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Research Article

Automorphisms of a certain subalgebra of the upper triangular matrix algebra

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Abstract

For a commutative ring R with unity, the R-algebra of strictly upper triangular $n \times n$ matrices over R is denoted by $N_n(R)$, where n is a positive integer greater than 1. For the identity matrix I, $\alpha \in R$, $A \in N_n(R)$, the set of all elements $\alpha I + A$ is defined as the scalar upper triangular matrix algebra $ST_n(R)$ which is a subalgebra of the upper triangular matrices $T_n(R)$. In this paper, we investigate the R-algebra automorphisms of $ST_n(R)$. We extend the automorphisms of $N_n(R)$ to $ST_n(R)$ and classify all the automorphisms of $ST_n(R)$. We generalize the results of Cao and Wang and prove that not all automorphisms of $ST_n(R)$ can be extended to the automorphisms of $T_n(R)$.

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1. Introduction

Kezlan [3] proved that: "If R is a commutative ring with unity, then every R-algebra automorphism of the algebra of upper triangular $n \times n$ matrices over R is inner ". Coelho [2] extended Kezlan's result for any ring R. Cao and Wang [1] classified all the automorphisms of strictly upper triangular matrix algebras over a commutative ring R. In this work, we would like to generalize these results and determine how such central automorphisms exist on $ST_n(R)$ but not on $T_n(R)$.

Let R be a commutative ring with unity and $N_n(R)$ be the algebra of all $n \times n$ strictly upper triangular matrices over R, where n is a positive integer greater than 1.

Define $ST_n(R)$ with unity matrix I,

$$ST_n(R) = \{\alpha I + A : \alpha \in R,\ A \in N_n(R)\}.$$

If $X \in ST_n(R)$, we may write $X = \alpha I + A = \alpha I + \sum_{i<j} a_{ij} E_{ij}$ where $\alpha \in R$ and $A \in N_n(R)$. $ST_n(R)$ is a subalgebra of $T_n(R)$ and let us call the *scalar upper triangular matrix algebra*. The standard matrix units are the matrices E_{ij} with a 1 at (i,j) position and zero elsewhere. Clearly, the set of matrices $\{E_{ij} : 1 \le i < j \le n\}$ form a basis of

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$N_n(R)$. For $n \geq 4$, consider $c = (c_2, \ldots, c_{n-2}) \in R^{n-3}$. Y.Cao and J. Wang [1] defined the *central automorphisms* of $N_n(R)$, $\mu_c : N_n(R) \to N_n(R)$ such that for any $A \in N_n(R)$

$$\mu_c(A) = A + \left(\sum_{k=2}^{n-2} a_{k,k+1} c_k \right) E_{1n}.$$

With respect to the basis of $N_n(R)$, automorphisms can be written as:

$$\mu_c(E_{i,i+1}) = E_{i,i+1} + c_i E_{1n}$$

for $2 \leq i \leq n-2$ and

$$\mu_c(E_{ij}) = E_{ij}$$

for $j \neq i+1$. For any matrix $X \in ST_n(R)$, let $S_1, S_2, ..., S_{n-1}$ define the diagonals of $X = \alpha I + \sum_{i<j} a_{ij} E_{ij}$ as follows:

$$\begin{aligned} S_1 &= \{a_{12}, a_{23}, \ldots, a_{n-1,n}\}, \\ S_2 &= \{a_{13}, a_{24}, \ldots, a_{n-2,n}\}, \\ &\vdots \\ S_{n-1} &= \{a_{1n}\}. \end{aligned}$$

2. Automorphisms of $ST_n(R)$

There are no invertible elements in $N_n(R)$, hence one cannot define an inner automorphism by the usual terminology. For a fixed matrix $X \in T_n(R)$ with 1 on the main diagonal, the map $\Theta(Y) = XYX^{-1}$ is an inner automorphism [1]. The restriction of Θ on $N_n(R)$ is also an automorphism of $N_n(R)$.

