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Research Article

Existence Results for Systems of Quasi-Variational Relations

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ABSTRACT. The existence of solutions for a system of variational relations, in a general form, is studied using a fixed point result for contractions in metric spaces. As a particular case, sufficient conditions for the existence of solutions of a system of quasi-equilibrium problems are given.

Keywords: Variational relations problems, system of equilibrium problems, fixed points.

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1. INTRODUCTION AND PRELIMINARIES

For each $i \in I = \{1, \dots, n\}$, let X_i be a nonempty subset of a complete metric space (E_i, d_i) and $X = \prod_{i\in I} X_i$ a subset of the product space $E = \prod_{i\in I} E_i$. Let $S_i, Q_i : X \to 2^{X_i}$ be two set-valued maps with nonempty values. Let $R_i(x, y_i)$ be a relation between $x \in X$ and $y_i \in X_i$.

The general system of quasi-variational relations that we consider in this paper is:

$$(SQVR) \quad \text{Find } \bar{x} = (\bar{x}_1, \dots, \bar{x}_n) \in X \text{ such that for each } i \in I,$$
$$\bar{x}_i \in S_i(\bar{x}) \text{ and } R_i(\bar{x}, y_i) \text{ holds for all } y_i \in Q_i(\bar{x}).$$

Variational relations problems were considered for the first time by D.T. Luc in [11], as a general model that encompasses optimization problems, equilibrium problems or variational inclusion problems. Several authors continued the study of variational relations problems, see for instance the papers [10], [12], [9], [2], [1] and the references therein. Existence results for the solutions of variational relations problems are obtained mostly in two ways: by applying intersection results for set valued mappings (see [11]) or by using various fixed points theorems (see [11], [7], [4]).

The system $(SQVR)$ was introduced by L.J. Lin and Q.H. Ansari in [8], where the existence of a solution was established using a maximal element theorem for a family of set-valued maps. The same system was studied in [5] by a factorization method, that followed the ideas from [6].

In this paper, we will give sufficient conditions for the existence of solutions of the system $(SQVR)$, using a fixed-point theorem for set-valued mappings that are Reich-type contractions. The general result obtained for the system of variational relations will be applied in the last section to a system of equilibrium problems.

In the rest of this section, we present some notations and results needed in the paper.

The metric on the product space will be defined by $d : E \times E \to \mathbb{R}_+$,

$$d(x, y) = d_1(x_1, y_1) + \cdots + d_n(x_n, y_n),$$

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for $x = (x_1, \ldots, x_n) \in E$ and $y = (y_1, \ldots, y_n) \in E$.
For any nonempty sets $A, B \subset E$ and $x \in E$, denote by

$$\begin{aligned} D(x, B) &= \inf_{b \in B} d(x, b) \text{ and} \\ H(A, B) &= \max\{\sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b)\}. \end{aligned}$$

$H(A, B)$ is the generalized Hausdorff functional of A and B. Similarly, we will denote by $H_i(A_i, B_i)$ the Hausdorff distance induced by d_i, for A_i and B_i subsets of E_i.

Lemma 1.1. *For $x = (x_1, \ldots, x_n)$, $A = A_1 \times \cdots \times A_n$ and $B = B_1 \times \cdots \times B_n$, we have*

$$\begin{aligned} D(x, B) &= D_1(x_1, B_1) + \cdots + D_n(x_n, B_n), \\ H(A, B) &\leq H_1(A_1, B_1) + \cdots + H_n(A_n, B_n). \end{aligned}$$

Lemma 1.2. *(a) If $A, B \subset E$ are such that for each $a \in A$ there exists $b \in B$ such that $d(a, b) \leq c$ and for each $b \in B$ there exists $a \in A$ such that $d(a, b) \leq c$, then $H(A, B) \leq c$.*
(b) If $A, B \subset E$ and $\varepsilon > 0$, then for each $a \in A$ there exists $b \in B$ such that $d(a, b) \leq H(A, B) + \varepsilon$.

There is a vast literature on the existence of fixed points of generalized contractions, both single-valued and set-valued (see for instance [3], [14]). We will use the following:
A set-valued mapping $F : E \to 2^E$ is said to be a *Reich - type contraction* if there exist $a, b, c \geq 0$, with $a + b + c < 1$ such that $H(F(x), F(y)) \leq ad(x, y) + bD(x, F(x)) + cD(y, F(y))$, for each $x, y \in E$.

