Research Article

Ladders and fan graphs are cycle-antimagic

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Abstract

A simple graph \( G = (V, E) \) admits an \( H \)-covering if every edge in \( E \) belongs to at least one subgraph of \( G \) isomorphic to a given graph \( H \). The graph \( G \) admitting an \( H \)-covering is \((a, d)\)-\( H \)-antimagic if there exists a bijection \( f : V \cup E \to \{1, 2, \ldots, |V| + |E|\} \) such that, for all subgraphs \( H' \) of \( G \) isomorphic to \( H \), the \( H' \)-weights, \( wt_f(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e) \), form an arithmetic progression with the initial term \( a \) and the common difference \( d \). Such a labeling is called super if the smallest possible labels appear on the vertices. In this paper we prove the existence of super \((a, d)\)-\( H \)-antimagic labelings of fan graphs and ladders for \( H \) isomorphic to a cycle.

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1. Introduction

Let \( G = (V, E) \) be a finite simple graph. An edge-covering of \( G \) is a family of subgraphs \( H_1, H_2, \ldots, H_t \) such that each edge of \( E \) belongs to at least one of the subgraphs \( H_i \), \( i = 1, 2, \ldots, t \). Then it is said that \( G \) admits an \((H_1, H_2, \ldots, H_t)\)-edge covering. If every subgraph \( H_i \) is isomorphic to a given graph \( H \), then the graph \( G \) admits an \( H \)-covering. A bijective function \( f : V \cup E \to \{1, 2, \ldots, |V| + |E|\} \) is an \((a, d)\)-\( H \)-antimagic labeling of a graph \( G \) admitting an \( H \)-covering whenever, for all subgraphs \( H' \) isomorphic to \( H \), the \( H' \)-weights

\[
wt_f(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)
\]

form an arithmetic progression \( a, a + d, a + 2d, \ldots, a + (t - 1)d \), where \( a > 0 \) and \( d \geq 0 \) are two integers, and \( t \) is the number of all subgraphs of \( G \) isomorphic to \( H \). Such a labeling is called super if the smallest possible labels appear on the vertices. A graph that admits

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a (super) $(a, d)$-$H$-antimagic labeling is called (super) $(a, d)$-$H$-antimagic. For $d = 0$ it is called $H$-magic and $H$-supermagic, respectively.

The $H$-supermagic graphs were first studied by Gutíérrez and Lladó [10] as an extension of the edge-magic and super edge-magic graphs introduced by Kotzig and Rosa [14] and Enomoto et al. [9], respectively. Lladó and Moragas [15] studied the cycle-(super)magic behavior of several classes of connected graphs. Further results on $H$-supermagic labelings are available in [18–21].

The $(a, d)$-$H$-antimagic labeling was introduced by Inayah et al. [11]. In [12] the super $(a, d)$-$H$-antimagic labeling for some shackles of a connected graph $H$ was investigated. In [22] it was proved that wheels $W_n$, $n \geq 3$ are super $(a, d)$-$C_k$-antimagic for every $k = 3, 4, \ldots, n - 1, n + 1$ and $d = 0, 1, 2$.

The (super) $(a, d)$-$H$-antimagic labeling is related to a super $d$-antimagic labeling of type $(1, 1, 0)$ of a plane graph, that is the generalization of a face-magic labeling introduced by Lih [16]. A labeling of type $(1, 1, 0)$, that is a total labeling, of a plane graph is said to be $d$-antimagic if for every positive integer $s$ the set of weights of all $s$-sided faces is $W_s = \{a_s, a_s + d, a_s + 2d, \ldots, a_s + (f_s - 1)d\}$ for some integers $a_s$ and $d \geq 0$, where $f_s$ is the number of the $s$-sided faces. Note that, we allow different sets $W_s$ for different $s$. The weight of a face under a labeling of type $(1, 1, 0)$ is the sum of labels of all the edges and vertices surrounding that face. If $d = 0$ then Lih called such labeling as magic [16] and described magic (0-antimagic) labelings of type $(1, 1, 0)$ for wheels, friendship graphs and prisms. For further information on super $d$-antimagic labelings, we refer to [1, 4, 6].

For $H \cong K_2$, a (super) $(a, d)$-$H$-antimagic labeling is also called a (super) $(a, d)$-edge-antimagic total labeling, introduced in [23] and further studied in [3, 17]. The vertex version of these labelings for generalized pyramid graphs is given in [2].

The existence of super $(a, d)$-$H$-antimagic labelings for disconnected graphs is studied in [7]. In [5] it is shown that the disjoint union of multiple copies of a (super) $(a, 1)$-tree-antimagic graph is also a (super) $(b, 1)$-tree-antimagic. This leads to a natural question. Whether the similar result holds also for other differences and other $H$-antimagic graphs.

A fan graph $F_n$, $n \geq 2$, is a graph obtained by joining all the vertices of the path $P_n$ to a further vertex, called the centre. The vertices on the path are called path vertices, the edges adjacent to the central vertex are called spokes and the remaining edges are called path edges. Thus $F_n$ contains $n + 1$ vertices, say, $v_1, v_2, \ldots, v_{n + 1}$, and $2n - 1$ edges, say, $v_i v_{i + 1}$, $1 \leq i \leq n$, and $v_nv_1$, $1 \leq n - 1$.