Moreover, we define the *inner automorphism* θ on $ST_n(R)$, with a given invertible matrix $B \in T_n(R)$ and for any $A \in ST_n(R)$ as

$$\theta(A) = BAB^{-1}.$$

In this section, we are going to classify the automorphisms of $ST_n(R)$. We first investigate the central automorphisms of $ST_n(R)$ in the classification. It is not difficult to see that, for $n \geq 4$,

$$\mu_c : ST_n(R) \to ST_n(R)$$

is an automorphism of $ST_n(R)$. To avoid the details of a tedious proof, we use Kezlan's [3] notation. Let

$$\theta : T_n(R) \longrightarrow T_n(R)$$

be an R-algebra automorphism of $T_n(R)$. Define

$$\begin{aligned} \theta(E_{kk}) &= \left[e_{ij}^{(k)} \right] \text{ for } k = 1, 2, \ldots, n \\ \theta(E_{k,k+1}) &= \left[a_{ij}^{(k)} \right] \text{ for } k = 1, 2, \ldots, n-1. \end{aligned}$$

The results in [3] show that $a_{k,k+1}^{(k)}$ is a unit with $a_{k,k}^{(k)} = 0$, $e_{k,k}^{(k)} = 1$ and $e_{k,k}^{(q)} = 0$ for $q \neq k$.

Remark 2.1. In terms of the image of matrix units under θ, it is shown that, k^{th} diagonal entry of $\theta(E_{kk})$ is 1, while the other diagonal entries are 0. That is $\theta(E_{kk})$ is in the form $\theta(E_{kk}) = E_{kk} + J_k$ where J_k is strictly upper triangular. For $k = 1, \cdots, n$ we use the fact that $\theta(E_{kk})$ is the idempotent, that is $\theta(E_{kk}) = \theta(E_{kk})\theta(E_{kk})$. Thus as a consequence of [3], $\theta(E_{kk})$ has the following form:

$$\theta(E_{kk}) = \begin{bmatrix} 0 & \cdots & 0 & e^{(k)}_{1,k} & * & . & * \\ \vdots & \cdots & \vdots & \vdots & . & . & . \\ 0 & \cdots & 0 & e^{(k)}_{k-1,k} & * & . & * \\ 0 & \cdots & 0 & 1 & e^{(k)}_{k,k+1} & . & e^{(k)}_{k,n} \\ 0 & \cdots & 0 & 0 & 0 & . & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & . & 0 & 0 & 0 & . & 0 \end{bmatrix}. \tag{2.1}$$

The following proposition gives us the property of $\theta(E_{kk}) = \left[e^{(k)}_{ij}\right]$.

Proposition 2.2. *Assume that* $\theta(E_{kk}) = \left[e^{(k)}_{ij}\right]$ *for* $k = 1, \ldots, n$. *If* $i < j$ *then for* $i = 1, \ldots, n-1$

$$\sum_{k=i}^{j} e^{(k)}_{ij} = 0.$$

Proof. It was shown in [3] that

$$e^{(p)}_{kk} = \begin{cases} 1 \text{ if } p = k \\ 0 \text{ if } p \neq k. \end{cases}$$

Consider $\left(e^{(1)}_{ij}, e^{(2)}_{ij}, \ldots, e^{(r)}_{ij}\right)$ as an r-*tuple* and define r-*tuple sum* as $e^{(1)}_{ij} + e^{(2)}_{ij} + \ldots + e^{(r)}_{ij}$. To use induction on r, we start with the case $r = 2$. Using the multiplication of matrix units E_{pp} and $E_{p+1,p+1}$ we get

$$E_{pp}E_{p+1,p+1} = 0.$$

Since $\theta(0) = 0$, in our notation this result can be viewed as:

$$\left[e^{(p)}_{ij}\right]\left[e^{(p+1)}_{ij}\right] = 0.$$

For $p = 1, ..., n-1$,

$$\left(e^{(p)}_{p,p+1} + e^{(p+1)}_{p,p+1}\right) E_{p,p+1} = 0.$$

Then we obtain $e^{(p)}_{p,p+1} + e^{(p+1)}_{p,p+1} = 0$ for all 2-*tuples*. Assume our claim is true for the sum of $(r-1)$-*tuples*. That is

$$\sum_{i=p}^{p+r-2} e^{(i)}_{p,p+r-2} = 0 \text{ for } p = 1, \ldots, (n-r)+2.$$

