RESEARCH ARTICLE

ISOMETRIC IMMERSIONS IN 3-DIMENSIONAL EUCLIDEAN SPACE

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ABSTRACT

In this paper, we examine the image of geodesic curves of Riemann 2-manifolds under the isometric immersions in three dimensional Euclidean space. We show that the curvature of these curves is equal to the normal curvature of the manifold in the direction of tangent vector field of the geodesics. Moreover, we prove that if the parameter curves of the manifold are the line of curvature, then the geodesic torsion of geodesics is equal to the torsion of the image curve.

Keywords: Immersion, Geodesic, Riemann manifold

1. INTRODUCTION

In [1], the isometric immersions are considered in four and $n-$dimensional Euclidean space. The authors gave some results about Riemannian 2-manifolds through the geodesic curves such that these geodesic curves are considered as a curve with constant curvature (W-curve) in 4-dimensional Euclidean space. Note that the circles and right circular helices are the only W-curves in two and three dimensional Euclidean space, respectively. For a special case of Riemann 2-manifolds, Ferus and Schirrmacher [1] gave the characterization theorem in four dimensional Euclidean space, by the following:

Theorem 1.1 *Let M be a closed, connected Riemannian 2-manifold, and $f: M \to \mathbb{E}^4$ be an isometric immersion such that for each unit speed geodesic $\alpha: \mathbb{R} \to M$, the image curve $\gamma = f \circ \alpha$ be a $W-$curve. Then there are two possibilities:*

i) If M contains a non-periodic geodesic, then f covers a standard torus $S^1(r_1) \times S^1(r_2) \subset \mathbb{E}^4$.

ii) If all geodesics in M are periodic, then f is an isometry onto Euclidean $2-sphere$ $S^2(r) \subset \mathbb{E}^3$.

In [2], the calculations in proof of Theorem 1.1 are extended and interpreted, by elementary approaches.

In this study, we investigate the curvatures of curve-surface (manifold) pair, in terms of Darboux frame, in three dimensional Euclidean space. Especially, we consider the curvatures of Riemannian 2-manifolds and their geodesics.

2. PRELIMINARIES

Let us give the inner product of u and v in the $n-$dimensional Euclidean space by

$$u \cdot v = \sum_{i=1}^{n} u_i v_i$$

where $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$. Let M be a hypersurface, $P \in M$, and $u \in T_pM$. The normal curvature of M in the u −direction is

$$k(u) = S_p(u) \cdot u$$

where S_p is the shape operator of the surface at the point P.

Let γ be a curve on hypersurface M and $\{t(s), V(s), U(s)\}$ be the moving Darboux frame where $t(s)$ is the unit tangent vector field of the curve, $U(s)$ is the unit normal vector field of the surface which is restricted to γ, and $V(s) = U(s) \times t(s)$. Also, the derivative formulas of this frame is given by

$$\begin{aligned} t' &= \kappa_g V + \kappa_n U \\ V' &= -\kappa_g t + \tau_g U \\ U' &= -\kappa_n t - \tau_g V \end{aligned}$$

where κ_n, κ_g and τ_g are the normal curvature, the geodesic curvature and the geodesic torsion of the curve, respectively [3].

Besides, the geodesic torsion is given by

$$\tau_g = \tau + \frac{d\phi}{ds} \tag{2.1}$$

where τ is ordinary torsion of $\gamma(s)$ and ϕ is the angle between the osculating plane of the curve and the tangent plane to the surface.

Definition 2.1 *Let $\overline{\nabla}$ and ∇ be the Levi-Civita connections in Euclidean space $\mathbb{E}^n$ and hypersurface M respectively. The Gauss equation is given by*

$$\overline{\nabla}_T T = \nabla_T T + h(T,T)U$$

where h is the second fundemental form of the surface, $T \in \chi(M)$, and U is the unit normal vector field of M.

Let us give the *Euler's formula* by following:

Lemma 2.2 *Let M be a hypersurface and $u = cos\theta \, u_1 + sin\theta \, u_2$ be a unit principal vector that makes positive oriented angle θ with vector u_1, in tangent space of M. Then the normal curvature of M, in the direction of u is given by*

$$k(u) = k_1 cos^2\theta + k_2 sin^2\theta$$

for some $\theta \in (0, \pi/2)$ where k_1, k_2 principal curvatures of M.

Let us give the useful property in [4]:

Lemma 2.3 *Let the parameter curves of the hypersurface through the all $P \in M$ be a line of curvature and e_1 be a unit principal vector of the surface . If the curve α on M makes the positive oriented angle θ with e_1, then the geodesic torsion of (α, M) is given by*

$$\tau_g = (k_2 - k_1) \, sin\theta \, cos\theta$$

where k_1 and k_2 are the principal curvatures of the surface.

3. IMMERSIONS IN EUCLIDEAN SPACE

In this section we give our main results by following two theorems.

Theorem 3.1 *Let M be a closed, connected Riemannian 2-manifold and $f: M \to \mathbb{E}^3$ be a isometric immersion. Let us define the curve $\gamma = f \circ \alpha$ such that $\alpha: \mathbb{R} \to M$ is the unit speed geodesic curve. In this case, the curvature $\kappa(s)$ of γ is equal to the normal curvature of M in the direction of $\alpha'(s)$ at $\alpha(s)$.*

