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ON NEW BÉZIER BASES WITH SCHURER POLYNOMIALS AND CORRESPONDING RESULTS IN APPROXIMATION THEORY

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ABSTRACT. A new type Bézier bases with λ shape parameters have been defined [30, Ye et al., 2010]. We slightly modify these bases to establish new Bézier bases with Schurer polynomials and λ shape parameters. We construct a new type Schurer operators via defined new Bézier-Schurer bases. Also, we study statistical convergence properties of these operators and obtain an estimate for the rate of weighted A-statistical convergence. Moreover, we prove two Voronovskaja-type theorems including a Voronovskaja-type approximation theorem using weighted A-statistical convergence.

1. EXTENDED BÉZIER BASES

In computer aided geometric design and computer graphics parametric representations of surfaces and curves have extensively been used for modeling miscellaneous surfaces. It is important which basis functions are used if we want to preserve the shape of the curve or surface when we demonstrate a parametric surface or curve. This is why Bernstein-Bézier curve and surface representation have an important role in computer graphics. Bernstein basis functions are used to construct classical Bézier curves since they have a simple structure to use. They have also received attention for their utility in the meshing of curved geometries and the numerical solution of partial differential equations. We refer to [15, 22, 29] for recent computer graphics studies including Bézier curves or bases.

A new type Bézier bases with shape parameters λ were defined by Ye et al. in 2010 [30]. We slightly modify these bases to establish new Bézier bases with Schurer polynomials, which were defined in [25], and shape parameters λ.

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Let $d \geq 0$ be a given integer and shape parameters $\lambda \in [-1, 1]$. We define the following Bézier-Schurer bases

$$
\begin{aligned}
\tilde{s}_{n,0}(\lambda; x) &= s_{n,0}(x) - \frac{\lambda}{n+d+1} s_{n+1,1}(x), \\
\tilde{s}_{n,i}(\lambda; x) &= s_{n,i}(x) + \frac{\lambda}{(n+d)^2 - 1} \left[(n+d-2i+1) s_{n+1,i}(x) \right. \\
&\quad \left. -(n+d-2i-1) s_{n+1,i+1}(x) \right] \ (i = 1, 2 \ldots, n+d-1), \\
\tilde{s}_{n,n+d}(\lambda; x) &= s_{n,n+d}(x) - \frac{\lambda}{n+d+1} s_{n+1,n+d}(x),
\end{aligned} \tag{1}
$$

where fundamental Schurer polynomials $s_{n,i}(x)$ of degree $n+d$ defined as

$$
s_{n,i}(x) = \binom{n+d}{i} x^i \ (1-x)^{n+d-i} \quad (i = 0, 1, \ldots, n+d).
$$

Lemma 1. *New Bézier-Schurer bases have partition of unity property.*

Proof. It is enough to show the equality $\sum_{i=0}^{n+d} \tilde{s}_{n,i}(\lambda, x) = 1$ holds.

$$
\begin{aligned}
\sum_{i=0}^{n+d} \tilde{s}_{n,i}(\lambda, x) &= s_{n,0}(x) - \frac{\lambda}{n+d+1} s_{n+1,1}(x) + s_{n,n+d}(x) - \frac{\lambda}{n+d+1} s_{n+1,n+d}(x) \\
&\quad + \sum_{i=1}^{n+d-1} \left[s_{n,i}(x) + \lambda \left(\frac{n+d-2i+1}{(n+d)^2-1} s_{n+1,i}(x) \right.\right. \\
&\quad \left.\left. - \frac{n+d-2i-1}{(n+d)^2-1} s_{n+1,i+1}(x) \right) \right] \\
&= s_{n,0}(x) + s_{n,1}(x) + \cdots + s_{n,n+d}(x) \\
&\quad + \lambda \left(\frac{n+d-1}{(n+d)^2-1} s_{n+1,1}(x) - \frac{n+d-3}{(n+d)^2-1} s_{n+1,2}(x) \right) \\
&\quad + \lambda \left(\frac{n+d-3}{(n+d)^2-1} s_{n+1,2}(x) - \frac{n+d-5}{(n+d)^2-1} s_{n+1,3}(x) \right) + \cdots \\
&\quad + \lambda \left(-\frac{n+d-5}{(n+d)^2-1} s_{n+1,n+d-2}(x) + \frac{n+d-3}{(n+d)^2-1} s_{n+1,n+d-1}(x) \right) \\
&\quad + \lambda \left(-\frac{n+d-3}{(n+d)^2-1} s_{n+1,n+d-1}(x) + \frac{n+d-1}{(n+d)^2-1} s_{n+1,n+d}(x) \right) \\
&\quad - \frac{\lambda}{n+d+1} s_{n+1,1}(x) - \frac{\lambda}{n+d+1} s_{n+1,n+d}(x).
\end{aligned}
$$

Since Schurer operators satisfy the equality $\sum_{i=0}^{n+d} s_{n,i}(x) = 1$ we get the desired result. □

Rest of the paper is organized as follows: In Section 2, λ-Schurer operators are constructed and corresponding approximation results are obtained. In Section 3, some statistical approximation properties of defined operators are studied and an estimate for the rate of weighted A-statistical convergence is established. In Section 4, two Voronovskaja-type theorems including a Voronovskaja-type approximation theorem using weighted A-statistical convergence are proved. Final section of the paper is devoted to give some concluding remarks including some future studies.

2. λ-Schurer operators and corresponding results in approximation theory

A new type λ-Bernstein operators have been introduced by Cai et al. in [6] based on Bézier bases defined by Ye et al. in [30]. We refer to [5, 6, 20, 23, 26] for recent studies about λ-Bernstein type operators and [13, 14, 28] for some Schuer type operators.

Considering a given non-negative integer d, we introduce λ-Schurer operators $S^{\lambda}_{n,d}(f;x) : C[0,1+d] \longrightarrow C[0,1]$

$$S^{\lambda}_{n,d}(f;x) = \sum_{i=0}^{n+d} \tilde{s}_{n,i}(\lambda;x)\ f\left(\frac{i}{n}\right) \tag{2}$$

for any $n \in \mathbb{N}$, where new Bézier-Schurer bases $\tilde{s}_{n,i}(\lambda;x)$ are defined in (1).