Use induction on p. For $p = 1$,

$$e^{(1)}_{1r} + e^{(2)}_{1r} + \ldots + e^{(r-1)}_{1r} + e^{(r)}_{1r} = 0.$$

Now consider each case for $t = 1, \ldots, (r-1)$. We know that $E_{tt}E_{rr} = 0$. Apply the automorphism θ to $E_{tt}E_{rr}$, we get

$$\theta(E_{tt}E_{rr}) = 0.$$

For $t = 1, \ldots, (r-1)$ the following terms can be obtained. If $t = 1$,

$$\left(e^{(1)}_{1r} + e^{(r)}_{1r} + \sum_{i=2}^{r-1} e^{(1)}_{1i}e^{(r)}_{ir}\right) E_{1r} = 0.$$

For $1 < t \leq r-1$,

$$\left(e_{1r}^{(t)} + \sum_{i=2}^{r-1} e_{1i}^{(t)} e_{ir}^{(r)}\right) E_{1r} = 0.$$

By using the sum

$$\theta(E_{11}E_{rr} + E_{22}E_{rr} + \ldots + E_{(r-1),(r-1)}E_{rr}) = 0,$$

we get

$$e_{1r}^{(1)} + e_{1r}^{(2)} + \ldots + e_{1r}^{(r-1)} + e_{1r}^{(r)} + \sum_{j=2}^{r-1}\left(\sum_{i=1}^{r-1} e_{1j}^{(i)} e_{jr}^{(r)}\right) = 0.$$

By the assumption we have, for $j = 2, \ldots, r-1$, that

$$\sum_{j=2}^{r-1}\left(\sum_{i=1}^{r-1} e_{1j}^{(i)} e_{jr}^{(r)}\right) = 0.$$

Also the calculations on $\theta(E_{ii})$, in (2.1), can be used to deduce the result,

$$e_{1r}^{(1)} + e_{1r}^{(2)} + \ldots + e_{1r}^{(r-1)} + e_{1r}^{(r)} = 0.$$

Assume that our assumption is true for $p = q$, that is

$$e_{q,q+r-1}^{(q)} + e_{q,q+r-1}^{(q+1)} + \ldots + e_{q,q+r-1}^{(q+r-2)} + e_{q,q+r-1}^{(q+r-1)} = 0.$$

Now consider the case $p = q+1$. We want to prove that

$$e_{q+1,q+r}^{(q+1)} + e_{q+1,q+r}^{(q+2)} + \ldots + e_{q+1,q+r}^{(q+r-1)} + e_{q+1,q+r}^{(q+r)} = 0.$$

Apply the automorphism θ to $E_{tt}E_{q+1,q+1}$ for $t = 1, \ldots, q$ we get

$$\theta(E_{tt}E_{q+1,q+1}) = 0.$$

For $t = q+1$,

$$\left(e_{q+1,q+r}^{(q+1)} + e_{q+1,q+r}^{(q+r)} + \sum_{i=q+2}^{q+r-1} e_{q+1,i}^{(q+1)} e_{i,q+r}^{(q+r)}\right) E_{q+1,q+r} = 0.$$

For $q+1 < t < q+r$,

$$\left(e_{q+1,q+r}^{(t)} + \sum_{i=t}^{q+r-1} e_{q+1,i}^{(t)} e_{i,q+r}^{(q+r)}\right) E_{q+1,q+r} = 0.$$

