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A fixed point theorem for mappings with an F-contractive iterate

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Abstract

In this paper, we introduce the notion of F-contraction in the setting of complete metric space and we prove a fixed point theorem for F-contractive iteration.

Keywords: F-contraction; Fixed point; Complete metric space; Contractive iteration.
2010 MSC: 47H10, 54H25.

1. Introduction and Preliminaries

Banach fixed-point theorem [8] has plenty of extension. One of them is the following theorem, in the setting of contractive iterate, given by Bryant.

Theorem 1.1. *[10] If f is a mapping of a complete metric space into itself and if, for some positive integer k, f^k is a contraction, then f has a unique fixed point.*

It is clear that the iterate of f^k is necessarily continuous. On the other hand, continuity of the iterate f^k does not imply the continuity of f. The example of Bryant illustrate this observation:

Example 1.2. *[10] Let $T : [0, 2] \to [0, 2]$ be defined by*

$$T(x) = \begin{cases} 0 & \text{if } x \in [0, 1], \\ 1 & \text{if } x \in (1, 2]. \end{cases}$$

Then 2nd iteration of T is equal to 0 for all $x \in [0, 2]$ although T is not continuous.

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This interesting result of Bryant [10] was improved by Sehgal [18] by proposing the idea of the "contractive iterate at each point".

The significant result of Sehgal is the following:

Theorem 1.3. *[18] Let (X,d) be a complete metric space, $q \in [0,1)$ and $T : X \to X$ be a continuous mapping. If for each $x \in X$ there exists a positive integer $k = k(x)$ such that*

$$d(T^{k(x)}x, T^{k(x)}y) \leq qd(x,y) \tag{1}$$

for all $y \in X$, then T has a unique fixed point $u \in X$. Moreover,for any $x \in X$, $u = \lim_{n\to\infty} T^n x$.

Following this result, number of authors research on iteration of the mapping, see e.g., [2, 3, 4, 9, 11, 12, 16].

Another interesting extension of the Banach contraction mapping principle was given by Wardowski [19] in 2012. Roughly speaking, he transformed the contraction inequality by using an auxiliary function.

First of all we shall fix the basic notations: Throughout the paper, $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive integers and the set of nonnegative integers. Similarly, let $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ represent the set of reals, positive reals and the set of nonnegative reals, respectively. Throughout the paper, all consider set X is non-empty.

We start with the definition of auxiliary function that was used by Wardowski [19] to define the new type contraction.

Definition 1.4. *[19] Let $F : \mathbb{R}^+ \to \mathbb{R}$ and we are considering the following conditions:*

($F1$) *F is strictly increasing, that is, for all $\xi, \eta \in \mathbb{R}_+$ if $\xi < \eta$ then $F(\xi) < F(\eta)$.*

($F2$) *For every sequence $\{t_n\}_{n=1}^{\infty}$ of positive real numbers*

$$\lim_{n\to\infty} t_n = 0 \text{ if and only if } \lim_{n\to\infty} F(t_n) = -\infty.$$

($F3$) *There is $k \in (0,1)$ such that $\lim_{t\to 0^+} \left(t^k F(t)\right) = 0$.*

Example 1.5. *[19] Let $F_i : \mathbb{R}^+ \to \mathbb{R}$ where $i = 1,2,3,4$ define by*

(e1) $F_1(t) = \ln t$,

(e2) $F_2(t) = t + \ln t$,

(e3) $F_3(t) = -1/\sqrt{t}$,

(e4) $F_4(t) = \ln(t^2 + t)$.

Then $F_1, F_2, F_3, F_4 \in \mathcal{F}$.

Definition 1.6. *[19] Let (X,d) be a metric space. A map $T : X \to X$ is said to be an $F-$contraction on (X,d) if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$*

$$d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y)) \tag{2}$$

From ($F1$) and ($F2$) easily conclude that every F-contraction is a contractive mapping, that is, for all $x, y \in X$ with $Tx \neq Ty$, we have

$$d(Tx,Ty) < d(x,y)$$

Theorem 1.7. *[19] Let T be a self-mapping on a complete metric space (X,d). If T forms an $F-$contraction, then it possesses a unique fixed point u. Moreover, for any $x \in X$ the sequence $\{T^n x\}$ is convergent to u.*

Remark 1.8. *From* $(F1)$ *and (2) it follows that*

$$\begin{aligned} F(d(Tx,Ty)) &\leq F(d(x,y)) - \tau < F(d(x,y)) \Rightarrow \\ &\Rightarrow d(Tx,Ty) < d(x,y) \end{aligned}$$

for all $x, y \in X$ *such that* $Tx \neq Ty$*. Also,* T *is a continuous operator.*

There are many result regarding F-contraction, see e.g.[1, 5, 7, 13].

