Research Article

Existence Results for Systems of Quasi-Variational Relations

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ABSTRACT. The existence of solutions for a system of variational relations, in a general form, is studied using a fixed point result for contractions in metric spaces. As a particular case, sufficient conditions for the existence of solutions of a system of quasi-equilibrium problems are given.

Keywords: Variational relations problems, system of equilibrium problems, fixed points.

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1. INTRODUCTION AND PRELIMINARIES

For each $i \in I = \{1, \ldots, n\}$, let $X_i$ be a nonempty subset of a complete metric space $(E_i, d_i)$ and $X = \prod_{i \in I} X_i$ a subset of the product space $E = \prod_{i \in I} E_i$. Let $S_i, Q_i : X \to 2^{X_i}$ be two set-valued maps with nonempty values. Let $R_i(x, y_i)$ be a relation between $x \in X$ and $y_i \in X_i$.

The general system of quasi-variational relations that we consider in this paper is:

\begin{equation}
\text{(SQVR)} \quad \text{Find } \bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in X \text{ such that for each } i \in I,
\bar{x}_i \in S_i(\bar{x}) \text{ and } R_i(\bar{x}, y_i) \text{ holds for all } y_i \in Q_i(\bar{x}).
\end{equation}

Variational relations problems were considered for the first time by D.T. Luc in [11], as a general model that encompasses optimization problems, equilibrium problems or variational inclusion problems. Several authors continued the study of variational relations problems, see for instance the papers [10], [12], [9], [2], [1] and the references therein. Existence results for the solutions of variational relations problems are obtained mostly in two ways: by applying intersection results for set valued mappings (see [11]) or by using various fixed points theorems (see [11], [7], [4]).

The system (SQVR) was introduced by L.J. Lin and Q.H. Ansari in [8], where the existence of a solution was established using a maximal element theorem for a family of set-valued maps. The same system was studied in [5] by a factorization method, that followed the ideas from [6].

In this paper, we will give sufficient conditions for the existence of solutions of the system (SQVR), using a fixed-point theorem for set-valued mappings that are Reich-type contractions. The general result obtained for the system of variational relations will be applied in the last section to a system of equilibrium problems.

In the rest of this section, we present some notations and results needed in the paper.

The metric on the product space will be defined by $d : E \times E \to \mathbb{R}_+$,

$$d(x, y) = d_1(x_1, y_1) + \cdots + d_n(x_n, y_n).$$

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for \( x = (x_1, \ldots, x_n) \in E \) and \( y = (y_1, \ldots, y_n) \in E \).

For any nonempty sets \( A, B \subset E \) and \( x \in E \), denote by

\[
D(x, B) = \inf_{b \in B} d(x, b) \quad \text{and} \quad H(A, B) = \max \{\sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b)\}.
\]

\( H(A, B) \) is the generalized Hausdorff functional of \( A \) and \( B \). Similarly, we will denote by \( H_i(A_i, B_i) \) the Hausdorff distance induced by \( d_i \), for \( A_i \) and \( B_i \) subsets of \( E_i \).

**Lemma 1.1.** For \( x = (x_1, \ldots, x_n) \), \( A = A_1 \times \cdots \times A_n \) and \( B = B_1 \times \cdots \times B_n \), we have

\[
D(x, B) = D_1(x_1, B_1) + \cdots + D_n(x_n, B_n), \quad H_i(A, B) \leq H_1(A_1, B_1) + \cdots + H_n(A_n, B_n).
\]

**Lemma 1.2.**

(a) If \( A, B \subset E \) are such that for each \( a \in A \) there exists \( b \in B \) such that \( d(a, b) \leq c \) and for each \( b \in B \) there exists \( a \in A \) such that \( d(a, b) \leq c \), then \( H(A, B) \leq c \).

(b) If \( A, B \subset E \) and \( \varepsilon > 0 \), then for each \( a \in A \) there exists \( b \in B \) such that \( d(a, b) \leq H(A, B) + \varepsilon \).