Consider the following sum:

$$\theta(E_{q+1,q+1}E_{q+r,q+r} + E_{q+2,q+2}E_{q+r,q+r} + \ldots + E_{q+r-1,q+r-1}E_{q+r,q+r}) = 0,$$

then we have

$$e_{q+1,q+r}^{(q+1)} + e_{q+1,q+r}^{(q+2)} + \ldots + e_{q+1,q+r}^{(q+r-1)} + e_{q+1,q+r}^{(q+r)} + \sum_{j=q+2}^{q+r-1}\left(\sum_{i=q+1}^{q+r-1} e_{q+1,j}^{(i)} e_{j,q+r}^{(q+r)}\right) = 0.$$

By the assumption we have

$$\sum_{j=q+2}^{q+r-1}\left(\sum_{i=q+1}^{q+r-1} e_{q+1,j}^{(i)} e_{j,q+r}^{(q+r)}\right) = 0$$

and this implies

$$e_{q,q+r-1}^{(q)} + e_{q,q+r-1}^{(q+1)} + \ldots + e_{q,q+r-1}^{(q+r-2)} + e_{q,q+r-1}^{(q+r-1)} = 0.$$

Thus the proof is completed. □

Lemma 2.3. *Let R be a commutative ring with identity and θ be an R-algebra automorphism of $ST_n(R)$. If $n = 2, 3$ then θ is an inner automorphism η_D for some $D \in T_n(R)$.*

Proof. For the automorphism $\theta : ST_n(R) \to ST_n(R)$ as given in Remark 2.1, let

$$\begin{aligned} \theta(E_{kk}) &= \left[e_{ij}^{(k)}\right] \text{ for } k = 1, 2, \ldots, n \\ \theta(E_{k,k+1}) &= \left[a_{ij}^{(k)}\right] \text{ for } k = 1, 2, \ldots, n-1. \end{aligned}$$

Recall that all the diagonal entries $a_{ii}^{(k)}$ of $\theta(E_{k,k+1})$ are zero and

$$e_{qq}^{(p)} = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{otherwise.} \end{cases}$$

Also $a_{12}^{(1)}, a_{23}^{(2)}, \ldots, a_{n-1,n}^{(n-1)}$ are all units.
For $n = 2$, consider $A \in ST_n(R)$ where

$$A = \begin{bmatrix} \alpha & b \\ 0 & \alpha \end{bmatrix}$$

for $\alpha, b \in R$. Every matrix $A \in ST_2(R)$ can be written as linear combination of matrix units. That is,

$$A = \alpha E_{11} + bE_{12} + \alpha E_{22}.$$

Applying θ to A, we get

$$\theta(A) = \alpha\theta(E_{11}) + b\theta(E_{12}) + \alpha\theta(E_{22}).$$

By using the notation of Remark 2.1,

$$\begin{aligned} \theta(\alpha E_{11}) &= \begin{pmatrix} \alpha & \alpha e_{12}^{(1)} \\ 0 & 0 \end{pmatrix} \\ \theta(bE_{12}) &= \begin{pmatrix} 0 & ba_{12}^{(1)} \\ 0 & 0 \end{pmatrix} \\ \theta(\alpha E_{22}) &= \begin{pmatrix} 0 & \alpha e_{12}^{(1)} \\ 0 & \alpha \end{pmatrix}. \end{aligned}$$

As a result,

$$\theta(A) = \begin{bmatrix} \alpha & \alpha e_{12}^{(1)} + ba_{12}^{(1)} + \alpha e_{12}^{(2)} \\ 0 & \alpha \end{bmatrix}$$

for some invertible element $a_{12}^{(1)} \in R$. By Proposition 2.2, we have that $e_{12}^{(1)} + e_{12}^{(2)} = 0$ and

$$\theta(A) = \begin{bmatrix} \alpha & ba_{12}^{(1)} \\ 0 & \alpha \end{bmatrix}$$

for some invertible element $a_{12}^{(1)} \in R$. Choosing $D = \begin{bmatrix} 1 & 0 \\ 0 & \left(a_{12}^{(1)}\right)^{-1} \end{bmatrix} \in T_2(R)$, we can obtain that

$$\theta(A) = DAD^{-1}$$

which means that all the automorphisms of $ST_2(R)$ are inner.

