &= \sum_{n\in\mathbb{J}}\langle P_{R(V_xU_{x^*})}(V_xU_{x^*})^{\dagger}Kf, x_n^*\rangle x_n\\ &= V_xU_{x^*}P_{R(V_xU_{x^*})}(V_xU_{x^*})^{\dagger}Kf\\ &= Kf.\end{aligned}$$

So $\{K^*(V_xU_{x^*})^{\dagger^*}P_{R(VU)}x_n^*\}_{n\in\mathbb{J}}$ is a K-dual pseudoframe.
Define $U_y : \mathcal{X} \longrightarrow l^2(\mathbb{J})$ by $U_yf = \{\langle f, y_n^*\rangle\}_{n\in\mathbb{J}}$. Now letting

$$\phi = U_y - U_{x^*}(V_xU_{x^*})^{\dagger}K,$$

one can see that ϕ is bounded and

$$\begin{aligned}V_x\phi f &= V_xU_yf - V_xU_{x^*}(V_xU_{x^*})^{\dagger}Kf\\ &= Kf - P_{R(V_xU_{x^*})}Kf = 0, \qquad (f \in \mathcal{X}).\end{aligned}$$

Moreover, since $U_{x^*}^*e_n = x_n^*, U_y^*e_n = y_n^*$ we have

$$\begin{aligned}K^*(V_xU_{x^*})^{\dagger^*}x_n^* &+ (U_y - U_{x^*}(V_xU_{x^*})^{\dagger}K)^*e_n\\ &= K^*(V_xU_{x^*})^{\dagger^*}x_n^* + U_y^*e_n - K^*(V_xU_{x^*})^{\dagger^*}U_{x^*}^*e_n\\ &= K^*(V_xU_{x^*})^{\dagger^*}x_n^* + y_n^* - K^*(V_xU_{x^*})^{\dagger^*}x_n^*\\ &= y_n^*.\end{aligned}$$

□

Proposition 3.11. *If $\{x_n\}_{n\in\mathbb{J}}$ is a minimal sequence and $\{x_n^*\}_{n\in\mathbb{J}}, \{y_n^*\}_{n\in\mathbb{J}}$ are two K-dual pseudoframes of $\{x_n\}_{n\in\mathbb{J}}$. Then $\{P_\mathcal{X}x_n^*\}_{n\in\mathbb{J}} = \{P_\mathcal{X}y_n^*\}_{n\in\mathbb{J}}$.*

Proof. If $\{y_n^*\}_{n\in\mathbb{J}}$ and $\{x_n^*\}_{n\in\mathbb{J}}$ are K-dual pseudoframes of $\{x_n\}_{n\in\mathbb{J}}$, then $\sum_{n\in\mathbb{J}}(\langle P_\mathcal{X}f, x_n^*\rangle - \langle P_\mathcal{X}f, y_n^*\rangle)x_n = 0$, for all $f \in \mathcal{H}$. So for all $f \in \mathcal{H}, n \in \mathbb{J}$, we have $\langle P_\mathcal{X}f, x_n^*\rangle = \langle P_\mathcal{X}, y_n^*\rangle$. Thus $\{P_\mathcal{X}x_n^*\}_n = \{P_\mathcal{X}y_n^*\}_n$. □

Corollary 3.12. *Let $\{x_n\}_{n\in\mathbb{J}}$ be a minimal K-pseudoframe with respect to $\{x_n^*\}_{n\in\mathbb{J}}$ and for some x_m, $x_m \neq 0$, $\{x_n\}_{n\neq m}$ is a K-pseudoframe with respect to $\{x_n^*\}_{n\neq m}$. Then $P_\mathcal{X}x_m = 0$. Moreover, for every K-dual pseudoframe $\{y_n^*\}_n$, $P_\mathcal{X}y_m^* = 0$.*