There is a vast literature on the existence of fixed points of generalized contractions, both single-valued and set-valued (see for instance [3], [14]). We will use the following:

A set-valued mapping \( F : E \to 2^E \) is said to be a Reich-type contraction if there exist \( a, b, c \geq 0 \), with \( a + b + c < 1 \) such that \( H(F(x), F(y)) \leq ad(x, y) + bD(x, F(x)) + cD(y, F(y)) \), for each \( x, y \in E \).

**Theorem 2.1** ([13]). Let \((E, d)\) be a complete metric space and let \( F : E \to 2^E \) be a Reich-type contraction. Suppose also that \( F(x) \) is a closed set, for every \( x \in E \). Then, \( F \) has at least a fixed point.

2. An existence result for a system of variational relations

We give in what follows sufficient conditions for the existence of solutions of the system \((SQR)\) formulated in the previous section.

For \( x = (x_1, \ldots, x_n) \in X \) and \( i \in I \), fixed, we denote

\[
\Gamma_i(x) = \{z_i \in S_i(x) \mid R_i(x_1, \ldots, z_i, \ldots, x_n; t_i) \text{ holds for all } t_i \in Q_i(x)\}
\]

and we define the function \( \Gamma : X \to 2^X \) by \( \Gamma(x) = \Gamma_1(x) \times \cdots \times \Gamma_n(x) \). It is easy to see that any fixed point of \( \Gamma \) is a solution of \((SQR)\).

**Theorem 2.2.** Suppose that for any \( i \in I \), the set \( X_i \) is nonempty, closed and:

(i) for any \( x \in X \), \( \Gamma_i(x) \) is nonempty;

(ii) there exists \( q_i \in ]0, 1[ \) such that, for every \( x^1, x^2 \in X_i \), if \( \epsilon_i \in \Gamma_i(x_i) \), there exists \( z^2_i \in \Gamma_i(x_i^2) \) such that

\[
d_i(z^1_i, z^2_i) \leq q_i H_i(S_i(x_i^1), S_i(x_i^2));
\]

(iii) there exist \( a_i, b_i, c_i \in ]0, 1[ \), with \( \max_i a_i + \max_i b_i + \max_i c_i < 1 \) such that, for every \( x^1, x^2 \in X_i \),

\[
H_i(S_i(x_i^1), S_i(x_i^2)) \leq a_i d_i(x_i^1, x_i^2) + b_i D_i(x_i^1, S_i(x_i^1)) + c_i D_i(x_i^2, S_i(x_i^2));
\]

(iv) for any \( x \in X \), the set \( S_i(x) \) is closed;

(v) the relation \( R_i \) is closed in the i-th variable, that is: for any sequence \((z_i^k)_{k \in \mathbb{N}} \subset X_i \) such that \( z_i^k \to z_i \) when \( k \to \infty \), if \( R_i(x_i, \ldots, z_i^k, \ldots, x_n; t_i) \) holds, then \( R_i(x_i, \ldots, z_i, \ldots, x_n; t_i) \) holds too.

Then, \((SQR)\) admits at least a solution.
Proof. We will prove that $\Gamma : X \to 2^X$ is a Reich-type contraction and we will use Theorem 1.1 to obtain the existence of a fixed point of $\Gamma$. Since $X$ is closed and $(E, d)$ is complete, the space $(X, d)$ is complete too.

For each $i \in I$ and $x \in X$, hypotheses (iv) and (v) imply that $\Gamma_i(x)$ is closed. Then $\Gamma(x)$ is closed too.

Let $x^1 = (x_1^1, \ldots, x_n^1) \in X$ and $x^2 = (x_1^2, \ldots, x_n^2) \in X$. Let $z_i^1 \in \Gamma_i(x^1)$. According to (ii), there exists $z_i^2 \in \Gamma_i(x^2)$ such that

$$d_i(z_i^1, z_i^2) \leq q_i H_i(S_i(x^1), S_i(x^2)).$$

Similarly, for any $z_i^2 \in \Gamma_i(x^2)$ there exists $z_i^1 \in \Gamma_i(x^1)$ such that (2.1) holds. From Lemma 1.2, we have