For the case $n = 3$, we proceed with the steps similar to the case $n = 2$ in order to determine the invertible matrix B.

Let $A = \begin{bmatrix} \alpha & b & c \\ 0 & \alpha & d \\ 0 & 0 & \alpha \end{bmatrix} \in ST_n(R)$. By using matrix units, we get

$$\theta(E_{11}) = \begin{pmatrix} 1 & e_{12}^{(1)} & e_{13}^{(1)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\theta(E_{22}) = \begin{pmatrix} 0 & e_{12}^{(2)} & e_{13}^{(2)} \\ 0 & 1 & e_{23}^{(2)} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\theta(E_{33}) = \begin{pmatrix} 0 & 0 & e_{13}^{(3)} \\ 0 & 0 & e_{23}^{(3)} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\theta(E_{12}) = \begin{pmatrix} 0 & a_{12}^{(1)} & a_{13}^{(1)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\theta(E_{23}) = \begin{pmatrix} 0 & 0 & a_{13}^{(2)} \\ 0 & 0 & a_{23}^{(2)} \\ 0 & 0 & 0 \end{pmatrix}$$

and we obtain

$$\theta(A) = \begin{bmatrix} \alpha & ba_{12}^{(1)} & ca_{12}^{(1)}a_{23}^{(1)} + ba_{13}^{(1)} + da_{13}^{(2)} + \alpha\left(e_{13}^{(1)} + e_{13}^{(2)} + e_{13}^{(3)}\right) \\ 0 & \alpha & da_{23}^{(2)} \\ 0 & 0 & \alpha \end{bmatrix}.$$

Similarly, it can be deduced from Proposition 2.2,

$$e_{13}^{(1)} + e_{13}^{(2)} + e_{13}^{(3)} = 0 \Rightarrow \alpha\left(e_{13}^{(1)} + e_{13}^{(2)} + e_{13}^{(3)}\right) = 0.$$

Hence,

$$\theta(A) = \begin{bmatrix} \alpha & ba_{12}^{(1)} & ca_{12}^{(1)}a_{23}^{(1)} + ba_{13}^{(1)} + da_{13}^{(2)} \\ 0 & \alpha & da_{23}^{(2)} \\ 0 & 0 & \alpha \end{bmatrix}.$$

It is now easy to define the invertible matrix $B \in T_3(R)$ as,

$$B = \begin{pmatrix} 1 & a_{13}^{(1)}\left(a_{12}^{(1)}a_{23}^{(2)}\right)^{-1} & 0 \\ 0 & \left(a_{12}^{(1)}\right)^{-1} & -a_{13}^{(1)}\left(a_{12}^{(1)}\right)^{2}\left(a_{23}^{(2)}\right)^{-1} \\ 0 & 0 & \left(a_{12}^{(1)}\right)^{-1}\left(a_{23}^{(2)}\right)^{-1} \end{pmatrix}$$

so that

$$\theta(A) = \eta_B = BAB^{-1}.$$

□

Now, we can state the main theorem of this paper.

Theorem 2.4. *Let θ be an R-algebra automorphism of $ST_n(R)$ and R be a commutative ring with identity. For $n \geq 4$,*

$$\theta = \eta_D \mu_c \lambda_P$$

where η_D, λ_P are inner automorphisms and μ_c is a central automorphism of $ST_n(R)$.