Lemma 2. *We have following results for λ-Schurer operators:*

$$S^{\lambda}_{n,d}(t;x) = \frac{n+d}{n}x + \frac{1-2x+x^{n+d+1}-(1-x)^{n+d+1}}{n(n+d-1)}\lambda;$$

$$S^{\lambda}_{n,d}(t^2;x) = \frac{(n+d)^2}{n^2}x^2 + \frac{n+d}{n^2}x(1-x) + \frac{2(n+d)x-1-4(n+d)x^2+(2(n+d)+1)x^{n+d+1}+(1-x)^{n+d+1}}{n^2(n+d-1)}\lambda;$$

Proof. Using definition of operators (2) and Bézier-Schurer bases $\tilde{s}_{n,i}(\lambda;x)$ (1), we write

$$\begin{aligned} S^{\lambda}_{n,d}(t;x) &= \sum_{i=0}^{n+d} \frac{i}{n}\tilde{s}_{n,i}(\lambda;x) \\ &= \frac{n+d}{n}s_{n,n+d}(x) - \frac{n+d}{n}\frac{\lambda}{n+d+1}s_{n+1,n+d}(x) \\ &+ \sum_{i=0}^{n+d-1} \frac{i}{n}\left[s_{n,i}(x) + \lambda\left(\frac{n+d-2i+1}{(n+d)^2-1}s_{n+1,i}(x) - \frac{n+d-2i-1}{(n+d)^2-1}s_{n+1,i+1}(x)\right)\right] \\ &= \sum_{i=0}^{n+d}\frac{i}{n}s_{n,i}(x) + \lambda\left(\varphi_1(n,d,x) - \varphi_2(n,d,x)\right), \end{aligned}$$

where

$$\varphi_1(n,d,x)=\sum_{i=0}^{n+d}\frac{i}{n}\frac{n+d-2i+1}{(n+d)^2-1}s_{n+1,i}(x);$$
$$\varphi_2(n,d,x)=\sum_{i=1}^{n+d-1}\frac{i}{n}\frac{n+d-2i-1}{(n+d)^2-1}s_{n+1,i+1}(x).$$

Now we compute the expressions $\varphi_1(n,d,x)$ and $\varphi_2(n,d,x)$.

$$\begin{aligned}\varphi_1(n,d,x)&=\frac{1}{n+d-1}\sum_{i=0}^{n+d}\frac{i}{n}s_{n+1,i}(x)-\frac{2}{(n+d)^2-1}\sum_{i=0}^{n+d}\frac{i^2}{n}s_{n+1,i}(x)\\&=\frac{x(n+d+1)}{n(n+d-1)}\sum_{i=0}^{n+d-1}s_{n,i}(x)-\frac{2x}{n(n+d-1)}\sum_{i=0}^{n+d-1}s_{n,i}(x)\\&-\frac{2x^2(n+d)}{n(n+d-1)}\sum_{i=0}^{n+d-2}s_{n-1,i}(x)\\&=-\frac{(1-x^{n+d})(x(n+d)+x-2x)}{n(n+d-1)}-\frac{2x^2(n+d)(1-x^{n+d-1})}{n(n+d-1)}\\&=\frac{x-x^{n+d+1}}{n}-\frac{2(n+d)(x^2-x^{n+d+1})}{n(n+d-1)}.\end{aligned}$$

$$\begin{aligned}\varphi_2(n,d,x)&=\frac{n+d-1}{n((n+d)^2-1)}\sum_{i=1}^{n+d-1}i\ s_{n+1,i+1}(x)\\&-\frac{2}{n((n+d)^2-1)}\sum_{i=1}^{n+d-1}i^2\ s_{n+1,i+1}(x)\\&=\frac{2x}{n(n+d-1)}\sum_{i=1}^{n+d-1}s_{n,i}(x)-\frac{2x^2(n+d)}{n(n+d-1)}\sum_{i=0}^{n+d-2}s_{n-1,i}(x)\\&+\frac{x}{n}\sum_{i=1}^{n+d-1}s_{n,i}(x)\\&-\frac{2}{n((n+d)^2-1)}\sum_{i=1}^{n+d-1}s_{n+1,i+1}(x)-\frac{1}{n(n+d+1)}\sum_{i=1}^{n+d-1}s_{n+1,i+1}(x)\\&=\frac{2x-2x(1-x)^{n+d}-x^{n+d+1}}{n(n+d-1)}-\frac{2(n+d)(x^2-x^{n+d+1})}{n(n+d-1)}\\&-\frac{2-(1-x)^{n+d+1}-2x(n+d+1)(1-x)^{n+d}-2x^{n+d+1}}{n((n+d)^2-1)}\end{aligned}$$

$$+\frac{x-x^{n+d+1}}{n}-\frac{1-(1-x)^{n+d+1}-x(n+d+1)(1-x)^{n+d}-x^{n+d+1}}{n(n+d-1)}.$$

We obtain the result for $S^{\lambda}_{n,d}(t;x)$ combining the results obtained for $\varphi_1(n,d,x)$ and $\varphi_2(n,d,x)$ since Schurer operators are linear, and Schurer operators and fundamental Schurer bases satisfy the following equality:

$$\sum_{i=1}^{n+d}\frac{i}{n}s_{n,i}(x)=\left(1+\frac{d}{n}\right)x.$$

We again use the definition of operators (2), Bézier-Schurer bases $\tilde{s}_{n,i}(\lambda;x)$ (1) and the following relations to prove the second part of the lemma:

$$S^{\lambda}_{n,d}(t^2;x)=\sum_{i=0}^{n+d}\frac{i^2}{n^2}\tilde{s}_{n,i}(\lambda;x)=\frac{(n+d)^2}{n^2}s_{n,n+d}(x)-\frac{(n+d)^2}{n^2}\frac{\lambda}{n+d+1}s_{n+1,n+d}(x)$$
$$+\sum_{i=0}^{n+d-1}\frac{i^2}{n^2}\left[s_{n,i}(x)+\lambda\left(\frac{n+d-2i+1}{(n+d)^2-1}s_{n+1,i}(x)-\frac{n+d-2i-1}{(n+d)^2-1}s_{n+1,i+1}(x)\right)\right]$$
$$=\sum_{i=0}^{n+d}\frac{i^2}{n^2}s_{n,i}(x)+\lambda\left(\varphi_3(n,d,x)-\varphi_4(n,d,x)\right),$$

where

$$\varphi_3(n,d,x)=\sum_{i=0}^{n+d}\frac{i^2}{n^2}\frac{n+d-2i+1}{(n+d)^2-1}s_{n+1,i}(x);$$
$$\varphi_4(n,d,x)=\sum_{i=1}^{n+d-1}\frac{i^2}{n^2}\frac{n+d-2i-1}{(n+d)^2-1}s_{n+1,i+1}(x).$$

