Available online: November 20, 2019

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 69, Number 1, Pages 369–375 (2020)
DOI: 10.31801/cfsuasmas.440575
ISSN 1303–5991 E-ISSN 2618-6470

http://communications.science.ankara.edu.tr/index.php?series=A1

MORE IDENTITIES FOR FIBONACCI AND LUCAS QUATERNIONS

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Abstract. In this paper, we define the associate matrix as

$$F = \begin{pmatrix} 1+i+2j+3k & i+j+2k \\ i+j+2k & 1+j+k \end{pmatrix}.$$

By the means of the matrix F, we give several identities about Fibonacci and Lucas quaternions by matrix methods. Since there are two different determinant definitions of a quaternion square matrix (whose entries are quaternions), we obtain different Cassini identities for Fibonacci and Lucas quaternions apart from Cassini identities that given in the papers [5] and [7].

1. Introduction

The quaternions were described by Irish mathematicians Sir William and Rowan Hamilton as a extension of a complex number. The set of quaternion is defined by

$$H = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_n \in \mathbb{Z},\ n = 0, 1, 2, 3\}$$

where $i^2 = j^2 = k^2 = -1 = ijk$. This imply that $ij = k = -ji,\ jk = i = -kj$ and $ki = j = -ik$. The set of all quaternions form are associate but not commutative algebra. We can write

$$q = a_0 + u$$

where $u = ia_1 + ja_2 + ka_3$. The conjugate of the quaternion q is denoted by q^* and defined by $q^* = q - u$. Namely, the conjugate of the quaternion q is $q^* = a_0 - ia_1 - ja_2 - ka_3$.

For $n \geq 2$, the Fibonacci and Lucas sequences are defined as

$$F_n = F_{n-1} + F_{n-2},\ F_0 = 0,\ F_1 = 1$$

and

$$L_n = L_{n-1} + L_{n-2},\ L_0 = 2,\ L_1 = 1,$$

Received by the editors: July 04, 2018, Accepted: October 31, 2019.
2010 *Mathematics Subject Classification.* 11B39, 11R52.
Key words and phrases. Fibonacci quaternions, Lucas quaternions, recurrence relations.

respectively. There are lots of amazing identities belongs to Fibonacci and Lucas numbers. For the details, we refer the book of T. Koshy ([1]).

A. F. Horadam [2] defined nth Fibonacci and Lucas quaternions as follows,

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}$$

and

$$K_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3}.$$

The conjugates of these quaternions are given by

$$\widetilde{Q}_n = F_n - iF_{n+1} - jF_{n+2} - kF_{n+3}$$

and

$$\widetilde{K}_n = L_n - iL_{n+1} - jL_{n+2} - kL_{n+3}.$$

There are several researchers who focus on this Fibonacci and Lucas quaternions. Swamy [4] gave interesting identities for Fibonacci quaternions. Iyer [3] established some relations about Fibonacci and Lucas quaternions. Binet formula and generating functions of Fibonacci quaternions was given by Halıcı [5]. Akyiğit et al. [6] gave the definition of split Fibonacci quaternions together with their properties. Afterwards, they gave Fibonacci generalized quaternions and they used the well-known identities related to the Fibonacci and Lucas numbers to obtain the relations regarding these quaternions in [7]. Another type generalization was given by Tan et. al. [8], [9]. They defined bi-periodic Fibonacci and Lucas quaternions.

We use the determinant of quaternion matrix whose entries are quaternions. Since the set of all quaternions are not commutative, we give the definitions of the determinant of a quaternions matrix. Let A be a quaternion square matrix. Denote A by,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where $a_{ij} \in H$ for $i = 1, 2$ and $j = 1, 2$. The determinant of A, $\det A$, is defined by

$$\begin{aligned} \det A &= \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ &= a_{11}a_{22} - a_{12}a_{21}. \end{aligned} \tag{1}$$

The above definition is called rule "multiplication from above to down below". Since the set of all quaternion is not commutative, another product direction can be defined. Namely, the definition

$$\begin{aligned} \det A &= \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ &= a_{22}a_{11} - a_{21}a_{12} \end{aligned} \tag{2}$$

is called the rule "multiplication from down below to above" (For details, see the book [11], section 9.11.)