Proof. First, apply θ to each E_{ii} for $i = 1, \ldots, (n-1)$. The S_2 diagonal only contains the elements $e^{(p)}_{i,i+1}$. Thus we obtain

$$\sum_{p=i}^{i+1} e^{(p)}_{i,i+1} E_{i,i+1} \text{ for } p = 1, \ldots, n-1.$$

By Proposition 2.2, we can get

$$\sum_{p=i}^{i+1} e^{(p)}_{i,i+1} = 0.$$

A consequence of above sum allows us just to see the image of $E_{i,i+1}$ under θ. We examine

$$\theta(E_{i,i+1}) = a^{(i)}_{i,i+1} E_{i,i+1} \text{ for } i = 1, \ldots, n-1,$$

We want to show that $a^{(k)}_{p,p+1} = 0$ if $k \neq p$. On the contrary, assume that $a^{(k)}_{p,p+1} \neq 0$ and take $k = 1$ to get a contradiction. Apply θ to the equality $E_{p,p+1}E_{23} = 0$, we obtain

$$\begin{aligned} \theta(E_{p,p+1}E_{23}) &= \theta(E_{p,p+1})\theta(E_{23}) \\ \left[a^{(p)}_{ij}\right]\left[a^{(2)}_{ij}\right] &= 0. \end{aligned}$$

Consider the term $a^{(p)}_{12} a^{(2)}_{23} E_{13}$ on the left side of the above equality. Since $a^{(p)}_{12} a^{(2)}_{23} = 0$ and $a^{(2)}_{23}$ is a unit by [3], then $a^{(p)}_{12} = 0$. If $k > 1$, apply θ to the equality $E_{k,k+1}E_{p+1,p+2}$ to get

$$\left[a^{(k)}_{ij}\right]\left[a^{(p+1)}_{ij}\right] = 0.$$

Consider the term $a^{(k)}_{p,p+1} a^{(p+1)}_{p+1,p+2} E_{p,p+2} = 0$ on the left side of the above equality. Since $a^{(k)}_{p,p+1} a^{(p+1)}_{p+1,p+2} = 0$ and $a^{(p+1)}_{p+1,p+2}$ is a unit then $a^{(k)}_{p,p+1} = 0$ but, this is a contradiction. Now defining the diagonal matrix $D \in T_n(R)$ with diagonal entries from the set $\left\{1, (a^{(1)}_{12})^{-1}, (a^{(1)}_{12}.a^{(2)}_{23})^{-1}, \ldots, (a^{(1)}_{12}.a^{(2)}_{23} \ldots a_{n-1,n})^{-1}\right\}$ we have the following result on the diagonal S_2,

$$\eta_D^{-1}\theta(E_{i,i+1}) = E_{i,i+1} \text{ for } i = 1, \ldots, n-1.$$

We are going to use induction on t to prove that there exist inner automorphisms η_{P_t} with $P_t \in T_n(R)$ and

$$\eta_{P_t}^{-1}\eta_D^{-1}\theta(E_{i,i+1}) = E_{i,i+1} \text{ for } i = 1, \ldots, n-1 \text{ and } t = 2, \ldots, n-1.$$

Assume that we have

$$\eta_{P_t}^{-1}\eta_D^{-1}\theta(E_{i,i+1}) = E_{i,i+1}$$

for $i = 1, \ldots, n-1$, on the diagonals $S_1, \ldots, S_t$. What about on the diagonal S_{t+1}? Apply θ to E_{ii}, we can obtain the sum

$$\sum_{p=i}^{t} e^{(p)}_{i,i+t} E_{i,i+t} \text{ for } i = 1, ..., n-t+1.$$

By Proposition 2.2, we have

$$\sum_{p=i}^{t} e^{(p)}_{i,i+t} E_{i,i+t} = 0.$$

Then we just consider the rest, that is:

$$\eta_{P_t}^{-1}\eta_D^{-1}\theta(E_{i,i+1}) = E_{i,i+1} + \sum_{j=1}^{n-t} b_{j,i}^{(i)} E_{j,j+t}.$$