Now we compute the expressions $\varphi_3(n,d,x)$ and $\varphi_4(n,d,x)$.

$$\begin{aligned}\varphi_3(n,d,x)&=\frac{1}{n+d-1}\sum_{i=0}^{n+d}\frac{i^2}{n^2}s_{n+1,i}(x)-\frac{2}{(n+d)^2-1}\sum_{i=0}^{n+d}\frac{i^3}{n^2}s_{n+1,i}(x)\\&=\frac{(n+d)(n+d+1)x^2}{n^2(n+d-1)}\sum_{i=0}^{n+d-2}s_{n-1,i}(x)+\frac{x}{n^2}\sum_{i=0}^{n+d-1}s_{n,i}(x)\\&-\frac{2(n+d)x^3}{n^2}\sum_{i=0}^{n+d-3}s_{n-2,i}(x)-\frac{6(n+d)x^2}{n^2(n+d-1)}\sum_{i=0}^{n+d-2}s_{n-1,i}(x)\\&=\frac{(n+d)(n+d+1)(x^2-x^{n+d+1})}{n^2(n+d-1)}+\frac{x-x^{n+d+1}}{n^2}\\&-\frac{2(n+d)(x^3-x^{n+d+1})}{n^2}-\frac{6(n+d)(x^2-x^{n+d+1})}{n^2(n+d-1)}\end{aligned}$$

$$= \frac{2(n+d)(x^{n+d+1}-x^3)}{n^2} + \frac{x-x^{n+d+1}}{n^2} + \frac{(n+d)^2-5(n+d)(x^2-x^{n+d+1})}{n^2(n+d-1)}.$$

$$\begin{aligned}
\varphi_4(n,d,x) &= \frac{1}{n+d+1}\sum_{i=1}^{n+d-1}\frac{i^2}{n^2}\, s_{n+1,i+1}(x) - \frac{2}{(n+d)^2-1}\sum_{i=1}^{n+d-1}\frac{i^3}{n^2}\, s_{n+1,i+1}(x)\\
&= \frac{x^2(n+d)}{n^2}\sum_{i=0}^{n+d-2} s_{n-1,i}(x) - \frac{x}{n^2}\sum_{i=1}^{n+d-1} s_{n,i}(x)\\
&+ \frac{1}{n^2(n+d+1)}\sum_{i=1}^{n+d-1} s_{n+1,i+1}(x) + \frac{2(n+d)x^3}{n^2}\sum_{i=0}^{n+d-3} s_{n-2,i}(x)\\
&+ \frac{2x}{n^2(n+d-1)}\sum_{i=1}^{n+d-1} s_{n,i}(x) - \frac{2}{n^2((n+d)^2-1)}\sum_{i=1}^{n+d-1} s_{n+1,i+1}(x)\\
&= \frac{x^2(n+d)(1-x^{n+d-1})}{n^2} - \frac{x(1-x^{n+d})}{n^2}\\
&+ \frac{1-(1-x)^{n+d+1}-x(n+d+1)(1-x)^{n+d}-x^{n+d+1}}{n^2(n+d-1)}\\
&- \frac{2(n+d)x^3(1-x^{n+d-2})}{n^2} - \frac{2x(1-x^{n+d})}{n^2(n+d-1)}\\
&+ \frac{2-2(1-x)^{n+d+1}-2x(n+d+1)(1-x)^{n+d}-2x^{n+d+1}}{n^2((n+d)^2-1)}.
\end{aligned}$$

We get $S_{n,d}^{\lambda}(t^2;x)$ combining $\varphi_3(n,d,x)$ and $\varphi_4(n,d,x)$ since Schurer operators and fundamental Schurer bases satisfy the following equality:

$$\sum_{i=1}^{n+d}\frac{i^2}{n^2}s_{n,i}(x) = \frac{n+d}{n^2}\left\{(n+d)x^2+x(1-x)\right\}.$$

□

Corollary 3. *We have the following relations for $S_{n,d}^{\lambda}(t-x;x)$ and $S_{n,d}^{\lambda}((t-x)^2;x)$:*

$$\begin{aligned}
S_{n,d}^{\lambda}(t-x;x) &= \frac{d}{n}x + \frac{1-2x+x^{n+d+1}-(1-x)^{n+d+1}}{n(n+d-1)}\lambda;\\
S_{n,d}^{\lambda}((t-x)^2;x) &= \frac{d^2}{n^2}x^2 + \frac{n+d}{n^2}x(1-x) - \frac{2x^{n+d+2}-2x(1-x)^{n+d+1}}{n(n+d-1)}\lambda\\
&+ \frac{2dx-1-4dx^2+(2(n+d)+1)x^{n+d+1}+(1-x)^{n+d+1}}{n^2(n+d-1)}\lambda;
\end{aligned}$$

Corollary 4. *We have the following relations for* $S_{n,d}^{\lambda}(t-x;x)$ *and* $S_{n,d}^{\lambda}((t-x)^2;x)$*:*

$$\lim_{n\to\infty} n\ S_{n,d}^{\lambda}(t-x;x) = dx;$$
$$\lim_{n\to\infty} n\ S_{n,d}^{\lambda}((t-x)^2;x) = x(1-x).$$

Remark 5. *We have the following results for* λ*-Schurer operators and Bézier-Schurer bases:*

- *If we take* $d=0$*, Bézier-Schurer bases (1) reduce to the classical Bézier bases defined in* [30].
- *If we take* $\lambda=0$*,* λ*-Schurer operators (2) reduce to the classical Schurer operators defined in* [25].
- *If we take* $d,\ \lambda=0$*,* λ*-Schurer operators (2) with Bézier-Schurer bases (1) reduce to the classical Bernstein operators defined in* [3].

The following theorem gives the uniform convergence property of λ-Schurer operators (2) by the well-known Bohman-Korovkin-Popoviciu theorem:

Theorem 6. *Let* $f\in C[0,1+d]$*, then we have*

$$\lim_{n\to\infty} S_{n,d}^{\lambda}(f;x) = f(x)$$

uniformly on $[0,1]$*, where* $C[0,1+d]$ *denotes the space of all real-valued continuous functions on* $[0,1+d]$ *endowed with the norm* $\|f\|_{C[0,1]} = \sup_{x\in[0,1+d]}|f(x)|$.

We achieve a global approximation formula in terms of Ditzian-Totik uniform modulus of smoothness of first and second order for λ-Schurer operators (2), and give a local direct estimate of the rate of convergence by Lipschitz-type function involving two parameters.