Theorem 1.1 ([13]). *Let (E, d) be a complete metric space and let $F : E \to 2^E$ be a Reich-type contraction. Suppose also that $F(x)$ is a closed set, for every $x \in E$. Then, F has at least a fixed point.*

2. An existence result for a system of variational relations

We give in what follows sufficient conditions for the existence of solutions of the system $(SQVR)$ formulated in the previous section.
For $x = (x_1, \ldots, x_n) \in X$ and $i \in I$ fixed, we denote

$$\Gamma_i(x) = \{z_i \in S_i(x) \mid R_i(x_1, \ldots, z_i, \ldots, x_n; t_i) \text{ holds for all } t_i \in Q_i(x)\}$$

and we define the function $\Gamma : X \to 2^X$ by $\Gamma(x) = \Gamma_1(x) \times \cdots \times \Gamma_n(x)$. It is easy to see that any fixed point of Γ is a solution of (SQVR).

Theorem 2.2. *Suppose that for any $i \in I$, the set X_i is nonempty, closed and:*
(i) for any $x \in X$, $\Gamma_i(x)$ is nonempty;
(ii) there exists $q_i \in]0, 1[$ such that, for every $x^1, x^2 \in X$, if $z_i^1 \in \Gamma_i(x^1)$, there exists $z_i^2 \in \Gamma_i(x^2)$ such that

$$d_i(z_i^1, z_i^2) \leq q_i H_i(S_i(x^1), S_i(x^2));$$

(iii) there exist $a_i, b_i, c_i \in]0, 1[$, with $\max_{i \in I} a_i + \max_{i \in I} b_i + \max_{i \in I} c_i < 1$ such that, for every $x^1, x^2 \in X$,

$$H_i(S_i(x^1), S_i(x^2)) \leq a_i d_i(x_i^1, x_i^2) + b_i D_i(x_i^1, S_i(x^1)) + c_i D_i(x_i^2, S_i(x^2));$$

(iv) for any $x \in X$, the set $S_i(x)$ is closed;
(v) the relation R_i is closed in the $i - th$ variable, that is: for any sequence $(z_i^k)_{k \in \mathbb{N}} \subset X_i$ such that $z_i^k \to z_i$ when $k \to \infty$, if $R_i(x_i, \ldots, z_i^k, \ldots x_n; t_i)$ holds, then $R_i(x_i, \ldots, z_i, \ldots x_n; t_i)$ holds too.
Then, (SQVR) admits at least a solution.

Proof. We will prove that $\Gamma : X \to 2^X$ is a Reich-type contraction and we will use Theorem 1.1 to obtain the existence of a fixed point of Γ. Since X is closed and (E, d) is complete, the space (X, d) is complete too.

For each $i \in I$ and $x \in X$, hypotheses (iv) and (v) imply that $\Gamma_i(x)$ is closed. Then $\Gamma(x)$ is closed too.

Let $x^1 = (x_1^1, \ldots, x_n^1) \in X$ and $x^2 = (x_1^2, \ldots, x_n^2) \in X$. Let $z_i^1 \in \Gamma_i(x^1)$. According to (ii), there exists $z_i^2 \in \Gamma_i(x^2)$ such that

$$d_i(z_i^1, z_i^2) \leq q_i H_i(S_i(x^1), S_i(x^2)). \tag{2.1}$$

Similarly, for any $z_i^2 \in \Gamma_i(x^2)$ there exists $z_i^1 \in \Gamma_i(x^1)$ such that (2.1) holds. From Lemma 1.2, we have

$$H_i(\Gamma_i(x^1), \Gamma_i(x^2)) \leq q_i H_i(S_i(x^1), S_i(x^2)). \tag{2.2}$$

Further, using Lemma 1.1, (2.2), (iii), and the inclusion $\Gamma_i(x) \subseteq S_i(x)$, for any $x \in X$, follows

$$\begin{aligned} H(\Gamma(x^1), \Gamma(x^2)) &\leq \sum_{i=1}^n H_i(\Gamma_i(x^1), \Gamma_i(x^2)) \leq \sum_{i=1}^n q_i H_i(S_i(x^1), S_i(x^2)) \\ &\leq \sum_{i=1}^n (q_i a_i d_i(x_i^1, x_i^2) + q_i b_i D_i(x_i^1, S_i(x^1)) + q_i c_i D_i(x_i^2, S_i(x^2))) \\ &\leq qad(x^1, x^2) + qb \sum_{i=1}^n D_i(x_i^1, \Gamma_i(x^1)) + qc \sum_{i=1}^n D_i(x_i^2, \Gamma_i(x^2)) \\ &= qa\, d(x^1, x^2) + qb\, D(x^1, \Gamma(x^1)) + qc\, D(x^2, \Gamma(x^2)), \end{aligned}$$