Proof. Let $\overline{\nabla}$ and ∇ be the Levi-Civita connections in Euclidean space $\mathbb{E}^3$ and M, respectively. $f: M \to \mathbb{E}^3$ be a isometric immersion and $\alpha: \mathbb{R} \to M$ be a unit speed geodesic curve. It is obvious that

$$\gamma'(s) = (f \circ \alpha)'(s) = T(s)$$

where T is the unit tangent vector field of $\gamma(s)$. Since f is an isometric immersion, the unit tangent vector field of α and T can be considered as the same vectors. It follows from Definition 2.1 that

$$\overline{\nabla}_T T = \nabla_T T + h(T, T)U$$

Since α is geodesic, $\nabla_T T = 0$ and so

$$\overline{\nabla}_T T = \gamma''(s) = \langle S(T(s)), T(s) \rangle U \tag{3.1}$$

It follows from (3.1) that

$$\kappa(s) = \|\langle S(T(s)), T(s) \rangle U\|. \tag{3.2}$$

Let us state the principal curvatures of M as (k_1, k_2) and the unit principal vectors (e_1, e_2) that correspond to (k_1, k_2) respectively. Thus, we write $S(e_1) = k_1 e_1$ and $S(e_2) = k_2 e_2$. Moreover, the unit tangent vector field of the curve is written by $T(s) = \cos s\, e_1 + \sin s\, e_2$ for suitable $s \in (0, \pi/2)$. Since S is linear operator, we get $S(T(s)) = k_1 \cos s\, e_1 + k_2 \sin s\, e_2$. Hence, we find

$$\langle S(T(s)), T(s) \rangle = k_1 \cos^2 s + k_2 \sin^2 s. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\kappa(s) = k_1 \cos^2 s + k_2 \sin^2 s.$$

From Lemma 2.2, the normal curvature of M in the direction of T is

$$k(T(s)) = k_1 \cos^2 s + k_2 \sin^2 s = \kappa(s)$$

which is intended.

Theorem 3.2 *Let M be a closed, connected Riemannian 2-manifold and $f: M \to \mathbb{E}^3$ be a isometric immersion. Let us define the curve $\gamma = f \circ \alpha$ such that $\alpha: \mathbb{R} \to M$ is the unit speed geodesic curve. If the parameter curves of the surface through the all $P \in M$ are the line of curvature, then the geodesic torsion of (α, M) is equal to the torsion of γ.*

Proof. Let us state the principal curvatures of M as (k_1, k_2) and unit principal vectors (e_1, e_2) that correspond to (k_1, k_2) respectively. The unit tangent vector field of any curve on M is written by

$$T(s) = \cos s\, e_1 + \sin s\, e_2$$

for suitable $s \in (0, \pi/2)$. Let the parameter curves be a line of curvature that through all $P \in M$. It follows from Lemma 2.3 that

$$\tau_g(s) = (k_2 - k_1)\, sins\, coss$$

Besides, the orthonormal Frenet frame of γ is given by $\{V_1, V_2, V_3\}$ such that

$$V_1(s) = T(s) = E_1(s)$$
$$V_2(s) = \frac{E_2(s)}{\|E_2(s)\|} = \frac{\langle S(T(s)), T(s)\rangle U}{\|\langle S(T(s)), T(s)\rangle U\|}$$
$$V_3(s) = \frac{E_3(s)}{\|E_3(s)\|}$$

where $E_3(s) = \gamma'''(s) - \frac{\langle \gamma'''(s), E_1(s)\rangle}{\langle E_1(s), E_1(s)\rangle} E_1(s) - \frac{\langle \gamma'''(s), E_2(s)\rangle}{\langle E_2(s), E_2(s)\rangle} E_2(s)$.

It follows from ([1], p.61) that

$$\tau(s) = \langle V_2'(s), V_3(s)\rangle = (k_2 - k_1)\, sins\, coss = \tau_g(s)$$

This completes the proof.

Corollary 3.3 In terms of Theorem 3.2, the angle between the osculating plane of γ and the tangent plane to the M at any point is fixed.

Proof. It follows from (2.1) and Theorem 3.2 that

$$\frac{d\phi}{ds} = 0 \tag{3.4}$$

where ϕ is the angle between the osculating plane of γ and the tangent plane to the manifold at any point. It is obvious from (3.4) that ϕ is fixed.

Example 3.4 Let us consider the 2-dimensional circular cylinder such that $\varphi: \mathbb{R}^2 \to \mathbb{R}^3$,

$$\varphi(x^1, x^2) = (cosx^2, sinx^2, x^1).$$

The unit speed circular helix on φ can be given as

$$\gamma(s) = \left(cos\frac{s\sqrt{2}}{2}, sin\frac{s\sqrt{2}}{2}, \frac{s\sqrt{2}}{2} \right).$$

It is easy to see that γ is a geodesic on the surface. The partial derivatives of φ are,

$$\frac{d\varphi}{dx^1} = \varphi_1 = (0,0,1)\,;\; \frac{d\varphi}{dx^2} = \varphi_2 = (-sinx^2, cosx^2, 0).$$

Since the unit normal vector $\vec{n} \circ \varphi = \frac{\varphi_1 \times \varphi_2}{\|\varphi_1 \times \varphi_2\|}$,

$$\vec{n} \circ \varphi = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ -sinx^2 & cosx^2 & 0 \end{vmatrix} = (-cosx^2, -sinx^2, 0).$$

Therefore,

$$U = \vec{n}(\gamma(s)) = \vec{n}\left(\varphi\left(\frac{s\sqrt{2}}{2}, \frac{s\sqrt{2}}{2} \right) \right) = \left(-cos\frac{s\sqrt{2}}{2}, -sin\frac{s\sqrt{2}}{2}, 0 \right).$$

From the derivative formulas we get,

$$\kappa_n(\gamma'(s)) = \langle \gamma'', U \rangle = \frac{1}{2}$$

Also the curvature of the curve $\kappa(s) = \|\gamma''(s)\| = \frac{1}{2}$. Thus, $\kappa_n(\gamma'(s)) = \kappa(s)$ which realise the Theorem 3.1.

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