Definition 7. *Global approximation formula in terms of Ditzian-Totik uniform modulus of smoothness of first and second order defined by*

$$\omega_{\xi}(f,\zeta) := \sup_{0<|h|\le\zeta}\ \sup_{x,x+h\xi(x)\in[0,1]} \{|f(x+h\xi(x))-f(x)|\};$$

$$\omega_2^{\tau}(f,\zeta) := \sup_{0<|h|\le\zeta}\ \sup_{x,x\pm h\tau(x)\in[0,1]} \{|f(x+h\tau(x))-2f(x)+f(x-h\tau(x))|\},$$

respectively, where τ *is an admissible step-weight function on* $[a,b]$*, i.e.* $\tau(x) = [(x-a)(b-x)]^{1/2}$ *if* $x\in[a,b]$, [7]. *We write AC for absolutely continuous functions, then* K*-functional is*

$$K_{2,\tau(x)}(f,\zeta) = \inf_{g\in W^2(\tau)} \left\{ ||f-g||_{C[0,1]} + \zeta||\tau^2 g''||_{C[0,1]} : g\in C^2[0,1+d] \right\},$$

where $\zeta>0$*,* $W^2(\tau) = \{g\in C[0,1+d] : g'^2 g'' \in C[0,1+d]\}$ *and* $C^2[0,1+d] = \{g\in C[0,1+d] : g', g''\in C[0,1+d]\}$.

Remark 8. *It is known by* [9] *that there exists an absolute constant* $C>0$*, such that*

$$C^{-1}\omega_2^{\tau}(f,\sqrt{\zeta})\le K_{2,\tau(x)}(f,\zeta)\le C\omega_2^{\tau}(f,\sqrt{\zeta}). \tag{3}$$

First we obtain global approximation formula in terms of Ditzian-Totik uniform modulus of smoothness of first and second order.

Theorem 9. *Let* $f\in C[0,1+d]$*,* $x\in[0,1]$ *and* $\lambda\in[-1,1]$*. Then for* $C>0$*, λ-Schurer operators (2) verify*

$$|S_{n,d}^{\lambda}(f;x)-f(x)|\le C\omega_2^{\tau}\Big(f,\frac{\sqrt{\alpha_{n,\lambda}(x)+\beta_{n,\lambda}^2(x)}}{2\tau(x)}\Big)+\omega_{\xi}\Big(f,\frac{\beta_{n,\lambda}(x)}{\xi(x)}\Big),$$

where $\beta_{n,\lambda}(x)=S_{n,d}^{\lambda}(t-x;x)$ *and* $\alpha_{n,\lambda}(x)=S_{n,d}^{\lambda}((t-x)^2;x)$ *are given in Corollary 3, and* $\tau(x)$ $(\tau\neq 0)$ *is an admissible step-weight function of Ditzian-Totik modulus of smoothness such that* τ^2 *is concave.*

Proof. Let $f\in C[0,1+d]$, $x\in[0,1]$ and $\lambda\in[-1,1]$. Defining the operators

$$\breve{S}_{n,d}^{\lambda}(f;x)=S_{n,d}^{\lambda}(f;x)+f(x)-f\Big(x+\frac{d}{n}x+\frac{1-2x+x^{n+d+1}-(1-x)^{n+d+1}}{n(n+d-1)}\lambda\Big) \tag{4}$$

we see that $\breve{S}_{n,d}^{\lambda}(1;x)=1$ and $\breve{S}_{n,d}^{\lambda}(t;x)=x$, that is $\breve{S}_{n,d}^{\lambda}(t-x;x)=0$.

Let $u=\rho x+(1-\rho)t$, $\rho\in[0,1]$. Since τ^2 is concave on $[0,1]$, it follows that $\tau^2(u)\ge\rho\tau^2(x)+(1-\rho)\tau^2(t)$ and

$$\frac{|t-u|}{\tau^2(u)}\le\frac{\rho|x-t|}{\rho\tau^2(x)+(1-\rho)\tau^2(t)}\le\frac{|t-x|}{\tau^2(x)}. \tag{5}$$

Hence the following inequalities hold:

$$\begin{aligned}|\breve{S}_{n,d}^{\lambda}(f;x)-f(x)|&\le|\breve{S}_{n,d}^{\lambda}(f-g;x)|+|\breve{S}_{n,d}^{\lambda}(g;x)-g(x)|+|f(x)-g(x)|\\&\le 4\|f-g\|_{C[0,1+d]}+|\breve{S}_{n,d}^{\lambda}(g;x)-g(x)|.\end{aligned} \tag{6}$$

Applying Taylor's formula we obtain

$$\begin{aligned}&|\breve{S}_{n,d}^{\lambda}(g;x)-g(x)|\\&\le S_{n,d}^{\lambda}\Big(\Big|\int_x^t|t-u|\ |g''(u)|du\Big|;x\Big)+\Big|\int_x^{x+\beta_{n,\lambda}(x)}\big|x+\beta_{n,\lambda}(x)-u\big|\ |g''(u)|\ du\Big|\\&\le\|\tau^2g''\|_{C[0,1+d]}S_{n,d}^{\lambda}\Big(\Big|\int_x^t\frac{|t-u|}{\tau^2(u)}du\Big|;x\Big)\\&\le+\|\tau^2g''\|_{C[0,1+d]}\ \Big|\int_x^{x+\beta_{n,\lambda}(x)}\frac{|x+\beta_{n,\lambda}(x)-u|}{\tau^2(u)}\ du\Big|\\&\le\tau^{-2}(x)\|\tau^2g''\|_{C[0,1+d]}S_{n,d}^{\lambda}((t-x)^2;x)+\tau^{-2}(x)\|\tau^2g''\|_{C[0,1+d]}\beta_{n,\lambda}^2(x).\end{aligned} \tag{7}$$

By definition of K-functional with relation (3) and inequalities (6)-(7), we have

$$|\breve{S}_{n,d}^{\lambda}(f;x)-f(x)| \leq 4\|f-g\|_{C[0,1+d]}+\tau^{-2}(x)\|\tau^2 g''\|_{C[0,1+d]}\big(\alpha_{n,\lambda}(x)+\beta_{n,\lambda}^2(x)\big)$$
$$\leq C\omega_2^{\tau}\Bigg(f,\frac{\sqrt{\alpha_{n,\lambda}(x)+\beta_{n,\lambda}^2(x)}}{2\tau(x)}\Bigg).$$