where $q = \max_{i \in I} q_i$, $a = \max_{i \in I} a_i$, $b = \max_{i \in I} b_i$, $c = \max_{i \in I} c_i$ and $qa + qb + qc < 1$. Applying Reich's theorem follows the existence of a fixed point for Γ and consequently of a solution of (SQVR). □

By making a change in hypothesis (ii), we can obtain a second existence result:

Theorem 2.3. *Suppose that for any $i \in I$, the set X_i is nonempty, closed and:*

(i) for any $x \in X$, $\Gamma_i(x)$ is nonempty;

(ii) there exists $q_i \in]0, 1[$ such that, for every $x^1, x^2 \in X$, for every $z_i^1 \in \Gamma_i(x^1)$ and $z_i^2 \in \Gamma_i(x^2)$,

$$d_i(z_i^1, z_i^2) \leq q_i H_i(S_i(x^1), S_i(x^2));$$

(iii) there exist $a_i, b_i, c_i \in]0, 1[$, with $\max_{i \in I} a_i + \max_{i \in I} b_i + \max_{i \in I} c_i < 1$ such that, for every $x^1, x^2 \in X$,

$$H_i(S_i(x^1), S_i(x^2)) \leq a_i d_i(x_i^1, x_i^2) + b_i D_i(x_i^1, S_i(x^1)) + c_i D_i(x_i^2, S_i(x^2));$$

Then, (SQVR) admits a solution.

Proof. It can be noticed that for any $x \in X$ and $i \in I$, the set $\Gamma_i(x)$ contains only one element. Indeed, if $\zeta_i, \xi_i \in \Gamma_i(x)$, according to (ii), we get

$$d_i(\zeta_i, \xi_i) \leq q_i H_i(S_i(x), S_i(x)) = 0,$$

so $\zeta_i = \xi_i$. Since $\Gamma_i(x)$ is a singleton, it is a closed set. The rest of the proof is the same as for Theorem 2.2. □

Starting with another definition for the "partial" problem, we can obtain a new existence result, with different conditions.

For $x = (x_1, \ldots, x_n) \in X$ and $i \in I$ fixed, we denote

$$T_i(x) = \{z_i \in X_i \mid \quad z_i \in S_i(x_1, \ldots, z_i, \ldots, x_n) \text{ and } R_i(x_1, \ldots, z_i, \ldots, x_n; \theta_i) \\ \text{holds for all } \theta_i \in Q_i(x_1, \ldots, z_i, \ldots, x_n)\}$$

and we define the function $T : X \to 2^X$ by $T(x) = T_1(x) \times \cdots \times T_n(x)$. It is easy to see that any fixed point of T is a solution of (SQVR).

Theorem 2.4. *Suppose that for any $i \in I$, the set X_i is nonempty, closed and:*
(i) for any $x \in X$, $T_i(x)$ is nonempty;
(ii) there exists $q_i \in]0, 1[$ such that, for every $x^1, x^2 \in X$, if $z_i^1 \in T_i(x^1)$, there exists $z_i^2 \in T_i(x^2)$ such that

$$d_i(z_i^1, z_i^2) \le q_i H_i(S_i(x^1), S_i(x^2));$$

(iii) there exist $a_i, b_i, c_i \in]0, 1[$, with $\max_{i \in I} a_i + \max_{i \in I} b_i + \max_{i \in I} c_i < 1$ such that, for every $x^1, x^2 \in X$,

$$H_i(S_i(x^1), S_i(x^2)) \le a_i d_i(x_i^1, x_i^2) + b_i D_i(x_i^1, S_i(x^1)) + c_i D_i(x_i^2, S_i(x^2));$$

(iv) for any sequence $(z_i^k)_{k \in \mathbb{N}} \subset X_i$ such that $z_i^k \to z_i$ when $k \to \infty$, if $z_i^k \in S_i(x_1, \ldots, z_i^k, \ldots, x_n)$ for any $k \in \mathbb{N}$, then $z_i \in S_i(x_1, \ldots, z_i, \ldots, x_n)$;
(v) for any sequence $(z_i^k)_{k \in \mathbb{N}} \subset X_i$ such that $z_i^k \to z_i$ when $k \to \infty$, if $R_i(x_i, \ldots, z_i^k, \ldots x_n; \theta_i)$ holds for any $\theta_i \in Q_i(x_1, \ldots, z_i^k, \ldots, x_n)$, then the relation $R_i(x_i, \ldots, z_i, \ldots x_n; t_i)$ holds for any $t_i \in Q_i(x_1, \ldots, z_i, \ldots, x_n)$.
Then, (SQVR) admits at least a solution.