A ladder $P_2 \square P_2$, $n \geq 2$, is a Cartesian product of a path on $n$ vertices and a path on 2 vertices. The ladder $P_2 \square P_2$ contains $2n$ vertices, say, $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$, and $3n - 2$ edges $v_i u_{i + 1}, 1 \leq i \leq n - 1, u_i u_{i + 1}, 1 \leq i \leq n - 1$ and $v_i u_i, 1 \leq i \leq n$.

In this paper we investigate the super $(a, d)$-$H$-antimagic labelings of fan graphs and ladders for $H$ isomorphic to a cycle. We also prove some results on the existence of super $(a, d)$-cycle-antimagic labelings of fan graphs for several values of parameter $d$.

In [8] Bača et al. proved the following:

**Proposition 1.1.** [8] The fan graph $F_n$, $n \geq 3$, admits a super $(a, d)$-$C_k$-antimagic labeling for $k = 3, 4, \ldots, \left\lceil \frac{d}{2} \right\rceil + 2$ and $d \in \{1, 3, k - 7, k + 1, 2k - 5, 2k - 1, 3k - 9, 3k - 1\}$.

They showed that for a fan graph $F_n$, $n \geq 4$, there exists a super $(a, d)$-$C_3$-antimagic labeling for $d = 0, 1, 2, 3, 4, 5, 6, 8$ and a super $(a, d)$-$C_4$-antimagic labeling for $d = 0, 1, 2, 3, 4, 5, 6, 7, 11$ and they proposed the following open problem.

**Open Problem 1.** [8] Find a super $(a, d)$-$C_k$-antimagic labeling of the fan graph $F_n$ for $d \neq 1, 3, k - 7, k + 1, 2k - 5, 2k - 1, 3k - 9, 3k - 1$.

In [13] the following proposition was proved.

**Proposition 1.2.** [13] The fan graph $F_n$, $n \geq 3$, admits a super $(a, d)$-$C_k$-antimagic labeling for $k = 3, 4, \ldots, \left\lceil \frac{d}{2} \right\rceil + 2$ and $d \in \{1, 2, k - 5, k - 4, \ldots, k + 2, 2k - 5, 2k - 1\}$. 
In this paper we prove that the fan graphs are super $\langle a, d \rangle$-$C_k$-antimagic for $d \in \{0, 4, 5, 7\}$, $k = 3, 4, \ldots, \left\lceil \frac{n}{2} \right\rceil + 2$. Further we use the known results for fan graphs to obtain super cycle-antimagic labelings of ladders.

2. Fan graphs

Let $n, r$ be positive integers, $2 \leq r \leq n - 1$. Let $n = (t - 1)r + s$, where $2 \leq r \leq n - 1$ and $0 \leq s < r$. Thus $t = \lceil \frac{n}{r} \rceil$. The set of integers $\{1, 2, \ldots, n\}$, $n \geq 2$, can be arranged as a sequence $\{a_i^{r,n}\}_{i=1}^n$ such that the $i^{th}$ element $a_i^{r,n}$, $i = ar + \beta$, $0 \leq \alpha \leq t - 1$, $1 \leq \beta \leq r$, is defined as follows:

$$a_i^{r,n} = \begin{cases} 
(\beta - 1)t + \alpha + 1, & \text{if } 0 \leq \alpha \leq t - 1, 1 \leq \beta \leq s, \\
(\beta - 1)(t - 1) + s + \alpha + 1, & \text{if } 0 \leq \alpha \leq t - 2, s + 1 \leq \beta \leq r.
\end{cases}$$

That is

$$\{a_i^{r,n}\}_{i=1}^n = \{1, t + 1, \ldots, (s - 1)t + 1, st + 1, (s + 1)t, \ldots, (r - 1)t - r + s + 2, 2, t + 2, \ldots, (s - 1)t + 2, st + 2, (s + 1)t + 1, \ldots, (r - 1)t - r + s + 3, \ldots, n, t - 1, 2t - 1, \ldots, st - 1, (s + 1)t - 1, (s + 2)t - 2, \ldots, n, 2t, \ldots, st\}.$$  

In [13] it was proved that the sequence $\{a_i^{r,n}\}_{i=1}^n$ has the property that the sum of any $r$ consecutive members of this sequence differs by 1 from the sum of the foregoing $r$ consecutive members of this sequence.

**Proposition 2.1.** [13] Let $n, r$ be positive integers, $2 \leq r \leq n - 1$. The elements of $\{1, 2, \ldots, n\}$ can be arranged as a sequence $\{a_i^{r,n}\}_{i=1}^n$ such that $\sum_{j=t+1}^{t+r} a_j^{r,n} - \sum_{j=1}^{t+r-1} a_j^{r,n} = 1$ for $1 \leq i \leq n - r + 1$.

We immediately obtain that for $i = 1, 2, \ldots, n - r + 1$

$$1 = \sum_{j=t+1}^{i+r} a_j^{r,n} - \sum_{j=1}^{i+r-1} a_j^{r,n} = a_{i+r}^{r,n} - a_i^{r,n},$$

$$\sum_{j=i}^{i+r-1} a_j^{r,n} = r(st + n - t + 3) - s(n + 1) \frac{n}{2} - 1 + j.$$

Next theorem shows that the fan graph admits super cycle-antimagic labelings for differences $d = 0, 4, 5$ and $d = 7$.