Also, by Ditzian-Totik uniform modulus of smoothness of first order we have

$$|f(x+\beta_{n,\lambda}(x))-f(x)| = \left|f\left(x+\xi(x)\frac{\beta_{n,\lambda}(x)}{\xi(x)}\right)-f(x)\right| \leq \omega_{\xi}\left(f,\frac{\beta_{n,\lambda}(x)}{\xi(x)}\right).$$

Therefore, following inequality, which completes the proof, holds:

$$|S_{n,d}^{\lambda}(f;x)-f(x)| \leq |\breve{S}_{n,d}^{\lambda}(f;x)-f(x)|+\big|f(x+\beta_{n,\lambda}(x))-f(x)\big|$$
$$\leq C\omega_2^{\tau}\Bigg(f,\frac{\sqrt{\alpha_{n,\lambda}(x)+\beta_{n,\lambda}^2(x)}}{2\tau(x)}\Bigg)+\omega_{\xi}\left(f,\frac{\beta_{n,\lambda}(x)}{\xi(x)}\right).$$

□

Theorem 10. *The following inequality holds:*

$$|S_{n,d}^{\lambda}(f;x)-f(x)| \leq |\beta_{n,\lambda}(x)|\ |f'(x)|+2\sqrt{\alpha_{n,\lambda}(x)}w\big(f',\sqrt{\alpha_{n,\lambda}(x)}\ \big)$$

for $f\in C^1[0,1+d]$ and $x\in[0,1]$, where $\alpha_{n,\lambda}(x)$ and $\beta_{n,\lambda}(x)$ are given in Theorem 9.

Proof. For any $t\in[0,1]$ and $x\in[0,1]$ we have

$$f(t)-f(x)=(t-x)f'(x)+\int_x^t(f'(u)-f'(x))du.$$

Applying operators $S_{n,d}^{\lambda}(f;x)$ to both sides of (**??**), we have

$$S_{n,d}^{\lambda}(f(t)-f(x);x)=f'(x)S_{n,d}^{\lambda}(t-x;x)+S_{n,d}^{\lambda}\left(\int_x^t(f'(u)-f'(x))du;x\right).$$

The following inequality holds for any $\zeta>0$, $u\in[0,1]$ and $f\in C[0,1+d]$:

$$|f(u)-f(x)| \leq w(f,\zeta)\left(\frac{|u-x|}{\zeta}+1\right),$$

With above inequality we get

$$\left|\int_x^t(f'(u)-f'(x))du\right| \leq w(f',\zeta)\left(\frac{(t-x)^2}{\zeta}+|t-x|\right).$$

Hence we have

$$\begin{aligned}|S_{n,d}^{\lambda}(f;x)-f(x)| \quad \leq \quad & |f'(x)|\ |S_{n,d}^{\lambda}(t-x;x)| \\ & +w(f',\zeta)\left\{\frac{1}{\zeta}S_{n,d}^{\lambda}((t-x)^2;x)+S_{n,d}^{\lambda}(t-x;x)\right\}.\end{aligned}$$

Applying Cauchy-Schwarz inequality on the right hand side of above inequality, we have

$$|S_{n,d}^{\lambda}(f;x)-f(x)| \leq f'(x)|\beta_{n,\lambda}(x)| + w(f',\zeta)\left\{\frac{1}{\zeta}\sqrt{S_{n,d}^{\lambda}((t-x)^2;x)}+1\right\}\sqrt{S_{n,d}^{\lambda}(|t-x|;x)}.$$

□

3. Some statistical approximation theorems

In this section, we use weighted mean matrix method to establish statistical approximation properties of λ-Schurer operators. We also give an estimate for the rate of weighted A-statistical convergence of λ-Schurer operators.

Statistical convergence was first introduced in [8] and [27]. A new characterization in terms of weighted regular matrix and a Korovkin type approximation theorem through statistically weighted A-summable sequences of real or complex numbers have been given by Mohiuddine et al. [16, 17]. For further results in weighted statistical approximation theory we refer to [11, 12] and for statistical approximation papers to [2] .

All the following notions, notations and definitions which can be found in [2, 8, 11, 12, 17, 27] are needed for the results of this part.

Definition 11. *Natural density of K is denoted by $\zeta(K)=\lim_{n\to\infty}\frac{1}{n}|K_n|$ provided that limit exists, where $K_n=\{k\leq n : k\in K\}$, $K\subseteq\mathbb{N}_0:=\mathbb{N}\cup\{0\}$ and vertical bars denote cardinality of the enclosed set. A sequence $x=(x_n)$ of numbers is called statistically convergent to a number L, denoted by st-$\lim_{n\to\infty}x=L$, if, for each $\epsilon>0$, $\zeta\{n : n\in\mathbb{N}$ and $|x_n-L|\geqq\epsilon\}=0$.*

Definition 12. *A-transform of x denoted by $Ax:=\{(Ax)_n\}$ is defined as $(Ax)_n=\sum_{k=0}^{\infty}a_{nk}x_k$ for a given non-negative infinite summability matrix $A=(a_{nk}), n,k\in\mathbb{N}$. It is provided defined series converges for every $n\in\mathbb{N}_0$. If $\lim_{n\to\infty}(Ax)_n=L$ whenever $\lim_{n\to\infty}x_n=L$, we say that A is a regular method. Then sequence $x=(x_n)$ is said to be A-statistically convergent to L, denoted by st_A-$\lim_{n\to\infty}x_n=L$, provided that for each $\epsilon>0$, $\lim_{n\to\infty}\sum_{k:|x_k-L|\geqq\epsilon}a_{nk}=0$.*

Remark 13. *We have the following results for A-statistical convergence concept:*

- *If we take $A=(C_1)$, the Cesaro matrix of order 1, A-statistical convergence becomes ordinary statistical convergence which was introduced in* [10].
- *If we take $A=I$, the identity matrix, A-statistical convergence becomes classical convergence.*
- *Every convergent sequence is statistically convergent to the same limit but not conversely.*

Definition 14. [16] *Assume that $q=(q_n)$ is a sequence of non-negative numbers so that $q_0>0$ and $Q_n=\sum_{k=0}^{n}q_k\to\infty$ as $n\to\infty$. Then $x=(x_n)$ is called* *weighted A-statistically convergent to L, if, for every* $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{Q_n} \sum_{k=0}^{n} q_k \sum_{m:|x_m - L| \geqq \epsilon} a_{km} = 0.$$

In this case, we write $S_{\mathbf{A}}^{\widetilde{N}} - \lim_{n\to\infty} x_n = L$.

Remark 15. [16] *The weighted A-statistical convergence generalizes A-statistical convergence, which we recover by putting* $q_n = 1$ *for all* $n \in \mathbb{N}$.