Proof. For all $f \in \mathcal{H}$ we have

$$KP_\mathcal{X}f = \sum_{n\in\mathbb{J}}\langle P_\mathcal{X}f, x_n^*\rangle x_n = \sum_{n\neq m}\langle P_\mathcal{X}f, x_n^*\rangle x_n.$$

So $\langle P_\mathcal{X}f, x_m^*\rangle = 0$. Thus for all $f \in \mathcal{H}$, $\langle f, P_\mathcal{X}x_m^*\rangle = 0$. This implies that $P_\mathcal{X}x_m^* = 0$.
Also by Proposition 3.11, for any K-dual pseudoframe $\{y_n^*\}_{n\in\mathbb{J}}$, $P_\mathcal{X}y_n^* = 0$. □

A sequence $\{x_n\}_{n\in\mathbb{J}} \subseteq \mathcal{H}$ is called complete if $\langle f, x_n\rangle = 0$, for all $f \in \mathcal{H}$ implies that $f = 0$. Note that $\mathcal{N}(V_x) = \{\{c_n\}_{n\in\mathbb{J}} \in l^2(\mathbb{J}) : V_x(\{c_n\}_{n\in\mathbb{J}}) = 0\}$.

Lemma 3.13. *Let $\{x_n\}_{n\in\mathbb{J}}$ is a K-pseudoframe for $\mathcal{X}$ with respect to $\{x_n^*\}_{n\in\mathbb{J}}$ and U_{x^*}, V_x defined by* (2.1), (2.2) *such that $R(U_{x^*}) \subseteq R(V_x^*)$. If $f \in \mathcal{X}$ and $Kf = \sum_{n\in\mathbb{J}} c_n x_n$ for some scaler coefficients $\{c_n\}_{n\in\mathbb{J}}$, then*

$$\begin{aligned} \textstyle\sum_{n\in\mathbb{J}} |c_n|^2 \quad &= \quad \textstyle\sum_{n\in\mathbb{J}} |\langle f, K^*(V_xU_{x^*})^{\dagger^*} x_n^*\rangle|^2 \\ &\quad + \textstyle\sum_{n\in\mathbb{J}} |c_n - \langle f, K^*(V_xU_{x^*})^{\dagger^*} x_n^*\rangle|^2. \end{aligned} \tag{3.5}$$

Proof. First we note that the condition $R(U_{x^*}) \subseteq R(V_x^*)$ implies that $\mathcal{N}(V_x) \subseteq R(U_{x^*})^\perp$. Suppose that $Kf = \sum_{n\in\mathbb{J}} c_n x_n$. We have

$$\begin{aligned} \{c_n\}_{n\in\mathbb{J}} = \{c_n\}_{n\in\mathbb{J}} &- \{\langle f, K^*(V_xU_{x^*})^{\dagger^*} P_{R(V_xU_{x^*})} x_n^*\rangle\}_{n\in\mathbb{J}} \\ &+ \{\langle f, K^*(V_xU_{x^*})^{\dagger^*} P_{R(V_xU_{x^*})} x_n^*\rangle\}_{n\in\mathbb{J}}. \end{aligned}$$

On the other hand

$$\sum_{n\in\mathbb{J}} (c_n - \langle f, K^*(V_xU_{x^*})^{\dagger^*} P_{R(V_xU_{x^*})} x_n^*\rangle) x_n = 0.$$

So

$$\{c_n\}_{n\in\mathbb{J}} - \{\langle f, K^*(V_xU_{x^*})^{\dagger^*} P_{R(V_xU_{x^*})} x_n^*\rangle\}_{n\in\mathbb{J}} \in \mathcal{N}(V_x) \subseteq R(U_{x^*})^\perp.$$

Now by the fact that $\{\langle f, K^*(V_xU_{x^*})^{\dagger^*} P_{R(V_xU_{x^*})} x_n^*\rangle\}_{n\in\mathbb{J}}$ belongs to $R(U_{x^*})$, we obtain (3.5). □