**Theorem 2.2.** Let $n \geq 3$ be a positive integer and $3 \leq k \leq \left\lceil \frac{n}{2} \right\rceil + 2$. Then the fan graph $F_n$ admits a super $\langle a, d \rangle$-$C_k$-antimagic labeling for $d = 0, 4, 5, 7$.

**Proof.** Let $n \geq 3$ be a positive integer and $3 \leq k \leq \left\lceil \frac{n}{2} \right\rceil + 2$. We consider the total labelings $f_m$, $m = 0, 4, 5, 7$, of $F_n$ defined in the following way. The vertices are labeled with the numbers $1, 2, \ldots, n + 1$ such that

$$f_m(v_i) = a_i^{k-1,n},$$

$$f_5(v_i) = n + 1 - a_i^{k-1,n},$$

$$f_m(v_{n+1}) = n + 1,$$  

where $a_i^{k-1,n}$ is an element of the sequence $\{a_i^{k-1,n}\}_{i=1}^n$, see (2.1).
For $f_m$, $m = 0, 4, 5, 7$, we assign numbers $n + 2, n + 3, \ldots, 3n$ to the edges of $F_n$ in the following way:

\[
\begin{align*}
f_m(v_i v_{i+1}) &= a_i^{k-2,n-1} + n + 1, & \text{for } i = 1, 2, \ldots, n - 1 \text{ and } m = 0, 4, \\
f_m(v_i v_{i+1}) &= 2a_i^{k-2,n-1} + n + 1, & \text{for } i = 1, 2, \ldots, n - 1 \text{ and } m = 5, 7, \\
f_0(v_i v_{n+i}) &= 3n + 1 - i, & \text{for } i = 1, 2, \ldots, n, \\
f_4(v_i v_{n+i}) &= 2n + i, & \text{for } i = 1, 2, \ldots, n, \\
f_m(v_i v_{n+i}) &= n + 2i, & \text{for } i = 1, 2, \ldots, n \text{ and } m = 5, 7,
\end{align*}
\]

where $a_i^{k-2,n-1}$ is an element of the sequence $\{a_i^{k-2,n-1}\}_{i=1}^{n-1}$, see (2.1). It is easy to see that $f_m$, $m = 0, 4, 5, 7$, is a bijection.

Every cycle $C_k$ on $k$ vertices in a fan graph $F_n$ is of the form $C_k = v_j v_{j+1} v_{j+2} \ldots v_{j+k-2} v_{n+1} v_j$, where $j = 1, 2, \ldots, n - k + 2$. As $k = 3, 4, \ldots, \lfloor \frac{n}{2} \rfloor + 2$ every edge of $F_n$ belongs at least to one cycle $C_k$.

For the $C_k$-weight of the cycle $C_k^j$, $j = 1, 2, \ldots, n - k + 2$, under a total labeling $f_m$, $m = 0, 4, 5, 7$, we get

\[
wt_{f_m}(C_k^j) = \sum_{v \in V(C_k^j)} f_m(v) + \sum_{e \in E(C_k^j)} f_m(e)
\]

\[= \sum_{s=0}^{k-2} f_m(v_{j+s}) + f_m(v_{n+1}) + \sum_{s=0}^{k-3} f_m(v_{j+s}v_{j+s+1}) + f_m(v_{j+k-2}v_{n+1}) + f_m(v_j v_{n+1}).\]

We consider the difference of weights of cycles $C_k^{j+1}$ and $C_k^j$ for $j = 1, 2, \ldots, n - k + 1$

\[
wt_{f_m}(C_k^{j+1}) - wt_{f_m}(C_k^j) = f_m(v_{j+k-1}) + f_m(v_{j+k-2}v_{j+k-1}) + f_m(v_{j+1}v_{n+1}) - f_m(v_{j+k-1}v_{n+1}) - f_m(v_j v_{j+1}) - f_m(v_j v_{n+1}).
\]

Using (2.2) and (2.3) we obtain the following:

\[
wt_{f_0}(C_k^{j+1}) - wt_{f_0}(C_k^j) = a_j^{k-1,n} + (a_j^{k-2,n-1} + n + 1) + (3n + 1 - (j + 1)) + (3n + 1 - (j + k - 1)) - a_j^{k-1,n} - (a_j^{k-2,n-1} + n + 1) - (3n + 1 - j) - (3n + 1 - (j + k - 2)) = 0.
\]

Thus under the labeling $f_0$ all the $C_k$-weights are the same.

\[
wt_{f_4}(C_k^{j+1}) - wt_{f_4}(C_k^j) = a_j^{k-1,n} + (a_j^{k-2,n-1} + n + 1) + (2n + (j + 1)) + (2n + (j + k - 1)) - a_j^{k-1,n} - (a_j^{k-2,n-1} + n + 1) - (2n + j) - (2n + (j + k - 2)) = 4.
\]

Thus, under the labeling $f_4$ the $C_k$-weights form the arithmetic sequence with the difference 4.