Definition 1.9. *A* simulation function *is a mapping* $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ *satisfying the following conditions:*

(ζ_1) $\zeta(t, s) < s - t$ *for all* $t, s > 0$*;*

(ζ_2) *if* $\{t_n\}, \{s_n\}$ *are sequences in* $(0, \infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$*, then*

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0. \quad (3)$$

In [15], there was an additional condition, $\zeta(0, 0) = 0$*, that was dropped since this condition is superfluous. Let* $\mathcal{Z}$ *denote the family of all simulation functions* $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$*, that is, verifying* ($\zeta_1$) *and* ($\zeta_2$)*.*

On account (ζ_1)*, we have*

$$\zeta(t, t) < 0 \text{ for all } t > 0. \quad (4)$$

The following example is derived from [6, 15, 17].

Example 1.10. *Let* $\mu_i : [0, \infty) \to [0, \infty)$ *be continuous functions such that* $\mu_i(t) = 0$ *if and only if,* $t = 0$*. For* $i = 1, 2, 3, 4, 5, 6$*, we define the mappings* $\zeta_i : [0, \infty) \times [0, \infty) \to \mathbb{R}$*, as follows*

(i) $\zeta_1(t, s) = \mu_1(s) - \mu_2(t)$ *for all* $t, s \in [0, \infty)$*, where* $\mu_1, \mu_2 : [0, \infty) \to [0, \infty)$ *are two continuous functions such that* $\mu_1(t) = \mu_2(t) = 0$ *if and only if* $t = 0$ *and* $\mu_1(t) < t \leq \mu_2(t)$ *for all* $t > 0$*.*

(ii) $\zeta_2(t, s) = s - \frac{f(t,s)}{g(t,s)} t$ *for all* $t, s \in [0, \infty)$*, where* $f, g : [0, \infty)^2 \to (0, \infty)$ *are two continuous functions with respect to each variable such that* $f(t, s) > g(t, s)$ *for all* $t, s > 0$*.*

(iii) $\zeta_3(t, s) = s - \mu_3(s) - t$ *for all* $t, s \in [0, \infty)$*.*

(iv) $\zeta_4(t, s) = s\varphi(s) - t$ *for all* $s, t \in [0, \infty)$*, where* $\varphi : [0, \infty) \to [0, 1)$ *is a function such that* $\limsup_{t\to r^+} \varphi(t) < 1$ *for all* $r > 0$*.*

(v) $\zeta_5(t, s) = \eta(s) - t$ *for all* $s, t \in [0, \infty)$*, where* $\eta : [0, \infty) \to [0, \infty)$ *is an upper semi-continuous mapping such that* $\eta(t) < t$ *for all* $t > 0$ *and* $\eta(0) = 0$*.*

(vi) $\zeta_6(t, s) = s - \int_0^t \mu(u)du$ *for all* $s, t \in [0, \infty)$*, where* $\mu : [0, \infty) \to [0, \infty)$ *is a function such that* $\int_0^\varepsilon \mu(u)du$ *exists and* $\int_0^\varepsilon \mu(u)du > \varepsilon$*, for each* $\varepsilon > 0$*.*

It is clear that each function ζ_i $(i = 1, 2, 3, 4, 5, 6)$ *forms a simulation function.*

Inspired from the results in [14], we combine the approaches of Seghal [18] with the notions of F-contraction and simulation functions to get a more general contraction type mappings. We investigate the existence and uniqueness of a fixed point for such contractions.

2. Main results

Now we are ready to state our main theorem that is the extension of Theorem 1.3

Theorem 2.1. *Let (X, d) be a complete metric space and $T : X \to X$ a mapping which satisfies the condition: If there exists $F \in \mathcal{F}$ and $\tau > 0$ such that for each $x \in X$ there is a positive integer $n(x)$ such that for all $y \in X$*

$$d(T^{n(x)}(x), T^{n(x)}(y)) > 0 \Rightarrow \zeta(F(d(x, y)), \tau + F(d(T^{n(x)}(x), T^{n(x)}(y)))) \geq 0. \quad (5)$$

Then, T has a unique fixed point $z \in X$ and $T^n(x_0) \to z$ for each $x_0 \in X$, as $n \to \infty$.

Proof. We shall built a recursive sequence $\{x_k\}$ as follows: For the chooses arbitrary point $x_0 \in X$ with $n_0 = n(x_0)$, we set $x_1 = T^{n_0}x_0$ and inductively we get

$$x_{i+1} = T^{n_i}x_i \text{ with } n_i = n(x_i).$$

We assert that $x_i \neq x_{i+1}$ for all $i \in \mathbb{N}_0$. Suppose, on the contrary, there exists $i_0 \in \mathbb{N}_0$ such that $x_{i_0} = x_{i_0+1} = T^{n_{i_0}}x_{i_0}$. Then, x_{i_0} turns to be a fixed point of $T^{n_{i_0}}$. On the other hand,

$$Tx_{i_0} = T(T^{n_{i_0}}x_{i_0}) = T^{n_{i_0}}(Tx_{i_0}).$$

Thus, Tx_{i_0} form a fixed point of $T^{n_{i_0}}$. If $Tx_{i_0} = x_{i_0}$, then we conclude that T has a fixed point and that terminate the proof. Suppose, on the contrary, that $Tx_{i_0} \neq x_{i_0}$ and hence $d(T^{n_{i_0}}(Tx_{i_0}), T^{n_{i_0}}(x_{i_0})) > 0$. Then, by (5) we have