Theorem 3.14. *Let $\{x_n\}_{n\in\mathbb{J}}$ be a K-pseudoframe for $\mathcal{X}$ with respect to $\{x_n^*\}_{n\in\mathbb{J}}$ for a closed range operator $K \in B(\mathcal{H})$ and $R(U_{x^*}) \subseteq R(V_x)$. If $\langle K^\dagger P_{R(K)} x_j, x_j^*\rangle = 1$, then $\{x_n^*\}_{n\neq j}$ is not complete.*

Proof. Choose an arbitrary $j \in \mathbb{J}$. We know that

$$P_{R(K)} x_j = KK^\dagger P_{R(K)} x_j = \sum_{n\in\mathbb{J}} \langle K^\dagger P_{R(K)} x_j, x_n^*\rangle x_n,$$

so

$$P_{R(K)} x_j = P_{R(K)}^2 x_j = \sum_{n\in\mathbb{J}} \langle K^\dagger P_{R(K)} x_j, x_n^*\rangle P_{R(K)} x_n.$$

On the other hand we have

$$P_{R(K)} x_j = \sum_{n\in\mathbb{J}} \delta_{nj} P_{R(K)} x_n.$$

Now by Lemma 3.13, we obtain

$$\begin{aligned} 1 = \sum_{n\in\mathbb{J}} |\delta_{jn}|^2 = &\sum_{n\in\mathbb{J}} |\langle K^\dagger P_{R(K)} x_j, x_n^*\rangle|^2 + \sum_{n\in\mathbb{J}} |\langle K^\dagger P_{R(K)} x_j, x_n^*\rangle - \delta_{jn}|^2 \\ = &|\langle K^\dagger P_{R(K)} x_j, x_j^*\rangle|^2 + \sum_{n\neq j} |\langle K^\dagger P_{R(K)} x_j, x_n^*\rangle|^2 \\ &+ |\langle K^\dagger P_{R(K)} x_j, x_j^*\rangle - \delta_{jj}|^2 + \sum_{n\neq j} |\langle K^\dagger P_{R(K)} x_j, x_n^*\rangle|^2. \end{aligned}$$

So $\sum_{n\neq j} |\langle K^\dagger P_{R(K)} x_j, x_n^*\rangle|^2 = 0$. This implies that for all $n \neq j$, $|\langle K^\dagger P_{R(K)} x_j, x_n^*\rangle|^2 = 0$, which shows that $K^\dagger P_{R(K)} x_j$ is orthogonal to $x_n^*, n \neq j$. Thus $\{x_n^*\}_{n\neq j}$ is not complete. □

4. Pseudoatomic systems

In this section, we introduce the concept of the pseudoatomic systems for a bounded operator K and its relation with K-pseudoframe is studied.

Definition 4.1. Let $\mathcal{X}$ is a closed subspace of $\mathcal{H}$. A sequence $\{x_n\}_{n\in\mathbb{J}} \subset \mathcal{H}$ is called a pseudoatomic system for K, if the following conditions are satisfied

(i) $\{x_n\}_{n\in\mathbb{J}}$ is a Bessel sequence;
(ii) For any $f \in \mathcal{X}$, there exists $a_f = \{a_n\}_{n\in\mathbb{J}} \in l^2(\mathbb{J})$ such that $Kf = \sum_{n\in\mathbb{J}} a_n x_n$, where $\|a_f\|_{l^2(\mathbb{J})} \leq C\|f\|$, C is positive constant.

The following Theorem shows the relation between K-pseudoframe and pseudoatomic system for K for a closed subspace $\mathcal{X} \in \mathcal{H}$.