For $f_5$ we obtain

\[
wt_{f_5}(C_k^{j+1}) - wt_{f_5}(C_k^j) = (n + 1 - a_j^{k-1,n}) + (2a_j^{k-2,n-1} + n + 1) + (n + 2(j + 1)) + (n + 2(j + k - 1)) - (n + 1 - a_j^{k-1,n}) - (2a_j^{k-2,n-1} + n + 1) - (n + 2j) - (n + 2(j + k - 2)) = 5.
\]

which means that the $C_k$-weights form the arithmetic sequence with the difference 5.
For the labeling $f_7$ we get
\[
wt_{f_7}(C_j^{j+1}) - wt_{f_7}(C_j^j) = a_{j+k-1}^{k-1,n} + (2a_{j+k-2}^{k-2,n-1} + n + 1) + (n + 2(j + 1))
+ (n + 2(j + k - 1)) - a_{j}^{k-1,n} - (2a_{j}^{k-2,n-1} + n + 1)
- (n + 2j) - (n + 2(j + k - 2)) = 7.
\]
The $C_k$-weights under the total labeling $f_7$ form the arithmetic sequence with the difference $7$. This concludes the proof.

Figures 1 to 4 give a super $(a, d)$-$C_5$-antimagic labeling of the fan $F_6$ for $d = 0, 4, 5$ and $d = 7$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{A super $(85, 0)$-$C_5$-antimagic labeling of the fan $F_6$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{A super $(81,4)$-$C_5$-antimagic labeling of the fan $F_6$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{A super $(81,5)$-$C_5$-antimagic labeling of the fan $F_6$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{A super $(83,7)$-$C_5$-antimagic labeling of the fan $F_6$.}
\end{figure}

Immediately from Propositions 1.1, 1.2 and Theorem 2.2 we obtain the following result for cycle-antimagicness of fan graphs.
Theorem 2.3. The fan graph $F_n$, $n \geq 3$, admits a super $(a,d)$-$C_k$-antimagic labeling for $k = 3, 4, \ldots, \lceil \frac{n}{2} \rceil + 2$ and $d \in \{0, 1, 2, 3, 4, 5, 7, k - 7, k - 5, k - 4, \ldots, k + 2, 2k - 5, 2k - 1, 3k - 9, 3k - 1\}$.

We note that from the constructions of the given labelings described in the proofs of Propositions 1.1, 1.2 and Theorem 2.2, each of such labelings has the property, that the centre of the fan graph $F_n$ (the vertex $v_{n+1}$) is labeled with the number $n + 1$.

3. Ladders

In this section we deal with the super cycle-antimagicness of Cartesian product $P_n \square P_2$. We construct the required labelings from super cycle-antimagic labelings of fans.

Theorem 3.1. The ladder $P_n \square P_2$, $n \geq 3$, admits a super $(a,d)$-$C_{2(k-1)}$-antimagic labeling for $k = 3, 4, \ldots, \lceil \frac{n}{2} \rceil + 2$ and $d \in \{0, 2, 3, 4, 7, k - 7, k - 4, k - 3, k - 2, k, k + 1, k + 2, 2k - 5, 2k - 1, 3k - 9, 3k - 1\}$ having the following properties:

(P1) the vertex $v_{n+1}$ is labeled with the number $n + 1$;

(P2) it is possible to denote all subgraphs of $F_n$ isomorphic to $C_k$ by the symbols $C_k^i$, $i = 1, 2, \ldots, n - k + 2$, such that

\[ wt_{f_d}(C_k^i) = a_d + (i - 1)d; \]

(P3) and the weight of the cycle $C_k$ containing the vertex $v_1$ of $F_n$ is the smallest one, that is, $v_1 \in V(C_k^1)$.

We define the total labelings $g_1$ and $g_2$ of the $P_n \square P_2$ such that

\[
\begin{align*}
g_m(v_i) &= f_d(v_i), & \text{for } i = 1, 2, \ldots, n \text{ and } m = 1, 2, \\
g_1(u_i) &= n + i, & \text{for } i = 1, 2, \ldots, n - k + 2 \\
& \quad \text{and } v_i \in V(C_k^i), \\
g_2(u_i) &= 2n + 1 - i, & \text{for } i = 1, 2, \ldots, n - k + 2 \\
& \quad \text{and } v_i \in V(C_k^i), \\
g_1(u_i) &= n + i, & \text{for } i = n - k + 3, n - k + 4, \ldots, n, \\
g_2(u_i) &= 2n + 1 - i, & \text{for } i = n - k + 3, n - k + 4, \ldots, n, \\
g_m(v_iv_{i+1}) &= f_d(v_iv_{i+1}) + n - 1, & \text{for } i = 1, 2, \ldots, n - 1 \text{ and } m = 1, 2, \\
g_m(v_iu_i) &= f_d(v_iu_i) + n - 1, & \text{for } i = 1, 2, \ldots, n \text{ and } m = 1, 2, \\
g_1(u_iu_{i+1}) &= 6n - g_1(u_{i+1}), & \text{for } i = 1, 2, \ldots, n - 1, \\
g_2(u_iu_{i+1}) &= 6n - 1 - g_2(u_{i+1}), & \text{for } i = 1, 2, \ldots, n - 1.
\end{align*}
\]

According to Properties (P1) and (P3) the labeling $g_m$, $m = 1, 2$, is a bijective mapping from the set $\{1, 2, \ldots, 5n - 2\}$ and the smallest possible numbers are used to label the vertices of $P_n \square P_2$.