In this paper, we present some novel identities between Fibonacci and Lucas quaternions by using matrix method. Thanks to the this method, identities belongs to Fibonacci and Lucas quaternions can be obtained easily together with the properties of matrices. Before going further, we define the following two matrices,

$$U=\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},\ F=\begin{pmatrix} 1+i+2j+3k & i+j+2k \\ i+j+2k & 1+j+k \end{pmatrix}. \tag{3}$$

It is known that

$$U^n=\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}. \tag{4}$$

2. Main Results

First theorem is about the Cassini identity belongs to Fibonacci and Lucas quaternions. We obtain the two versions of Cassini identity since there are two different determinant definitions of quaternion matrix. The first type Cassini identity for Fibonacci quaternions was given by Halıcı ([5], Theorem 3.4) and Akyiğit et. al. [7] gave the first type Cassini identity for Fibonacci and Lucas generalized quaternions. For both identities, they used the determinant definition of (2). We get the second type Cassini identity for the definition of (1).

Theorem 1. *(First type Cassini identity) For* $n\geq 1$, *the identities*

$$Q_{n-1}Q_{n+1}-Q_n^2=(-1)^n(2Q_1-3k)$$

and

$$K_{n-1}K_{n+1}-K_n^2=5(-1)^{n-1}(2Q_1-3k).$$

Proof. By the matrices in (3) and (4), we obtain that

$$U^nF = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}\begin{pmatrix} 1+i+2j+3k & i+j+2k \\ i+j+2k & 1+j+k \end{pmatrix} \tag{5}$$

$$= \begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix}. \tag{6}$$

Using the second definition of determinant (2) for the equation (5), we get

$$\begin{aligned} Q_{n-1}Q_{n+1}-Q_n^2 &= \left(F_{n+1}F_{n-1}-F_n^2\right) \\ &\quad \times\left((1+j+k)(1+i+2j+3k)-(i+j+2k)^2\right) \\ &= (-1)^n(2Q_1-3k) \end{aligned}$$

as claimed. Since $F_{n-1}+F_{n+1}=L_n$, then we write

$$\begin{aligned} \left(U^{n-1}+U^{n+1}\right)F &= \begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix}\begin{pmatrix} 1+i+2j+3k & i+j+2k \\ i+j+2k & 1+j+k \end{pmatrix} \\ &= \begin{pmatrix} K_{n+1} & K_n \\ K_n & K_{n-1} \end{pmatrix}. \end{aligned}$$

Applying the equation (2) gives that

$$\begin{aligned} K_{n-1}K_{n+1}-K_n^2 &= \left(L_{n+1}L_{n-1}-L_n^2\right) \\ &\quad \times\left(\left(1+j+k\right)\left(1+i+2j+3k\right)-\left(i+j+2k\right)^2\right) \\ &= 5\left(-1\right)^{n-1}\left(2Q_1-3k\right). \end{aligned}$$

□

The first type Cassini identity for the generalized bi-periodic Fibonacci quaternions was given by Tan et. al [10].

Theorem 2. *(Second type Cassini identity) For $n \geq 1$, the identities*

$$Q_{n+1}Q_{n-1}-Q_n^2=\left(-1\right)^n\left(2+2j+5k\right)$$

and

$$K_{n+1}K_{n-1}-K_n^2=5\left(-1\right)^{n-1}\left(2+2j+5k\right).$$

Proof. Since the equation (5) holds, then the definition (1) yields that

$$\begin{aligned} Q_{n+1}Q_{n-1}-Q_n^2 &= \left(F_{n-1}F_{n+1}-F_n^2\right) \\ &\quad \times\left(\left(1+i+2j+3k\right)\left(1+j+k\right)-\left(i+j+2k\right)^2\right) \\ &= \left(-1\right)^n\left(2+2j+5k\right). \end{aligned}$$

Similarly, one can see that

$$K_{n+1}K_{n-1}-K_n^2=5\left(-1\right)^{n-1}\left(2+2j+5k\right)$$

holds. □

Theorem 3. *For $n, m \geq 1$ integers, then*

$$Q_{m+n}=F_{m+1}Q_n+F_mQ_{n-1}$$

follows.