We want to show $b_{j,i}^{(i)} = 0$ if $j \neq i$ and $j \neq i-t+1$. Assume that $b_{p,q}^{(q)} \neq 0$ with $p \neq q$ with $p \neq q+1-t$. If $p < n-t$ then we apply $\eta_{P_t}^{-1}\eta_D^{-1}\theta$ to $E_{q,q+1}E_{p+t,p+t+1}$, we get $\eta_{P_t}^{-1}\eta_D^{-1}\theta(E_{q,,q+1}E_{p+t,p+t+1}) = 0$. Thus, we obtain the following results:

$$\left[E_{q,q+1} + \sum_{j=1}^{n-t} b_{j,q}^{(q)} E_{j,j+t}\right]\left[E_{p+t,p+t+1} + \sum_{j=1}^{n-t} b_{j,p+t}^{(p+t)} E_{j,j+t}\right] = 0$$

and

$$b_{q+1,p+t}^{(p+t)} E_{q,p+t+1} + b_{p,q}^{(q)} E_{p,p+t+1} = 0.$$

Hence $b_{pq}^{(q)} = 0$, we get a contradiction. If $p = n-t$ then apply $\eta_{P_t}^{-1}\eta_D^{-1}\theta$ to $E_{p-1,p}E_{q,q+1}$ we get

$$\eta_X^{-1}\eta_D^{-1}\theta(E_{p-1,p}E_{q,q+1}) = 0$$

and

$$b_{pq}^{(q)} E_{p-1,p+t} + b_{q-t,p-1}^{(p-1)} E_{q-t,q+1} = 0.$$

Since $b_{pq}^{(q)} = 0$, we get a contradiction again. Hence, on the S_{t+1} diagonal we have

$$\eta_{P_t}^{-1}\eta_D^{-1}\theta(E_{i,i+1}) = E_{i,i+1} + b_{i-t+1,i}^{(i)} E_{i-t+1,i+1} + b_{i,i}^{(i)} E_{i,i+t}.$$

However, we want to get $\eta_{P_t}^{-1}\eta_D^{-1}\theta(E_{i,i+1}) = E_{i,i+1}$.
To prove this, we use induction again. Assume that there exists an inner automorphism $\lambda_{G_{k-1}}$such that on the S_{t+1} diagonal:

$$\lambda_{G_{k-1}}^{-1}\eta_{P_t}^{-1}\eta_D^{-1}\theta(E_{i,i+1}) = E_{i,i+1} \text{ for } i = 1,\dots,k-1,$$

and

$$\lambda_{G_{k-1}}^{-1}\eta_{P_t}^{-1}\eta_D^{-1}\theta(E_{i,i+1}) = E_{i,i+1} + d_{i-t+1,i}^{(i)} E_{i-t+1,i+1} + d_{i,i}^{(i)} E_{i,i+t} \text{ for } i = k,\dots,n-1.$$

Setting $Z = I + d_{k-t+1,k}^{(k)} E_{k-t+1,k} - d_{k,k}^{(k)} E_{k,k+t}$ and $G_k = G_{k-1}Z$ then for $1 \leq i < k$ we have

$$\lambda_{G_k}^{-1}\eta_{P_t}^{-1}\eta_D^{-1}\theta(E_{i,i+1}) = E_{i,i+1} + \delta_{i,k-t} d_{k+1-t,k}^{(k)} E_{i,k}$$

on the S_{t+1} diagonal. For $i = k-t$, applying $\lambda_{G_{k-1}}^{-1}\eta_{P_t}^{-1}\eta_D^{-1}\theta$ to the equality $E_{k-t,k-t+1}E_{k,k+1} = 0$, we obtain

$$d_{k+1-t,k}^{(k)} E_{k-t,k+1} = 0$$

on the diagonal S_{t+1}. Hence

$$d_{k+1-t,k}^{(k)} = 0.$$

We get

$$\lambda_{G_k}^{-1}\eta_{P_t}^{-1}\eta_D^{-1}\theta(E_{k,k+1}) = E_{k,k+1}$$

for $i = 1,\cdots,k$. For $i > k$, as stated on [1], we have

$$\lambda_{G_k}^{-1}\eta_{P_t}^{-1}\eta_D^{-1}\theta(E_{i,i+1}) = E_{k,k+1} + (d_{i-t+1,i+1}^{(i)} + \delta_{k+t,i} d_{kk}^{(k)}) E_{i+1-t,i+1} + d_{ii}^{(i)} E_{i,i+t}.$$