$$0 \leq \zeta(F(d(x_{i_0}, Tx_{i_0})), \tau + F(d(x_{i_0}, Tx_{i_0}))),$$

which is equivalent to

$$\tau + F(d(x_{i_0}, Tx_{i_0})) = \tau + F(d(T^{n_{i_0}}x_{i_0}, T^{n_{i_0}}Tx_{i_0})) \leq F(d(x_{i_0}, Tx_{i_0})), \quad (6)$$

a contradiction. Consequently, we deduce that

$$x_i \neq x_{i+1} \text{ for all } i \in \mathbb{N}_0. \quad (7)$$

Taking the expression (7) into account (5) implies that

$$d(x_{i+1}, x_i) > 0 \Rightarrow 0 \leq \zeta(F(d(x_i, x_{i-1})), \tau + F(d(x_{i+1}, x_i))),$$

turns to be

$$\tau + F(d(x_{i+1}, x_i)) \leq F(d(x_i, x_{i-1})),$$

which yields

$$F(d(x_{i+1}, x_i)) \leq F(\delta_{i-1}) - \tau \leq F(\delta_{i-1}) - 2\tau \leq \ldots \leq F(\delta_0) - i\tau, \quad (8)$$

where $\delta_j = d(T^{n_i}x_j, x_j)$ for all $j \in \mathbb{N}_0$.

As $i \to \infty$ the inequality above yields that $\lim_{i\to\infty} F(d(x_{i+1}, x_i)) = -\infty$. On account of axiom $(F2)$, we conclude that

$$\lim_{n\to\infty} d(x_{i+1}, x_i) = 0. \quad (9)$$

Taking the axiom $(F3)$ into the account, we find a $k \in (0, 1)$ such that

$$\lim_{i\to\infty} (d(x_{i+1}, x_i))^k F(d(x_{i+1}, x_i)) = 0. \quad (10)$$

On the other hand, by regarding (8), we find that

$$\begin{aligned}(d(x_{i+1}, x_i))^k F(d(x_{i+1}, x_i)) - (d(x_{i+1}, x_i))^k F(\delta_0) &\leq (d(x_{i+1}, x_i))^k (F(\delta_0) - i\tau) - (d(x_{i+1}, x_i))^k F(\delta_0) \\ &= -(d(x_{i+1}, x_i))^k i\tau \leq 0.\end{aligned} \quad (11)$$

Keeping, (9) and (10), in mind, by letting $n \to \infty$ in (11), we obtain that

$$\lim_{i\to\infty} i(d(x_{i+1}, x_i))^k = 0. \tag{12}$$

Here, (12) implies that there exists $n_1 \in \mathbb{N}$ such that $i\delta_i^k \le 1$ for all $i \ge n_1$. Attendantly, for all $i \ge n_1$, we find

$$(d(x_{i+1}, x_i))^k \le \frac{1}{i^{1/k}}. \tag{13}$$

After the estimation (13), we shall show that the recursive sequence $\{x_i\}$ is Cauchy. Consider $m, n \in \mathbb{N}$ such that $m > n \ge n_1$. On account of the estimation (13) together with the triangle inequality, we find that

$$d(x_m, x_n) \le \delta_{m-1} + \delta_{m-2} + ... + \delta_n < \sum_{j=n}^{\infty} \delta_j \le \sum_{j=n}^{\infty} \frac{1}{j^{1/k}}. \tag{14}$$

It is clear that the series $\sum_{j=n}^{\infty} \frac{1}{j^{1/k}}$ converges and hence we conclude that $\{x_i\}$ is a Cauchy sequence. Regarding the completeness of (X, d), there exists $u \in X$ such that $\lim_{i\to\infty} x_i = x^*$.

As a next step, we show that x^* is a fixed point of $T^{n(x^*)}$. Indeed, due to the continuity of T, we have

$$d(Tx^*, x^*) = \lim_{i\to\infty} d(Tx_i, x_i) = \lim_{n\to\infty} d(x_{i+1}, x_i) = 0,$$

For the proving the uniqueness of the fixed point let us consider x^* and y^* be two distinct fixed point and $n = n(x^*)$. So, we have $d(x^*, y^*) > 0$ and hence we get that

$$d(Tx^*, Ty^*) > 0 \Rightarrow 0 \le \zeta(F(d(x^*, y^*)), \tau + F(d(Tx^*, Ty^*)))$$

which is equivalent to

$$\tau + F(d(Tx^*, Ty^*)) \le F(d(x^*, y^*)), \tag{15}$$

a contradiction. □

It is clear that if we let $F(t) = \ln t$, then we deduce Theorem 1.3 that is the main result of Seghal[18]. Notice also that, for $n(x) = 1$ for all $x \in X$, Theorem 1.3 implies the well-known Banach contraction mapping principle.

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