$$H_i(\Gamma_i(x^1), \Gamma_i(x^2)) \leq q_i H_i(S_i(x^1), S_i(x^2)).$$

Further, using Lemma 1.1, (2.2), (iii), and the inclusion $\Gamma_i(x) \subseteq S_i(x)$, for any $x \in X$, follows

$$H(\Gamma(x^1), \Gamma(x^2)) \leq \sum_{i=1}^n H_i(\Gamma_i(x^1), \Gamma_i(x^2)) \leq \sum_{i=1}^n q_i H_i(S_i(x^1), S_i(x^2))$$

$$\leq \sum_{i=1}^n (q_i a_i d_i(x_1^1, x_1^2) + q_i b_i D_i(x_1^1, S_i(x^1)) + q_i c_i D_i(x_2^1, S_i(x^2)))$$

$$\leq q a d(x_1^1, x_2^1) + q b \sum_{i=1}^n D_i(x_1^1, \Gamma_i(x^1)) + q c \sum_{i=1}^n D_i(x_2^1, \Gamma_i(x^2))$$

$$= q a d(x_1^1, x_2^1) + q b D(x_1^1, \Gamma(x^1)) + q c D(x_2^1, \Gamma(x^2)),$$

where $q = \max_{i \in I} q_i$, $a = \max_{i \in I} a_i$, $b = \max_{i \in I} b_i$, $c = \max_{i \in I} c_i$ and $qa + qb + qc < 1$. Applying Reich’s theorem follows the existence of a fixed point for $\Gamma$ and consequently of a solution of (SVQR).

By making a change in hypothesis (ii), we can obtain a second existence result:

**Theorem 2.3.** Suppose that for any $i \in I$, the set $X_i$ is nonempty, closed and:

(i) for any $x \in X$, $\Gamma_i(x)$ is nonempty;

(ii) there exists $q_i \in [0, 1]$ such that, for every $x^1, x^2 \in X$, for every $z_i^1 \in \Gamma_i(x^1)$ and $z_i^2 \in \Gamma_i(x^2)$,

$$d_i(z_i^1, z_i^2) \leq q_i H_i(S_i(x^1), S_i(x^2));$$

(iii) there exist $a_i, b_i, c_i \in [0, 1]$ with $\max_{i \in I} a_i + \max_{i \in I} b_i + \max_{i \in I} c_i < 1$ such that, for every $x^1, x^2 \in X,$

$$H_i(S_i(x^1), S_i(x^2)) \leq a_i d_i(x_1^1, x_2^1) + b_i D_i(x_1^1, S_i(x^1)) + c_i D_i(x_2^1, S_i(x^2));$$

Then, (SVQR) admits a solution.

Proof. It can be noticed that for any $x \in X$ and $i \in I$, the set $\Gamma_i(x)$ contains only one element. Indeed, if $\zeta_i, \xi_i \in \Gamma_i(x)$, according to (ii), we get

$$d_i(\zeta_i, \xi_i) \leq q_i H_i(S_i(x), S_i(x)) = 0,$$

so $\zeta_i = \xi_i$. Since $\Gamma_i(x)$ is a singleton, it is a closed set. The rest of the proof is the same as for Theorem 2.2.

Starting with another definition for the “partial” problem, we can obtain a new existence result, with different conditions.
For \( x = (x_1, \ldots, x_n) \in X \) and \( i \in I \) fixed, we denote

\[
T_i(x) = \{ z_i \in X_i \mid z_i \in S_i(x_1, \ldots, z_i, \ldots, x_n) \text{ and } R_i(x_1, \ldots, z_i, \ldots, x_n; \theta_i) \text{ holds for all } \theta_i \in Q_i(x_1, \ldots, z_i, \ldots, x_n) \}
\]

and we define the function \( T : X \to 2^X \) by \( T(x) = T_1(x) \times \cdots \times T_n(x) \). It is easy to see that any fixed point of \( T \) is a solution of (SQVR).