Theorem 4.2. *Let K be a bounded operator. A sequence $\{x_n\}_{n\in\mathbb{J}}$ is a K-pseudoframe with respect to $\{x_n^*\}_{n\in\mathbb{J}}$ for $\mathcal{X}$ if and only if $\{x_n\}_{n\in\mathbb{J}}$ is a pseudoatomic system for K with respect to $\mathcal{X}$.*

Proof. By Definition 3.1, if $\{x_n\}_{n\in\mathbb{J}}$ is a K-pseudoframe for $\mathcal{X}$ with respect to $\{x_n^*\}_{n\in\mathbb{J}}$, then $\{x_n^*\}_{n\in\mathbb{J}}$ is a Bessel sequence with respect to $\mathcal{X}$ and $Kf = \sum_{n\in\mathbb{J}}\langle f, x_n^*\rangle x_n$ for all $f \in \mathcal{X}$. Thus the condition (ii) in Definition 4.1 holds. Also by Definition 3.1, $\{x_n\}_{n\in\mathbb{J}}$ is a Bessel sequence, so the condition (i) in Definition 4.1 is valid.

Conversely, by Definition 4.1, $\{x_n\}_{n\in\mathbb{J}}$ is a Bessel sequence and so there exists a bounded linear operator $T : l^2(\mathbb{J}) \longrightarrow \mathcal{H}$ such that $Te_n = x_n, n \in \mathbb{J}$. Since $Kf = \sum_{n\in\mathbb{J}} a_n x_n$, then $R(K) \subseteq R(T)$. Now by Theorem 2.6 there exists a bounded linear operator $M : \mathcal{H} \longrightarrow l^2(\mathbb{J})$ such that $K = TM$. Now set $a_n(f) = (Mf)_n$, where $(Mf)_n$ denotes the n^{th} component of Mf, we have

$$|a_n| \leq (\sum_{n\in\mathbb{J}} |a_n|^2)^{\frac{1}{2}} = \|a_f\|_{l^2(\mathbb{J})} \leq \|M\|\|f\|, \qquad (f \in \mathcal{X}).$$

Then by Riesz representation theorem, there exists x_n^* such that $a_n(f) = \langle f, x_n^*\rangle$. Hence for all $f \in \mathcal{X}$ we have

$$Kf = TMf = T(\{a_n\}_{n\in\mathbb{J}}) = \sum_{n\in\mathbb{J}}\langle f, x_n^*\rangle x_n.$$

Also for all $f \in \mathcal{X}$

$$\sum_{n\in\mathbb{J}} |\langle f, x_n^*\rangle|^2 = \sum_{n\in\mathbb{J}} |a_n|^2 \leq \|M\|^2\|f\|^2.$$

So $\{x_n^*\}_{n\in\mathbb{J}}$ is a Bessel with respect to $\mathcal{X}$. □

As an application of Theorem 4.2, we get a relation between K-exact and K-minimal pseudoframes.

Definition 4.3. Let $\{x_n\}_{n\in J}$ be K-pseudoframe for $\mathcal{X}$ with respect to $\{x_n^*\}_{n\in\mathbb{J}}$. We say $\{x_n\}_{n\in\mathbb{J}}$ is an K-exact pseudoframe with respect to $\{x_n^*\}_{n\in\mathbb{J}}$ if for every $j \in J$ the sequence $\{x_n\}_{i\neq j}$ is not a K-pseudoframe for $\mathcal{X}$.

Proposition 4.4. *Every K-exact pseudoframe is a K-minimal pseudoframe.*

Proof. Assume that $\{x_n\}_{n\in\mathbb{J}}$ is not a minimal pseudoframe. Let $x_i \neq 0$ for each i. Then there exists $\{c_n\}_{n\in\mathbb{J}}$ with $c_m \neq 0$ such that $x_m = \frac{-1}{c_m}\sum_{i\neq m} c_i x_i$, for some m. This implies that $\{x_i\}_{i\neq m}$ is a pseudoatomic system. Thus by Theorem 4.2, it is a K-pseudoframe. This shows that $\{x_n\}_{n\in\mathbb{J}}$ is not a K-exact pseudoframe. □

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