Proof. Hypotheses (iv) and (v) imply that for every $x \in X$, $T(x)$ is closed. The rest of the proof is identical to the one of Theorem 2.2. □

3. An existence result for a system of quasi-equilibrium problems

As a particular case of the system of quasi-variational relations, we consider

$$(SQEP) \quad \text{Find } \bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in X \text{ such that for each } i \in I, \\ \bar{x}_i \in S_i(\bar{x}) \text{ and } f_i(\bar{x}, t_i) \ge 0 \ \text{ for all } t_i \in S_i(\bar{x}).$$

The relation $R_i(x, t_i)$ holds iff $f_i(x, t_i) \ge 0$. In this section, we denote

$$\gamma_i(x) = \{z_i \in S_i(x) \mid f_i(x_1, \ldots, z_i, \ldots, x_n; t_i) \ge 0, \ \text{ for all } t_i \in S_i(x)\}$$

As a consequence of Theorem 2.3, we obtain:

Theorem 3.5. *Suppose that for any $i \in I$, the set X_i is nonempty, closed and:*
(a) for any $x \in X$, $\gamma_i(x)$ is nonempty;
(b) there exist $a_i, b_i, c_i \in]0, 1[$, with $\max_{i \in I} a_i + \max_{i \in I} b_i + \max_{i \in I} c_i < 1$ such that, for every $x^1, x^2 \in X$,

$$H_i(S_i(x^1), S_i(x^2)) \le a_i d_i(x_i^1, x_i^2) + b_i D_i(x_i^1, S_i(x^1)) + c_i D_i(x_i^2, S_i(x^2));$$

(c) there exists $m_i > 0$ such that for every $x = (x_1, \ldots, x_n) \in X$ and $t_i \in X_i$,

$$f_i(x_1, \ldots, x_i, \ldots, x_n; t_i) + f_i(x_1, \ldots, t_i, \ldots, x_n; x_i) \le -m_i d_i(x_i, t_i);$$

(d) f_i is lipschitz in the last variable, that is there exists $L_i > 0$ such that for every $x \in X$ and $t_i, \theta_i \in X_i$,

$$|f_i(x; t_i) - f_i(x; \theta_i)| \le L_i d_i(t_i, \theta_i),$$

(e) f_i is lipschitz in the $i-th$ variable, that is there exists $\lambda_i > 0$ such that for every $x \in X$ and $\zeta_i, \xi_i, t_i \in X_i$,

$$|f_i(x_1, \dots, \zeta_i, \dots x_n; t_i) - f_i(x_1, \dots, \xi_i, \dots x_n; t_i)| \leq \lambda_i d_i(\zeta_i, \xi_i);$$

(f) $L_i + \lambda_i < m_i$.

Then, (SQEP) admits a solution.

Proof. To apply Theorem 2.3, we just need to verify hypothesis (ii). Let $\varepsilon > 0$. Let $x^1, x^2 \in X$ and $z_i^1 \in \gamma(x^1)$, $z_i^2 \in \gamma(x^2)$.

Since $z_i^1 \in S_i(x^1)$, from Lemma 1.2, there exists $t_i^2 \in S_i(x^2)$ such that

$$d_i(z_i^1, t_i^2) \leq H_i(S_i(x^1), S_i(x^2)) + \varepsilon. \tag{3.3}$$

Similarly, since $z_i^2 \in S_i(x^2)$, there exists $t_i^1 \in S_i(x^1)$ such that

$$d_i(z_i^2, t_i^1) \leq H_i(S_i(x^1), S_i(x^2)) + \varepsilon. \tag{3.4}$$

From the definitions of $\gamma_i(x^1)$ and $\gamma_i(x^2)$, we get

$$f_i(x_1^1, \dots, z_i^1, \dots, x_n^1; t_i^1) \geq 0 \text{ and } f_i(x_1^2, \dots, z_i^2, \dots, x_n^2; t_i^2) \geq 0. \tag{3.5}$$