**Theorem 2.4.** Suppose that for any \( i \in I \), the set \( X_i \) is nonempty, closed and:

(i) for any \( x \in X \), \( T_i(x) \) is nonempty;

(ii) there exists \( q_i \in ]0, 1[ \) such that, for every \( x^1, x^2 \in X \), if \( z_i^1 \in T_i(x^1) \), there exists \( z_i^2 \in T_i(x^2) \) such that

\[
d_i(z_i^1, z_i^2) \leq q_i H_i(S_i(x^1), S_i(x^2));
\]

(iii) there exist \( a_i, b_i, c_i \in ]0, 1[ \) with \( \max_{i \in I} a_i + \max_{i \in I} b_i + \max_{i \in I} c_i < 1 \) such that, for every \( x^1, x^2 \in X \),

\[
H_i(S_i(x^1), S_i(x^2)) \leq a_i d_i(x_1^1, x_1^2) + b_i D_i(x_1^1, S_i(x^1)) + c_i D_i(x_2^1, S_i(x^2));
\]

(iv) for any sequence \( (z_i^k)_{k \in \mathbb{N}} \subset X_i \) such that \( z_i^k \to z_i \) when \( k \to \infty \), if \( z_i^k \in S_i(x_1, \ldots, z_i, \ldots, x_n) \) for any \( k \in \mathbb{N} \), then \( z_i \in S_i(x_1, \ldots, z_i, \ldots, x_n) \);

(v) for any sequence \( (z_i^k)_{k \in \mathbb{N}} \subset X_i \) such that \( z_i^k \to z_i \) when \( k \to \infty \), if \( R_i(x_1, \ldots, z_i, \ldots, x_n; \theta_i) \) holds for any \( \theta_i \in Q_i(x_1, \ldots, z_i, \ldots, x_n) \), then the relation \( R_i(x_1, \ldots, z_i, \ldots, x_n; t_i) \) holds for any \( t_i \in Q_i(x_1, \ldots, z_i, \ldots, x_n) \).

Then, (SQVR) admits at least a solution.

**Proof.** Hypotheses (iv) and (v) imply that for every \( x \in X \), \( T(x) \) is closed. The rest of the proof is identical to the one of Theorem 2.2. \( \square \)

### 3. An Existence Result for a System of Quasi-Equilibrium Problems

As a particular case of the system of quasi-variational relations, we consider

\[
(SQEP) \quad \text{Find } \vec{x} = (\vec{x}_1, \ldots, \vec{x}_n) \in X \text{ such that for each } i \in I, \\
\vec{x}_i \in S_i(\vec{x}) \text{ and } f_i(\vec{x}_i, t_i) \geq 0 \text{ for all } t_i \in S_i(\vec{x}).
\]

The relation \( R_i(x, t_i) \) holds iff \( f_i(x, t_i) \geq 0 \). In this section, we denote

\[
\gamma_i(x) = \{ z_i \in S_i(x) \mid f_i(x_1, \ldots, z_i, \ldots, x_n; t_i) \geq 0, \text{ for all } t_i \in S_i(x) \}
\]

As a consequence of Theorem 2.3, we obtain:

**Theorem 3.5.** Suppose that for any \( i \in I \), the set \( X_i \) is nonempty, closed and:

(a) for any \( x \in X \), \( \gamma_i(x) \) is nonempty;

(b) there exist \( a_i, b_i, c_i \in ]0, 1[ \) with \( \max_{i \in I} a_i + \max_{i \in I} b_i + \max_{i \in I} c_i < 1 \) such that, for every \( x^1, x^2 \in X \),

\[
H_i(S_i(x^1), S_i(x^2)) \leq a_i d_i(x_1^1, x_1^2) + b_i D_i(x_1^1, S_i(x^1)) + c_i D_i(x_2^1, S_i(x^2));
\]

(c) there exists \( m_i > 0 \) such that for every \( x = (x_1, \ldots, x_n) \in X \) and \( t_i \in X_i \),

\[
f_i(x_1, \ldots, x_i, \ldots, x_n; t_i) + f_i(x_1, \ldots, t_i, \ldots, x_n; x_i) \leq -m_i d_i(x_i, t_i);
\]

(d) \( f_i \) is lipschitz in the last variable, that is there exists \( L_i > 0 \) such that for every \( x \in X \) and \( t_i, \theta_i \in X_i \),

\[
|f_i(x; t_i) - f_i(x; \theta_i)| \leq L_i d_i(t_i, \theta_i),
\]
Next, adding these two inequalities, using (3.5) and hypothesis (d) follows that
\[ d_i(z_i^1, t_i^2) \leq H_i(S_i(x^1), S_i(x^2)) \leq \varepsilon. \]