We denote the $C_{2(k-1)}$ cycle $v_jv_{j+1} \ldots v_{j+k-2}v_{j+k-3} \ldots v_jv_j$, $j = 1, 2, \ldots, n - k + 2$, in $P_n \square P_2$ by the symbol $C^j_{2(k-1)}$ if $C^j_k = v_jv_{j+1}v_{j+2} \ldots v_{j+k-2}v_{j+k-1}v_j$ is the cycle in $F_n$. 
For the $C_{2(k-1)}$-weight of the cycle $C_{2(k-1)}^j$, $j = 1, 2, \ldots, n - k + 2$, in $P_n \square P_2$ under a total labeling $g_m$, $m = 1, 2$, we get

$$wt_{g_m}(C_{2(k-1)}^j) = \sum_{v \in V(C_{2(k-1)}^j)} g_m(v) + \sum_{e \in E(C_{2(k-1)}^j)} g_m(e)$$

$$= \sum_{s=0}^{k-2} g_m(v_{j+s}) + \sum_{s=0}^{k-2} g_m(u_{j+s}) + \sum_{s=0}^{k-3} g_m(v_{j+s}v_{j+s+1})$$

$$+ \sum_{s=0}^{k-3} g_m(u_{j+s}u_{j+s+1}) + g_m(v_j) + g_m(v_{j+k-2}u_{j+k-2}).$$

Using Property (P2) we obtain

$$wt_{g_1}(C_{2(k-1)}^j) = \sum_{s=0}^{k-2} g_1(v_{j+s}) + \sum_{s=0}^{k-2} g_1(u_{j+s}) + \sum_{s=0}^{k-3} g_1(v_{j+s}v_{j+s+1})$$

$$+ \sum_{s=0}^{k-3} g_1(u_{j+s}u_{j+s+1}) + g_1(v_j) + g_1(v_{j+k-2}u_{j+k-2})$$

$$= \sum_{s=0}^{k-2} f_d(v_{j+s}) + \sum_{s=0}^{k-2} g_1(u_{j+s})$$

$$+ \sum_{s=0}^{k-3} (f_d(v_{j+s}v_{j+s+1}) + n - 1) + \sum_{s=0}^{k-3} (6n - g_1(u_{j+s+1}))$$

$$+ (f_d(v_jv_{n+1}) + n - 1) + (f_d(v_{j+k-2}v_{n+1}) + n - 1)$$

$$= \sum_{s=0}^{k-3} f_d(v_{j+s}) + g_1(u_j) + \sum_{s=0}^{k-3} g_1(u_{j+s+1})$$

$$+ \sum_{s=0}^{k-3} f_d(v_{j+s}v_{j+s+1}) + (k - 2)(n - 1) + (k - 2)6n$$

$$- \sum_{s=0}^{k-3} g_1(u_{j+s+1}) + f_d(v_jv_{n+1}) + f_d(v_{j+k-2}v_{n+1})$$

$$+ 2n - 2$$

$$= \left( \sum_{s=0}^{k-2} f_d(v_{j+s}) + \sum_{s=0}^{k-3} f_d(v_{j+s}v_{j+s+1}) + f_d(v_jv_{n+1})$$

$$+ f_d(v_{j+k-2}v_{n+1}) \right) + g_1(u_j) + n(7k - 12) - k$$

$$= \left( wt_{f_d}(C_k^j) - (n + 1) \right) + g_1(u_j) + n(7k - 12) - k$$

$$= a_d + (j - 1)d + (n + j) + n(7k - 13) - k - 1$$

$$= a_d + n(7k - 12) - k - 1 - d + j(d + 1).$$
For the $C_{2(k-1)}$-weights under the labeling $g_2$ we get

$$wt_{g_2}(C_{2(k-1)}^j) = \sum_{s=0}^{k-2} g_2(v_{j+s}) + \sum_{s=0}^{k-2} g_2(u_{j+s}) + \sum_{s=0}^{k-3} g_2(v_{j+s}v_{j+s+1})$$

$$+ \sum_{s=0}^{k-3} g_2(u_{j+s}u_{j+s+1}) + g_2(v_{j}u_{j}) + g_2(v_{j+k-2}u_{j+k-2})$$

$$= \sum_{s=0}^{k-2} f_d(v_{j+s}) + \sum_{s=0}^{k-2} g_2(u_{j+s})$$

$$+ \sum_{s=0}^{k-3} (f_d(v_{j+s}v_{j+s+1}) + n - 1)$$

$$+ \sum_{s=0}^{k-3} (6n - 1 - g_2(u_{j+s+1}))$$

$$+ (f_d(v_{j}v_{n+1}) + n - 1) + (f_d(v_{j+k-2}v_{n+1}) + n - 1)$$

$$= \sum_{s=0}^{k-2} f_d(v_{j+s}) + g_2(u_{j}) + \sum_{s=0}^{k-3} g_2(u_{j+s+1})$$

$$+ \sum_{s=0}^{k-3} f_d(v_{j+s}v_{j+s+1}) + (k - 2)(n - 1) + (k - 2)(6n - 1)$$

$$- \sum_{s=0}^{k-3} g_2(u_{j+s+1}) + f_d(v_{j}v_{n+1}) + f_d(v_{j+k-2}v_{n+1})$$

$$+ 2n - 2$$

$$= \left( \sum_{s=0}^{k-2} f_d(v_{j+s}) + \sum_{s=0}^{k-3} f_d(v_{j+s}v_{j+s+1}) + f_d(v_{j}v_{n+1}) \right)$$