Proof. Since $U^{m+n}F=U^m\left(U^nF\right)$ holds, then we obtain that

$$\begin{aligned} U^{m+n}F &= \begin{pmatrix} Q_{m+n+1} & Q_{m+n} \\ Q_{m+n} & Q_{m+n-1} \end{pmatrix} \\ &= U^mU^nF \\ &= \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix}\begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix}. \end{aligned} \tag{7}$$

Equating the first row second column entries of two matrix in (7) is yields that

$$Q_{m+n}=F_{m+1}Q_n+F_mQ_{n-1}.$$

□

Since $F_{m+2}+F_m=F_{m+1}$ and $Q_{m+n+1}+Q_{m+n-1}=K_{m+n}$ holds, we get the following identity as a corollary.

Corollary 1. *For positive integers m and n, we get*

$$K_{m+n} = L_{m+1}Q_n + L_m Q_{n-1} \tag{8}$$

Theorem 4. *For $n, m \geq 1$ integers, we have*

$$Q_{m+1}Q_{n+1} + Q_m Q_n = Q_{m+n+1} + iQ_{m+n+2} + jQ_{m+n+3} + kQ_{m+n+4}.$$

Proof. The fact $F\left(U^{m+n}F\right) = \left(FU^m\right)\left(U^nF\right)$ yields that

$$\begin{pmatrix} 1+i+2j+3k & i+j+2k \\ i+j+2k & 1+j+k \end{pmatrix} \begin{pmatrix} Q_{m+n+1} & Q_{m+n} \\ Q_{m+n} & Q_{m+n-1} \end{pmatrix} \tag{9}$$

$$= \begin{pmatrix} Q_{m+1} & Q_m \\ Q_m & Q_{m-1} \end{pmatrix} \begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix} \tag{10}$$

If we equalize the pivot elements in the equation (9), we obtain claimed result. □

The equation (8) give the following system,

$$\begin{pmatrix} K_{m+n+1} & K_{m+n} \\ K_{m+n} & K_{m+n-1} \end{pmatrix} = \begin{pmatrix} L_{m+1} & L_m \\ L_m & L_{m-1} \end{pmatrix} \begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix}. \tag{11}$$

If we multiply the equation (11) with the matrix F from left side, we get that

$$\begin{aligned} &\begin{pmatrix} 1+i+2j+3k & i+j+2k \\ i+j+2k & 1+j+k \end{pmatrix} \begin{pmatrix} K_{m+n+1} & K_{m+n} \\ K_{m+n} & K_{m+n-1} \end{pmatrix} \\ = &\left(\begin{pmatrix} 1+i+2j+3k & i+j+2k \\ i+j+2k & 1+j+k \end{pmatrix} \begin{pmatrix} L_{m+1} & L_m \\ L_m & L_{m-1} \end{pmatrix}\right) \begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix} \\ = &\begin{pmatrix} K_{m+1} & K_m \\ K_m & K_{m-1} \end{pmatrix} \begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix}. \end{aligned}$$

Equating the first row and second column element, we obtain the following theorem.

Theorem 5. *For $m, n \geq 1$, we get*

$$K_{m+n+1} + iK_{m+n+2} + jK_{m+n+3} + kK_{m+n+4} = K_{m+1}Q_{n+1} + K_n Q_n.$$

Let define the conjugate matrix of F as

$$\widetilde{F} = \begin{pmatrix} 1-i-2j-3k & i-j-2k \\ i-j-2k & 1-j-k \end{pmatrix}.$$

We present the first and second type Cassini identities for the conjugate Fibonacci and Lucas quaternions.