Thus, there exists an inner automorphism $\lambda_{P_{n-1}}$ of $ST_n(R)$ such that

$$\lambda_{P_{n-1}}^{-1}\eta_D^{-1}\theta(E_{i,i+1}) = E_{i,i+1}$$

on the S_{n-1} diagonal. We want to know what happens on the S_n diagonal. Assume that, we have

$$\lambda_{P_{n-1}}^{-1}\eta_D^{-1}\theta(E_{i,i+1}) = E_{i,i+1} + c_i E_{1n} \text{ for } i = 1,\dots,n-1$$

on the S_n diagonal. If we apply $\lambda_{P_{n-1}}^{-1}\eta_D^{-1}\theta$ to E_{ii}, we get

$$(e_{1n}^{(1)} + e_{1n}^{(2)} + \ldots + e_{1n}^{(n-1)})E_{1n}.$$

By Proposition 2.2, $\left(e_{1n}^{(1)} + e_{1n}^{(2)} + \ldots + e_{1n}^{(n-1)}\right) = 0$ and if we take

$$Z = I + c_{n-1}E_{1,n-1} - c_1E_{2n}$$

as in [1], we get

$$\lambda_Z^{-1}\lambda_{p_{n-1}}^{-1}\eta_D^{-1}\theta(E_{i,i+1}) = E_{i,i+1} \text{ for } i = 1, n-1$$

and

$$\lambda_Z^{-1}\lambda_{P_{n-1}}^{-1}\eta_D^{-1}\theta(E_{i,i+1}) = E_{i,i+1} + c_iE_{1n} \text{ for } i = 2, \ldots, n-2.$$

Setting $c = (c_2, \ldots, c_{n-2})$, we get

$$\mu_c^{-1}\lambda_Z^{-1}\lambda_{p_{n-1}}^{-1}\eta_D^{-1}\theta(E_{i,i+1}) = E_{i,i+1} \text{ for } i = 1, \ldots, n.$$

So we have

$$\mu_c^{-1}\lambda_Z^{-1}\lambda_{p_{n-1}}^{-1}\eta_D^{-1}\theta = 1.$$

Thus

$$\theta = \eta_D\mu_c\lambda_P,$$

for some $P \in T_n(R)$. □

Remark 2.5. Let $N_n(R)$ be the strictly upper triangular matrix algebra over R, then

$$Aut\left(N_n\left(R\right)\right) \simeq Aut\left(ST_n\left(R\right)\right).$$

Example 2.6. Let $A \in ST_4(R)$ be defined as $\begin{bmatrix} a & b & c & d \\ 0 & a & e & f \\ 0 & 0 & a & r \\ 0 & 0 & 0 & a \end{bmatrix}$.

We can check that $\theta : ST_4(R) \to ST_4(R)$ defines an automorphism via

$$\theta\left(\begin{bmatrix} a & b & c & d \\ 0 & a & e & f \\ 0 & 0 & a & r \\ 0 & 0 & 0 & a \end{bmatrix}\right) = \begin{bmatrix} a & b & c & d-3e \\ 0 & a & e & f \\ 0 & 0 & a & r \\ 0 & 0 & 0 & a \end{bmatrix}.$$

But θ is not an inner automorphism. Notice that $Aut\left(ST_n\left(R\right)\right) \nsubseteq Aut\left(T_n\left(R\right)\right)$. Then all the automorphisms of $ST_n\left(R\right)$ cannot be extended to the automorphisms of $T_n(R)$.

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