From condition (c), we have

$$d_i(z_i^1, z_i^2) \leq -\frac{1}{m_i} f_i(x_1^1, \dots, z_i^1, \dots, x_n^1; z_i^2) - \frac{1}{m_i} f_i(x_1^1, \dots, z_i^2, \dots, x_n^1; z_i^1),$$

$$d_i(z_i^1, z_i^2) \leq -\frac{1}{m_i} f_i(x_1^2, \dots, z_i^1, \dots, x_n^2; z_i^2) - \frac{1}{m_i} f_i(x_1^2, \dots, z_i^2, \dots, x_n^2; z_i^1).$$

Next, adding these two inequalities, using (3.5) and hypothesis (d) follows that

$$\begin{aligned}
d_i(z_i^1, z_i^2) &\leq -\frac{1}{2m_i} f_i(x_1^1, \dots, z_i^1, \dots, x_n^1; z_i^2) + \frac{1}{2m_i} f_i(x_1^1, \dots, z_i^1, \dots, x_n^1; t_i^1) \\
&- \frac{1}{2m_i} f_i(x_1^1, \dots, z_i^2, \dots, x_n^1; z_i^1) - \frac{1}{2m_i} f_i(x_1^2, \dots, z_i^2, \dots, x_n^2; z_i^1) \\
&+ \frac{1}{2m_i} f_i(x_1^2, \dots, z_i^2, \dots, x_n^2; t_i^2) - \frac{1}{2m_i} f_i(x_1^2, \dots, z_i^1, \dots, x_n^2; z_i^2) \\
&\leq \frac{L_i}{2m_i} d_i(z_i^2, t_i^1) + \frac{L_i}{2m_i} d_i(z_i^1, t_i^2) \\
&- \frac{1}{2m_i} f_i(x_1^1, \dots, z_i^2, \dots, x_n^1; z_i^1) - \frac{1}{2m_i} f_i(x_1^2, \dots, z_i^1, \dots, x_n^2; z_i^2).
\end{aligned}$$

On the other hand, $z_i^1 \in \gamma(x^1)$ implies that $f_i(x_1^1, \dots, z_i^1, \dots, x_n^1; z_i^1) \geq 0$. Similarly, we have $f_i(x_1^2, \dots, z_i^2, \dots, x_n^2; z_i^2) \geq 0$. So it follows, using also condition (e), the previous inequality, (3.3) and (3.4) that

$$\begin{aligned}
d_i(z_i^1, z_i^2) &\leq \frac{L_i}{2m_i} d_i(z_i^2, t_i^1) + \frac{L_i}{2m_i} d_i(z_i^1, t_i^2) \\
&- \frac{1}{2m_i} f_i(x_1^1, \dots, z_i^2, \dots, x_n^1; z_i^1) - \frac{1}{2m_i} f_i(x_1^2, \dots, z_i^1, \dots, x_n^2; z_i^2) \\
&+ \frac{1}{2m_i} f_i(x_1^1, \dots, z_i^1, \dots, x_n^1; z_i^1) + \frac{1}{2m_i} f_i(x_1^2, \dots, z_i^2, \dots, x_n^2; z_i^2) \\
&\leq \frac{L_i}{m_i} H_i(S_i(x^1), S_i(x^2)) + \frac{L_i \varepsilon}{m_i} + \frac{\lambda_i}{m_i} d_i(z_i^1, z_i^2).
\end{aligned}$$

From here, we get

$$(1-\frac{\lambda_i}{m_i})d_i(z_i^1,z_i^2)\leq \frac{L_i}{m_i}H_i(S_i(x^1),S_i(x^2))+\frac{L_i\varepsilon}{m_i}.$$

When $\varepsilon\to 0$, the inequality becomes

$$d_i(z_i^1,z_i^2)\leq \frac{L_i}{m_i-\lambda_i}H_i(S_i(x^1),S_i(x^2)),$$

so $q_i=\frac{L_i}{m_i-\lambda_i}\in]0,1[$ as needed. □

We mention that sufficient conditions for the non-emptiness of the sets $\Gamma_i(x)$ or $T_i(x)$ can be given, for instance, by using intersection theorems of Ky Fan type (see [5], [4]).

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