We now give main results related to statistical approximation of operators in (2).

Theorem 16. *Let* $A = (a_{nk})$ *be a weighted non-negative regular summability matrix for* $n, k \in \mathbb{N}$ *and* $q = (q_n)$ *be a sequence of non-negative numbers such that* $q_0 > 0$ *and* $Q_n = \sum_{k=0}^{n} q_k \to \infty$ *as* $n \to \infty$. *For any* $f \in C[0, 1 + d]$, *we have*

$$S_{\mathbf{A}}^{\widetilde{N}} - \lim_{n\to\infty} \|S_{n,d}^{\lambda}(f) - f\|_{C[0,1]} = 0.$$

Proof. Consider sequence of functions $e_j(x) = x^j$, where $j \in \{0, 1, 2\}$ and $x \in [0, 1]$. It is sufficient to satisfy

$$S_{\mathbf{A}}^{\widetilde{N}} - \lim_{n\to\infty} \|S_{n,d}^{\lambda}(e_j; x) - e_j\|_{C[0,1]} = 0, \quad j = 0, 1, 2.$$

From Lemma 2, it is clear that

$$S_{\mathbf{A}}^{\widetilde{N}} - \lim_{n\to\infty} \|S_{n,d}^{\lambda}(e_0; x) - e_0\|_{C[0,1]} = 0. \tag{8}$$

We have

$$\begin{aligned} \|S_{n,d}^{\lambda}(e_1; x) - e_1\|_{C[0,1]} &= \sup_{x\in[0,1]} \left| \frac{d}{n} x + \frac{1 - 2x + x^{n+d+1} - (1-x)^{n+d+1}}{n(n+d-1)} \lambda \right| \\ &\leq \frac{d}{n} + \frac{4}{n(n+d-1)} \end{aligned}$$

by Lemma 2. We choose a number $\epsilon > 0$ for a given $\epsilon' > 0$ such that $\epsilon < \epsilon'$. If we define following sets:

$$\begin{aligned} \Delta &:= \left\{ n \in \mathbb{N} : \|S_{n,d}^{\lambda}(e_1; x) - e_1\|_{C[0,1]} \geqq \epsilon' \right\}, \\ \Delta_1 &:= \left\{ n \in \mathbb{N} : \frac{d}{n} + \frac{4}{n(n+d-1)} \geqq \epsilon - \epsilon' \right\}, \end{aligned}$$

we see that the inclusion $\Delta \subset \Delta_1$ holds and

$$\frac{1}{Q_n} \sum_{k=0}^{n} q_k \sum_{m\in\Delta} a_{km} \leq \frac{1}{Q_n} \sum_{k=0}^{n} q_k \sum_{m\in\Delta_1} a_{km} \qquad \text{for all } n \in \mathbb{N}. \tag{9}$$

So we have

$$S_{\mathbf{A}}^{\widetilde{N}} - \lim_{n\to\infty} \|S_{n,d}^{\lambda}(e_1; x) - e_1\|_{C[0,1]} = 0 \tag{10}$$

as $n \to \infty$ in (9). By Lemma 2 we have

$$\|S_{n,d}^{\lambda}(e_2;x) - e_2\|_{C[0,1]} = \sup_{x\in[0,1]} \left| \frac{2nd+d^2}{n^2}x^2 + \frac{n+d}{n^2}x(1-x) + \frac{2(n+d)x - 1 - 4(n+d)x^2 +}{n^2(n+d-1)}\lambda + \frac{(2(n+d)+1)x^{n+d+1} + (1-x)^{n+d+1}}{n^2(n+d-1)}\lambda \right| \leq \frac{2nd+d^2}{n^2} + \frac{2n+2d}{n^2} + \frac{8(n+d)+2}{n^2(n+d-1)}.$$

We also obtain

$$S_{\mathbf{A}}^{\widetilde{N}} - \lim_{n\to\infty} \|S_{n,d}^{\lambda}(e_2;x) - e_2\|_{C[0,1]} = 0 \tag{11}$$

since

$$S_{\mathbf{A}}^{\widetilde{N}} - \lim_{n\to\infty} \left[\frac{2nd+d^2}{n^2} + \frac{2n+2d}{n^2} + \frac{8(n+d)+2}{n^2(n+d-1)} \right] = 0.$$

Combining (8), (10) and (11), we get desired result. □

We now estimate rate of weighted A-statistical convergence of operators $S_{n,d}^{\lambda}(f;x)$.

Definition 17. *Let $A = (a_{nk})$ be a weighted non-negative regular summability matrix and let $q = (q_n)$ be a sequence of non-negative numbers such that $q_0 > 0$ and $Q_n = \sum_{k=0}^{n} q_k \to \infty$ as $n \to \infty$. Also let (u_n) be a positive non-decreasing sequence. We say that a sequence $x = (x_n)$ is weighted A-statistically convergent to L with the rate $o(u_n)$ if*

$$\lim_{n\to\infty} \frac{1}{u_n Q_n} \sum_{k=0}^{n} q_k \sum_{m:|x_m - L| \geqq \epsilon} a_{km} = 0.$$

This relation is denoted by $[stat_A, q_n] - o\ (u_n) = x_n - L$.

Theorem 18. *Let $A = (a_{nk})$ be a weighted non-negative regular summability matrix. Assume that following condition yields:*

$$w(f, h_n) = [stat_A, q_n] - o\ (u_n) \text{ on } [0,1], \text{ where } h_n = \sqrt{\|S_{n,d}^{\lambda}((s-x)^2;x)\|_{C[0,1+d]}}.$$

Then for every bounded $f \in C[0, 1+d]$ we have

$$\|S_{n,d}^{\lambda}(f) - f\|_{C[0,1]} = [stat_A, q_n] - o\ (u_n).$$

Proof. Let $f \in C[0, 1+d]$, then we have

$$|S_{n,d}^{\lambda}(f;x) - f(x)| \leq |S_{n,d}^{\lambda}\big(|f(t) - f(x)|;x\big) + A\ |S_{n,d}^{\lambda}(1;x) - 1|$$

$$\leq \omega(f,\zeta) S_{n,d}^{\lambda}\bigg(\frac{|t-x|}{\zeta}+1;x\bigg)$$
$$= \omega(f,\zeta) S_{n,d}^{\lambda}(1;x) + \omega(f,\zeta)\frac{1}{\zeta^2} S_{n,d}^{\lambda}\left((t-x)^2;x\right)$$

for any $x, s \in [0,1]$, where $A = \sup_{x\in[0,1]} |f(x)|$. Let $\zeta := h_n$ for all $n \in \mathbb{N}$. Taking supremum over $x \in [0,\infty)$ on both sides, we obtain

$$\|S_{n,d}^{\lambda}(f) - f\|_{C[0,1]} \leq \omega(f,h_n) + \omega(f,h_n)\frac{1}{h_n^2}\|S_{n,d}^{\lambda}((t-x)^2;x)\|_{C[0,1+d]} = 2\omega(f,h_n).$$