$$+ g_2(u_{j}) + n(7k - 12) - 2k + 2$$

$$= (ad + (j - 1)d) + (2n + 1 - j) + n(7k - 13) - 2k + 1$$

$$= ad + n(7k - 11) - 2k + 2 - d + j(d - 1).$$

As $d \in \{0, 2, 3, 4, 7, k - 7, k - 5, k - 4, k - 3, k - 2, k, k + 1, k + 2, 2k - 5, 2k - 1, 3k - 9, 3k - 1\}$, we immediately obtain that the ladder $P_n \square P_2$, $n \geq 3$, admits a super $(a, d^*)$-$C_{2(k-1)}$-antimagic labeling for $k = 3, 4, \ldots, \lceil \frac{d}{2} \rceil + 2$ and $d^* \in \{1, 2, 3, 4, 5, 6, 8, k - 8, k - 6, k - 5, \ldots, k + 3, 2k - 6, 2k - 4, 2k - 2, 2k, 3k - 10, 3k - 8, 3k - 2, 3k\}$. □

Figure 5 illustrates a super $(218, 1)$-$C_8$-antimagic labeling of the ladder $P_6 \square P_2$ obtained from $(85, 0)$-$C_5$-antimagic labeling of the fan $F_6$, (see Figure 1), by using the construction described in the previous theorem.
Let $d = 0, 10, 12, 14, 16$.

\textbf{Theorem 3.2.} Let $n \geq 3$ be a positive integer and $2 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1$. Then the ladder $P_n \square P_2$ admits a super $C_{2k}$-magic labeling.

\textbf{Proof.} Let $n \geq 3$ be a positive integer and $2 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1$. We consider the total labeling $f$ of ladder $P_n \square P_2$ defined in the following way.

$$
\begin{align*}
&f(v_i) = i, & \text{for } i = 1, 2, \ldots, n, \\
&f(u_i) = n + i, & \text{for } i = 1, 2, \ldots, n, \\
&f(v_iv_{i+1}) = 3n - i, & \text{for } i = 1, 2, \ldots, n - 1, \\
&f(u_iu_{i+1}) = 4n - 1 - i, & \text{for } i = 1, 2, \ldots, n - 1, \\
&f(v_iu_i) = 5n - 1 - i, & \text{for } i = 1, 2, \ldots, n.
\end{align*}
$$

It is easy to see that the labeling $f$ is a bijection and the vertices are labeled with numbers $1, 2, \ldots, 2n$.

Every cycle $C_{2k}$ on $2k$ vertices in $P_n \square P_2$ is of the form $C_{2k}^j = v_jv_{j+1}v_{j+2} \ldots v_{j+k-1}u_{j+k-1}u_{j+k-2} \ldots u_jv_j$, where $j = 1, 2, \ldots, n - k + 1$. As $k = 2, 3, \ldots, \lfloor \frac{n}{2} \rfloor + 1$ every edge of $P_n \square P_2$ belongs at least to one cycle $C_{2k}^j$.

For the $C_{2k}$-weight of the cycle $C_{2k}^j$, $j = 1, 2, \ldots, n - k + 1$, under the total labeling $f$ we get.

$$
\begin{align*}
wt_f(C_{2k}^j) &= \sum_{v \in V(C_{2k}^j)} f(v) + \sum_{e \in E(C_{2k}^j)} f(e) \\
&= \sum_{s=0}^{k-1} f(v_{j+s}) + \sum_{s=0}^{k-1} f(u_{j+s}) + \sum_{s=0}^{k-2} f(v_ju_{j+s+1}) \\
&\quad + \sum_{s=0}^{k-2} f(u_ju_{j+s+1}) + f(v_{j+k-1}u_{j+k-1}) + f(v_ju_j).
\end{align*}
$$

We consider the difference of weights of cycles $C_{2k}^{j+1}$ and $C_{2k}^j$ for $j = 1, 2, \ldots, n - k$

$$
\begin{align*}
wt_f(C_{2k}^{j+1}) - wt_f(C_{2k}^j) &= f(v_{j+k}) + f(v_{j+k}) + f(v_{j+k-1}v_{j+k}) \\
&\quad + f(u_{j+k-1}u_{j+k}) + f(v_{j+k+1}u_{j+k+1}) + f(v_{j+k}u_{j+k}) - f(v_j) \\
&\quad - f(u_j) - f(v_{j+1}v_{j+1}) - f(u_{j+1}u_{j+1}) - f(v_{j+1}u_j) \quad - f(v_{j+k-1}u_{j+k-1}) \\
&= (j + k) + (n + j + k) + (3n - (j + k - 1)) \\
&\quad + (4n - 1 - (j + k - 1)) + (5n - 1 - (j + 1)) \\
&\quad + (5n - 1 - (j + k)) - j - (n + j) - (3n - j) - (4n - 1 - j) \\
&\quad - (5n - 1 - j) - (5n - 1 - (j + k - 1)) = 0.
\end{align*}
$$

Thus the $C_{2k}$-weights are the same. This means that the labeling $f$ is super $C_{2k}$-magic. This concludes the proof. \qed
Figure 6 depicts a super $C_8$-magic labeling of the ladder $P_6 \Box P_2$.

![Image of a super $C_8$-magic labeling of the ladder $P_6 \Box P_2$.]