Similarly, since \( z_i^2 \in S_i(x^2) \), there exists \( t_i^1 \in S_i(x^1) \) such that
\[ d_i(z_i^2, t_i^1) \leq H_i(S_i(x^1), S_i(x^2)) + \varepsilon. \]

From the definitions of \( \gamma_i(x^1) \) and \( \gamma_i(x^2) \), we get
\[ f_i(x_1^1, \ldots, z_i^1, \ldots, x_n^1, t_i^1) \geq 0 \quad \text{and} \quad f_i(x_1^2, \ldots, z_i^2, \ldots, x_n^2, t_i^2) \geq 0. \]

From condition (c), we have
\[ d_i(z_i^1, z_i^2) \leq -\frac{1}{m_i} f_i(x_1^1, \ldots, z_i^1, \ldots, x_n^1, z_i^2) - \frac{1}{m_i} f_i(x_1^1, \ldots, z_i^2, \ldots, x_n^1, z_i^1), \]

\[ d_i(z_i^1, z_i^2) \leq \frac{1}{m_i} f_i(x_1^2, \ldots, z_i^1, \ldots, x_n^2, z_i^2) - \frac{1}{m_i} f_i(x_1^2, \ldots, z_i^2, \ldots, x_n^2, z_i^1). \]

Next, adding these two inequalities, using (3.5) and hypothesis (d) follows that
\[ d_i(z_i^1, z_i^2) \leq \frac{L_i}{2m_i} d_i(z_i^2, t_i^1) + \frac{L_i}{2m_i} d_i(z_i^1, t_i^2) \]
\[ - \frac{1}{m_i} f_i(x_1^1, \ldots, z_i^1, \ldots, x_n^1, z_i^2) - \frac{1}{m_i} f_i(x_1^2, \ldots, z_i^2, \ldots, x_n^2, z_i^1) \]
\[ + \frac{1}{m_i} f_i(x_1^2, \ldots, z_i^1, \ldots, x_n^2, z_i^1) - \frac{1}{m_i} f_i(x_1^2, \ldots, z_i^2, \ldots, x_n^2, z_i^2) \]
\[ \leq \frac{L_i}{m_i} H_i(S_i(x^1), S_i(x^2)) + \frac{L_i \varepsilon}{m_i} + \frac{\lambda_i}{m_i} d_i(z_i^1, z_i^2). \]

On the other hand, \( z_i^1 \in \gamma(x^1) \) implies that \( f_i(x_1^1, \ldots, z_i^1, \ldots, x_n^1, z_i^1) \geq 0 \). Similarly, we have \( f_i(x_1^2, \ldots, z_i^2, \ldots, x_n^2, z_i^2) \geq 0 \). So it follows, using also condition (e), the previous inequality, (3.3) and (3.4) that
\[ d_i(z_i^1, z_i^2) \leq \frac{L_i}{m_i} H_i(S_i(x^1), S_i(x^2)) + \frac{L_i \varepsilon}{m_i} + \frac{\lambda_i}{m_i} d_i(z_i^1, z_i^2). \]
From here, we get
\[(1 - \frac{\lambda_i}{m_i})d_i(z_1^i, z_2^i) \leq \frac{L_i}{m_i} H_i(S_i(x^1), S_i(x^2)) + \frac{L_i \varepsilon}{m_i}.
\]
When \(\varepsilon \to 0\), the inequality becomes
\[d_i(z_1^i, z_2^i) \leq \frac{L_i}{m_i - \lambda_i} H_i(S_i(x^1), S_i(x^2)),
\]
so \(q_i = \frac{L_i}{m_i - \lambda_i} \in ]0, 1[\) as needed.

We mention that sufficient conditions for the non-emptiness of the sets \(\Gamma_i(x)\) or \(T_i(x)\) can be given, for instance, by using intersection theorems of Ky Fan type (see [5], [4]).

\textbf{REFERENCES}