We define the following sets for a given $\epsilon > 0$:

$$\mathcal{S} = \left\{n : \|S_{n,d}^{\lambda}(f) - f\|_{C[0,1]} \geq \epsilon\right\} \text{ and } \mathcal{E} = \left\{n : \omega(f,h_n) \geq \frac{\epsilon}{2}\right\}.$$

It is easy to see the following inequality holds:

$$\frac{1}{u_n Q_n}\sum_{k=0}^{n}\sum_{m\in\mathcal{S}} q_k a_{km} \leq \frac{1}{u_n Q_n}\sum_{k=0}^{n}\sum_{m\in\mathcal{E}} q_k a_{km}.$$

Hence we are led to the fact that

$$\|S_{n,d}^{\lambda}(f) - f\|_{C[0,1]} = [stat_A, q_n] - o\,(u_n)$$

by the hypothesis, as asserted by Theorem 18. □

4. Voronovskaja-type approximation theorems

Two Voronovskaja-type theorems are established in this part: A quantitative Voronovskaja-type theorem and a Voronovskaja-type approximation theorem by $\bar{S}_{n,d}^{\lambda}(f;x)$ family of linear operators using the notion of weighted A-statistical convergence.

Theorem 19. *Let (x_n) be a sequence of real numbers such that $S_{\mathbf{A}}^{\widetilde{N}} - \lim_{n\to\infty} x_n = 0$, where $A = (a_{nk})$ is a weighted non-negative regular summability matrix. Also let $\bar{S}_{n,d}^{\lambda}(f;x)$ be a sequence of positive linear operators acting from $C_B[0,1+d]$ into $C[0,1+d]$ defined by*

$$\bar{S}_{n,d}^{\lambda}(f;x) = (1+x_n) S_{n,d}^{\lambda}(f;x).$$

Then for every $f \in C_B[0,1+d]$ we have

$$S_{\mathbf{A}}^{\widetilde{N}} - \lim_{n\to\infty} n\left\{\bar{S}_{n,d}^{\lambda}(f;x) - f(x)\right\} = xd\ f'(x) + \frac{x(1-x)}{2}\ f''(x),$$

where $f', f'' \in C_B[0,1+d]$.

Proof. Let $x \in [0,1]$ and $f'' \in C_B[0,1+d]$. Applying $\bar{S}_{n,d}^{\lambda}(f;x)$ to both sides of Taylor's expansion theorem, we have

$$\bar{S}_{n,d}^{\lambda}(f;x) - f(x) \quad = \quad f^{'}(x)\bar{S}_{n,d}^{\lambda}(t-x;x) + \frac{f^{''}(x)}{2}\bar{S}_{n,d}^{\lambda}((t-x)^2;x)$$

$$+\bar{S}_{n,d}^{\lambda}((t-x)^2 r_x(t);x),$$

which yields to

$$n\big\{\bar{S}_{n,d}^{\lambda}(f;x)-f(x)\big\}=nf^{'}(x)(1+x_n)S_{n,d}^{\lambda}(t-x;x)$$
$$+\frac{n}{2}f^{''}(x)(1+x_n)S_{n,d}^{\lambda}((t-x)^2;x)+n(1+x_n)S_{n,d}^{\lambda}((t-x)^2 r_x(t);x).$$

We also have from Corollary 3

$$S_{n,d}^{\lambda}(t-x;x)\leq\frac{d}{n}x+\frac{1+2x+x^{n+d+1}+(1-x)^{n+d+1}}{n(n+d-1)}:=E(n,d,x),$$

and again from Corollary 3

$$S_{n,d}^{\lambda}((t-x)^2;x)\leq\frac{d^2}{n^2}x^2+\frac{n+d}{n^2}x(1-x)+\frac{2x^{n+d+2}+2x(1-x)^{n+d+1}}{n(n+d-1)}$$
$$+\frac{2dx+1+4dx^2+(2(n+d)+1)x^{n+d+1}+(1-x)^{n+d+1}}{2n(n+d-1)}\lambda:=F(n,d,x).$$

Hence we have

$$\Bigg|n\big\{\bar{S}_{n,d}^{\lambda}(f;x)-f(x)\big\}-f^{'}(x)dx-f^{'}(x)\frac{1-2x+x^{n+d+1}-(1-x)^{n+d+1}}{n+d-1}\lambda$$
$$-\ f^{''}(x)\left(\frac{d^2}{2n}x^2+\frac{n+d}{2n}x(1-x)-\frac{x^{n+d+2}-x(1-x)^{n+d+1}}{n+d-1}\lambda\right.$$
$$\left.+\frac{2dx-1-4dx^2+(2(n+d)+1)x^{n+d+1}+(1-x)^{n+d+1}}{2n(n+d-1)}\lambda\right)\Bigg|$$
$$=nf^{'}(x)x_nS_{n,d}^{\lambda}(t-x;x)+\frac{n}{2}f^{''}(x)x_nS_{n,d}^{\lambda}((t-x)^2;x)$$
$$+n(1+x_n)S_{n,d}^{\lambda}((t-x)^2r_x(t);x)$$
$$\leq x_n\{f^{'}(x)\bar{E}(n,d,x)+\frac{f^{''}(x)}{2}\bar{F}(n,d,x)\}+n(1+x_n)S_{n,d}^{\lambda}((t-x)^2r_x(t);x)$$
$$\leq x_n\{\sup_{x\in[0,1]}|f'(x)|\bar{E}(n,d,x)+\frac{1}{2}\sup_{x\in[0,1]}|f''(x)|\bar{F}(n,d,x)\}$$
$$+n(1+x_n)S_{n,d}^{\lambda}((t-x)^2r_x(t);x),$$

where $\bar{E}(n,d,x)=n\ E(n,d,x)$ and $\bar{F}(n,d,x)=n\ F(n,d,x)$. Since we have

$$S_{\mathbf{A}}^{\widetilde{N}}-\lim_{n\to\infty}n\ (S_{n,d}^{\lambda}((t-x)^2r_x(t);x))=0$$

and $S_{\mathbf{A}}^{\widetilde{N}}-\lim_{n\to\infty}x_n=0$, we get desired result. □