**Figure 6.** A super $C_8$-magic labeling of the ladder $P_6 \Box P_2$.

**Theorem 3.3.** Let $n \geq 3$ be a positive integer and $2 \leq k \leq \lceil \frac{n}{2} \rceil + 1$. Then the ladder $P_n \Box P_2$ admits a super $(a,10)$-$C_{2k}$-antimagic labeling.

**Proof.** Let $n \geq 3$ be a positive integer and $2 \leq k \leq \lceil \frac{n}{2} \rceil + 1$. We consider the total labeling $f$ of ladder $P_n \Box P_2$ defined in the following way. The vertices are labeled with numbers $1, 2, \ldots, 2n$ such that

$$f(v_i) = 2a_i^{k,n}, \quad \text{for } i = 1, 2, \ldots, n,$$

$$f(u_i) = 2a_i^{k,n} - 1, \quad \text{for } i = 1, 2, \ldots, n,$$

where $a_i^{k,n}$ is an element of the sequence $\{a_i^{k,n}\}_{i=1}^{n}$, see (2.1).

The labeling $f$ assigns numbers $2n + 1, 2n + 2, \ldots, 5n - 2$ to the edges of $P_n \Box P_2$ in the following way:

$$f(v_iv_{i+1}) = 2a_i^{k-1,n-1} + 2n, \quad \text{for } i = 1, 2, \ldots, n - 1,$$

$$f(u_{i}u_{i+1}) = 2a_i^{k-1,n-1} + 2n - 1, \quad \text{for } i = 1, 2, \ldots, n - 1,$$

$$f(v_{i}u_{i}) = 4n - 2 + i, \quad \text{for } i = 1, 2, \ldots, n,$$

where $a_i^{k-1,n-1}$ is an element of the sequence $\{a_i^{k-1,n-1}\}_{i=1}^{n}$, see (2.1). It is easy to see that the labeling $f$ is a bijection.

As in the proof of the previous theorem we consider the difference of weights of cycles $C_{2k}^{j+1}$ and $C_{2k}^{j}$ for $j = 1, 2, \ldots, n - k$ and we use (2.2)

$$wt_f(C_{2k}^{j+1}) - wt_f(C_{2k}^{j}) = 2a_{j+k}^{k,n} + (2a_{j+k}^{k,n} - 1) + (2a_{j+k-1}^{k-1,n-1} + 2n)$$

$$+ (2a_{j+k-1}^{k-1,n-1} + 2n - 1) + (4n - 2 + (j + 1))$$

$$+ (4n - 2 + (j + k)) - 2a_j^{k,n} - (2a_j^{k,n} - 1) - (2a_{j-1}^{k-1,n-1} + 2n)$$

$$- (2a_j^{k-1,n-1} + 2n - 1) - (4n - 2 + j) - (4n - 2 + (j + k - 1)) = 10.$$

The $C_{2k}$-weights under the total labeling $f$ form the arithmetic sequence with the difference 10. This concludes the proof. \qed

Figure 7 illustrates a super $(210,10)$-$C_8$-antimagic labeling of the ladder $P_6 \Box P_2$. 

Let $n \geq 3$ be a positive integer and $2 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$. Then the ladder $P_n \boxtimes P_2$ admits a super $(a,d)$-$C_{2k}$-antimagic labeling for $d = 12, 14, 16$.

**Proof.** Let $n \geq 3$ be a positive integer and $2 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$. We consider the total labeling $f_m$, $m = 12, 14, 16$, of ladder $P_n \boxtimes P_2$ defined in the following way. The vertices are labeled with numbers $1, 2, \ldots, 2n$ such that

$$f_m(v_i) = a_i^{k,n}, \quad \text{for } i = 1, 2, \ldots, n \text{ and } m = 12, 14,$$

$$f_{16}(v_i) = 2a_i^{k,n}, \quad \text{for } i = 1, 2, \ldots, n,$$

$$f_{12}(u_i) = 2n + 1 - a_i^{k,n}, \quad \text{for } i = 1, 2, \ldots, n,$$

$$f_{14}(u_i) = a_i^{k,n} + n, \quad \text{for } i = 1, 2, \ldots, n,$$

$$f_{16}(u_i) = 2a_i^{k,n} - 1, \quad \text{for } i = 1, 2, \ldots, n,$$

where $a_i^{k,n}$ is an element of the sequence $\{a_i^{k,n}\}_{i=1}^{n}$, see (2.1).

The labeling $f_m$ assigns numbers $2n + 1, 2n + 2, \ldots, 5n - 2$ to the edges of $P_n \boxtimes P_2$ in the following way

$$f_m(v_iv_{i+1}) = 3a_i^{k-1,n-1} + 2n, \quad \text{for } i = 1, 2, \ldots, n - 1 \text{ and } m = 12, 14, 16,$$

$$f_m(u_iu_{i+1}) = 3a_i^{k-1,n-1} + 2n - 1, \quad \text{for } i = 1, 2, \ldots, n - 1 \text{ and } m = 12, 14, 16,$$

$$f_m(v_iu_i) = 2n - 2 + 3i, \quad \text{for } i = 1, 2, \ldots, n \text{ and } m = 12, 14, 16,$$

where $a_i^{k-1,n-1}$ is an element of the sequence $\{a_i^{k-1,n-1}\}_{i=1}^{n}$, see (2.1). It is easy to see that the labeling $f_m$ is a bijection.