Theorem 6. *The identities*

$$\begin{aligned} \widetilde{Q}_{n-1}\widetilde{Q}_{n+1} - \widetilde{Q}_n^2 &= (-1)^n (2-2j-5k) \\ \widetilde{Q}_{n+1}\widetilde{Q}_{n-1} - \widetilde{Q}_n^2 &= (-1)^n \left(2\widetilde{Q}_1 + 3k\right) \\ \widetilde{K}_{n-1}\widetilde{K}_{n+1} - \widetilde{K}_n^2 &= 5(-1)^{n-1} (2-2j-5k) \end{aligned}$$

and

$$\widetilde{K}_{n+1}\widetilde{K}_{n-1}-\widetilde{K}_n^2=5\left(-1\right)^{n-1}\left(2\widetilde{Q}_1+3k\right)$$

hold for $n\geq 1$.

Proof. The identity

$$\begin{aligned}\widetilde{F}U^n &= \begin{pmatrix} 1-i-2j-3k & i-j-2k \\ i-j-2k & 1-j-k \end{pmatrix}\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} \widetilde{Q}_{n+1} & \widetilde{Q}_n \\ \widetilde{Q}_n & \widetilde{Q}_{n-1} \end{pmatrix}.\end{aligned}$$

We get the first and second identities after taking the determinant under applying the rules (2) and (1), respectively. By the way of the proofs of Theorem 1 and Theorem 2, one can see the other identities easily. □

By using the conjugate matrix of F together with the matrix F, we get the following identities. We equalize the first row and second column element to obtain these identities,

Theorem 7. *For the integers* m *and* n,

1) The identity $\widetilde{F}\left(U^{m+n}F\right)=\left(\widetilde{F}U^m\right)\left(U^nF\right)$ yields that

$$Q_{m+n+1}-iQ_{m+n+2}-jQ_{m+n+3}-kQ_{m+n+4}=\widetilde{Q}_{m+1}Q_{n+1}+\widetilde{Q}_mQ_n.$$

2) The identity $\left(\widetilde{F}U^{m+n}\right)F=\left(\widetilde{F}U^m\right)\left(U^nF\right)$ gives

$$\widetilde{Q}_{m+n+1}+i\widetilde{Q}_{m+n+2}+j\widetilde{Q}_{m+n+3}+k\widetilde{Q}_{m+n+4}=\widetilde{Q}_{m+1}Q_{n+1}+\widetilde{Q}_mQ_n.$$

3) By the identities

$$\left(FU^{m+n}\right)\widetilde{F}=\left(FU^m\right)\left(U^n\widetilde{F}\right) \text{ and } F\left(Q^{m+n}\widetilde{F}\right)=\left(FQ^m\right)\left(Q^n\widetilde{F}\right),$$

$$\begin{aligned}\widetilde{Q}_{m+n+1}+i\widetilde{Q}_{m+n+2}+j\widetilde{Q}_{m+n+3}+k\widetilde{Q}_{m+n+4} &= Q_{m+1}\widetilde{Q}_{n+1}+Q_m\widetilde{Q}_n \\ Q_{m+n+1}-iQ_{m+n+2}-jQ_{m+n+3}-kQ_{m+n+4} &= Q_{m+1}\widetilde{Q}_{n+1}+Q_m\widetilde{Q}_n\end{aligned}$$

hold, respectively.

3. Open Question

There are several divisibility identities for Fibonacci and Lucas number. For m,n positive integers, the well-known identities are

$$n|m\Longleftrightarrow F_n|F_m$$

and

$$n=km\Leftrightarrow L_m|L_n \quad \text{where } k \text{ is odd integer.}$$

Which conditions are sufficient and necessary for the elements $\frac{Q_m}{Q_n}$ and $\frac{K_m}{K_n}$ to be Fibonacci and Lucas quaternions?

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