A quantitative Voronovskaja-type theorem for $S_{n,d}^{\lambda}(f;x)$ is established using Ditzian-Totik modulus of smoothness defined as

$$\omega_{\tau}(f,\zeta):=\sup_{0<|h|\leq\zeta}\left\{\left|f\left(x+\frac{h\tau(x)}{2}\right)-f\left(x-\frac{h\tau(x)}{2}\right)\right|,x\pm\frac{h\tau(x)}{2}\in[0,1]\right\},$$

where $\tau(x) = (x(1-x))^{1/2}$ and $f \in C[0,1+d]$, and corresponding Peetre's K-functional is defined by

$$K_\tau(f,\zeta) = \inf_{g\in W_\tau[0,1+d]} \big\{||f-g|| + \zeta||\tau g'^1[0,1+d], \zeta > 0\big\},$$

where $W_\tau[0,1+d] = \{g :\ g \in AC_{loc}[0,1+d],\ \|\tau g'\| < \infty\}$ and $AC_{loc}[0,1+d]$ is the class of absolutely continuous functions defined on $[a,b] \subset [0,1+d]$. There exists a constant $C > 0$ such that

$$K_\tau(f,\zeta) \le C\ \omega_\tau(f,\zeta).$$

Theorem 20. *Let $f, f', f'' \in C[0,1+d]$, then we have*

$$\left|S^\lambda_{n,d}(f;x) - f(x) - \beta_{n,\lambda}(x)f'(x) - \frac{\alpha_{n,\lambda}(x)+1}{2} f''(x)\right| \le \frac{C}{n}\tau^2(x)\omega_\tau\left(f'', \frac{1}{\sqrt{n}}\right)$$

for every $x \in [0,1]$ and sufficiently large n, where C is a positive constant, $\alpha_{n,\lambda}(x)$ and $\beta_{n,\lambda}(x)$ are defined in Theorem 9.

Proof. Consider following equality

$$f(t) - f(x) - (t-x)f'(x) = \int_x^t (t-u)f''(u)du$$

for $f \in C[0,1+d]$. It means we have

$$f(t) - f(x) - (t-x)f'(x) - \frac{f''(x)}{2}\big((t-x)^2+1\big) \le \int_x^t (t-u)[f''(u) - f''(x)]du. \tag{12}$$

Applying $S^\lambda_{n,d}(f;x)$ to both sides of (12), we obtain

$$\left|S^\lambda_{n,d}(f;x) - f(x) - S^\lambda_{n,d}((t-x);x)f'(x) - \frac{f''(x)}{2}\big(S^\lambda_{n,d}((t-x)^2;x) + S^\lambda_{n,d}(1;x)\big)\right|$$
$$\le S^\lambda_{n,d}\left(\Big|\int_x^t |t-u|\ |f''(u) - f''(x)|\ du\Big|; x\right). \tag{13}$$

The quantity in right hand side of (13) can be estimated as

$$\left|\int_x^t |t-u|\ |f''(u) - f''(x)|\ du\right| \le 2\|f''^2 + 2\|\tau g'^{-1}(x)|t-x|^3, \tag{14}$$

where $g \in W_\tau[0,1+d]$. There exists $C > 0$ such that

$$S^\lambda_{n,d}((t-x)^2;x) \le \frac{C}{2n}\tau^2(x) \quad \text{and} \quad S^\lambda_{n,d}((t-x)^4;x) \le \frac{C}{2n^2}\tau^4(x) \tag{15}$$

for sufficiently large n. Using Cauchy-Schwarz inequality, we have

$$\left|S^\lambda_{n,d}(f;x) - f(x) - S^\lambda_{n,d}((t-x);x)f'(x) - \frac{f''(x)}{2}\big(S^\lambda_{n,d}((t-x)^2;x) + S^\lambda_{n,d}(1;x)\big)\right|$$
$$\le 2\|f'' - g\|S^\lambda_{n,d}((t-x)^2;x) + 2\|\tau g'^{-1}(x)S^\lambda_{n,d}(|t-x|^3;x)$$

$$\leq \frac{C}{n}x(1-x)\|f''-g\|+2\|\tau g'^{-1}(x)\{S^{\lambda}_{n,d}((t-x)^2;x)\}^{1/2}\{S^{\lambda}_{n,d}((t-x)^4;x)\}^{1/2}$$
$$\leq \frac{C}{n}\tau^2(x)\Big\{\|f''^{-1/2}\|\tau g'\|\Big\}$$

by (13)–(15). Taking infimum on the right-hand side over all $g \in W_\tau[0, 1+d]$, we deduce

$$\left|S^{\lambda}_{n,d}(f;x)-f(x)-\beta_{n,\lambda}(x)f'(x)-\frac{\alpha_{n,\lambda}(x)+1}{2}f''(x)\right| \leq \frac{C}{n}\tau^2(x)\omega_\tau\left(f'', \frac{1}{\sqrt{n}}\right).$$

□

Finally we obtain the following theorem applying Taylor's expansion theorem and as an immediate consequence of Lemma (2), Corollary (3) and Corollary (4):

Theorem 21. *Let $f \in C_B[0, 1+d]$, then for each $x \in [0, 1]$*

$$\lim_{n\to\infty} n\big\{S^{\lambda}_{n,d}(f;x)-f(x)\big\} = xd\ f'(x)+\frac{x(1-x)}{2}\ f''(x)$$

uniformly on $[0, 1]$, where $f', f'' \in C_B[0, 1+d]$

As an immediate consequence of Theorem 20 we have the following result.

Corollary 22. *Let $f \in C[0, 1+d]$, then*

$$\lim_{n\to\infty} n\left[S^{\lambda}_{n,d}(f;x)-f(x)-\beta_{n,\lambda}(x)f'(x)-\frac{\alpha_{n,\lambda}(x)+1}{2}f''(x)\right]=0,$$

where $f', f'' \in C_B[0, 1+d]$, and $\alpha_{n,\lambda}(x)$ and $\beta_{n,\lambda}(x)$ are defined in Theorem 9.

5. Concluding Remarks

A Korovkin type approximation theorem via K_a-convergence on weighted spaces is studied by Yıldız et al. in [31] and a new concept, statistical e-convergence, is introduced by Sever and Talo in [18, 24, 32]. As a future work we may study the approximation properties of operators defined in this article and other Bernstein type operators using those convergence types. The results of the paper will also be extended to λ-Schurer-Kantorovich and λ-Schurer-Stancu operators using λ-Bézier-Schurer bases defined in (1).

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