Again, we consider the difference of weights of cycles $C_{2k}^{j+1}$ and $C_{2k}^j$ for $j = 1, 2, \ldots, n-k$, under the labelings $f_m$, $m = 12, 14, 16$.

For $m = 12$ we get

$$wt_{f_{12}}(C_{2k}^{j+1}) - wt_{f_{12}}(C_{2k}^j) = a_{j+k}^{k,n} + (2n + 1 - a_{j+k}^{k,n}) + (3a_{j+k-1}^{k-1,n-1} + 2n) + (3a_{j+k-1}^{k-1,n-1} + 2n - 1) + (2n - 2 + 3(j + 1)) + (2n - 2 + 3(j + k)) - a_j^{k,n} - (2n + 1 - a_j^{k,n}) - (3a_j^{k-1,n-1} + 2n) - (3a_j^{k-1,n-1} + 2n - 1) - (2n - 2 + 3j) - (2n - 2 + 3(j + k - 1)) = 12.$$

The $C_{2k}$-weights under the total labeling $f_{12}$ form the arithmetic sequence with the difference $12$. 

![Ladder and fan graphs are cycle-antimagic](image-url)
If $m = 14$ then

$$\begin{align*}
wt_{f_{14}}(C_{2k}^{j+1}) - wt_{f_{14}}(C_{2k}^j) &= a_{j+k}^{k,n} + (a_{j+k}^{k,n} + n) + (3a_{j+k-1}^{k-1,n-1} + 2n) \\
&+ (3a_{j+k-1}^{k-1,n-1} + 2n - 1) + (2n - 2 + 3(j + 1)) \\
&+ (2n - 2 + 3(j + k)) - a_{j}^{k,n} - (a_{j}^{k,n} + n) \\
&- (3a_{j}^{k-1,n-1} + 2n) - (3a_{j}^{k-1,n-1} + 2n - 1) \\
&- (2n - 2 + 3j) - (2n - 2 + 3(j + k - 1)) = 14.
\end{align*}$$

The $C_{2k}$-weights under the total labeling $f_{14}$ form the arithmetic sequence with the difference 14.

Finally, for the labeling $f_{16}$ we obtain

$$\begin{align*}
wt_{f_{16}}(C_{2k}^{j+1}) - wt_{f_{16}}(C_{2k}^j) &= 2a_{j+k}^{k,n} + (2a_{j+k}^{k,n} - 1) + (3a_{j+k-1}^{k-1,n-1} + 2n) \\
&+ (3a_{j+k-1}^{k-1,n-1} + 2n - 1) + (2n - 2 + 3(j + 1)) \\
&+ (2n - 2 + 3(j + k)) - 2a_{j}^{k,n} - (2a_{j}^{k,n} - 1) \\
&- (3a_{j}^{k-1,n-1} + 2n) - (3a_{j}^{k-1,n-1} + 2n - 1) \\
&- (2n - 2 + 3j) - (2n - 2 + 3(j + k - 1)) = 16.
\end{align*}$$

Thus the $C_{2k}$-weights under the total labeling $f_{16}$ form the arithmetic sequence with the difference 16. This concludes the proof. \hfill \Box

Figures 8, 9 and 10 give a super $(a, d)$-$C_8$-antimagic labeling of the ladder $P_6 \sqcup P_2$ for $d = 12, 14$ and $d = 16$.

![Figure 8](image1)

**Figure 8.** A super $(210, 12)$-$C_8$-antimagic labeling of the ladder $P_6 \sqcup P_2$.

![Figure 9](image2)

**Figure 9.** A super $(212, 14)$-$C_8$-antimagic labeling of the ladder $P_6 \sqcup P_2$. 
Ladders and fan graphs are cycle-antimagic

Figure 10. A super \((214, 16)\)-C\(_k\)-antimagic labeling of the ladder \(P_6 \square P_2\).

4. Conclusion

In this paper we prove the existence of super \((a, d)\)-cycle-antimagic labelings of fan graphs and ladders. Combining the results obtained in this paper and in \([8, 13]\) we get that the fan graph \(F_n, n \geq 3\), admits a super \((a, d)\)-\(C_k\)-antimagic labeling for \(k = 3, 4, \ldots, \lfloor \frac{n}{2} \rfloor + 2\) and \(d \in \{0, 1, 2, 3, 4, 5, 7, k-7, k-5, k-4, \ldots, k+2, 2k-5, 2k-1, 3k-9, 3k-1\}\). Further, we prove that the ladder \(P_n \square P_2, n \geq 3\), admits a super \((a, d)\)-\(C_{2(k-1)}\)-antimagic labeling for \(k = 3, 4, \ldots, \lfloor \frac{n}{2} \rfloor + 2\) and \(d \in \{0, 1, 2, 3, 4, 5, 6, 8, 10, 12, 14, 16, k-8, k-6, k-5, \ldots, k+3, 2k-6, 2k-4, 2k-2, 2k, 3k-10, 3k-8, 3k-3, k-2, 3k\}\).

Though there are results on super \((a, d)\)-cycle-antimagic labelings of fan graphs and ladders for various differences \(d\), still some values of \(d\) for which the problem of existence of the corresponding labelings is not solved is a scope for further research.

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References


