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Existence and uniqueness of solutions for nonlinear hybrid implicit Caputo-Hadamard fractional differential equations

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Abstract

In this paper, we use the contraction mapping principle to obtain the existence, interval of existence and uniqueness of solutions for nonlinear hybrid implicit Caputo-Hadamard fractional differential equations. We also use the generalization of Gronwall's inequality to show the estimate of the solutions.

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1. Introduction

Fractional differential equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors, see [1]–[12], [14]–[22] and the references therein.

Recently, Ahmad and Ntouyas [3] discussed the existence of solutions for the hybrid Hadamard differential equation

$$\begin{cases} {}^{H}D^{\alpha}\left(\dfrac{x(t)}{g\left(t,x\left(t\right)\right)}\right)=f\left(t,x\left(t\right)\right),\ t\in\left[1,T\right], \\ {}^{H}I^{\alpha}x\left(t\right)\big|_{t=1}=\eta, \end{cases}$$

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where ${}^{H}D^{\alpha}$ is the Hadamard fractional derivative of order $0 < \alpha \leq 1$. By employing the Dhage fixed point theorem, the authors obtained existence results.

The implicit fractional differential equation

$$\begin{cases} {}^{C}D^{\alpha}x(t) = f\left(t, x(t), {}^{C}D^{\alpha}x(t)\right), \\ x(0) = x_0, \end{cases}$$

has been investigated in [10], where ${}^{C}D^{\alpha}$ is the standard Caputo's fractional derivative of order $0 < \alpha < 1$. By using the contraction mapping principle, the existence, interval of existence and uniqueness of solutions has been established.

In [8], Dhaigude and Bhairat investigated the existence and stability of solutions of the following nonlinear implicit fractional differential equation

$$\begin{cases} \mathfrak{D}_1^{\alpha}x(t) = f\left(t, x(t), \mathfrak{D}_1^{\alpha}x(t)\right), \ t \in [1, b], \ b > 1, \\ x^{(k)}(1) = x_k \in \mathbb{R}^n, \ k = 0, 1, ..., m-1, \end{cases}$$

where $\mathfrak{D}_1^{\alpha}$ is the Caputo-Hadamard derivative of order $m-1 < \alpha \leq m$. By employing the modified version of contraction principle and the successive approximation method, the authors obtained existence and stability results.

In this paper, we are interested in the analysis of qualitative theory of the problems of the existence, interval of existence and uniqueness of solutions to nonlinear hybrid implicit Caputo-Hadamard fractional differential equations. Inspired and motivated by the works mentioned above and the references in this paper, we concentrate on the existence, interval of existence and uniqueness of solutions for the nonlinear hybrid implicit Caputo-Hadamard fractional differential equation

$$\begin{cases} \mathfrak{D}_1^{\alpha}\left(\dfrac{x(t)}{g(t, x(t))}\right) = f\left(t, x(t), \mathfrak{D}_1^{\alpha}\left(\dfrac{x(t)}{g(t, x(t))}\right)\right), \\ x(1) = \eta g(1, x(1)), \end{cases} \tag{1.1}$$

where $f : [1, T] \times \mathbb{R}^2 \to \mathbb{R}$ and $g : [1, T] \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$ are nonlinear continuous functions and $\mathfrak{D}_1^{\alpha}$ denotes the Caputo-Hadamard derivative of order $0 < \alpha < 1$. To show the existence, interval of existence and uniqueness of solutions of (1.1), we transform (1.1) into an integral equation and then use the contraction mapping principle. Further, by the generalization of Gronwall's inequality we obtain the estimate of solutions of (1.1).

2. Preliminaries

In this section we present some basic definitions, notations and results of fractional calculus [1, 9, 11, 14, 15, 19] which are used throughout this paper.

Definition 1 ([15]). *The Hadamard fractional integral of order $\alpha > 0$ for a continuous function $x : [1, +\infty) \to \mathbb{R}$ is defined as*

$$\mathfrak{I}_1^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} x(s) \frac{ds}{s}, \ \alpha > 0. \tag{2.1}$$

Definition 2 ([1, 11, 14]). *The Caputo-Hadamard fractional derivative of order α for a continuous function $x : [1, +\infty) \to \mathbb{R}$ is defined as*

$$\mathfrak{D}_1^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \delta^n(x)(s) \frac{ds}{s}, \ n-1 < \alpha < n, \tag{2.2}$$

where $\delta^n = \left(t\dfrac{d}{dt}\right)^n$, $n = [\alpha] + 1$.

Lemma 1 ([1, 11, 14])**.** *Let $\Re(\alpha) > 0$. Suppose $x \in C^{n-1}[1,+\infty)$ and $x^{(n)}$ exists almost everywhere on any bounded interval of $[1,+\infty)$. Then*

$$\mathfrak{I}_1^\alpha \left[\mathfrak{D}_1^\alpha x(t)\right] = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(1)}{\Gamma(k+1)} (\log t)^k .$$

In particular, when $0 < \Re(\alpha) < 1$, $\mathfrak{I}_1^\alpha \left[\mathfrak{D}_1^\alpha x(t)\right] = x(t) - x(1)$.

Lemma 2 ([15])**.** *For all $\mu > 0$ and $\nu > -1$, then*

$$\frac{1}{\Gamma(\mu)} \int_1^t \left(\log \frac{t}{s}\right)^{\mu-1} (\log s)^\nu \frac{ds}{s} = \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} (\log t)^{\mu+\nu} .$$

The following generalization of Gronwall's lemma for singular kernels plays an important role in obtaining our main results.

Lemma 3 ([13])**.** *Let $x : [1,T] \to [0,\infty)$ be a real function and w is a nonnegative locally integrable function on $[1,T]$. Assume that there is a constant $a > 0$ such that for $0 < \alpha < 1$*

$$x(t) \leq w(t) + a \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} x(s) \frac{ds}{s}.$$

Then, there exist a constant $K = K(\alpha)$ such that

$$x(t) \leq w(t) + Ka \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} w(s) \frac{ds}{s},$$

for every $t \in [1,T]$.

3. Main results

In this section, we give the equivalence of the initial value problem (1.1) and prove the existence, interval of existence, uniqueness and estimate of solutions of (1.1).

The proof of the following lemma is close to the proof of Lemma 6.2 given in [9].

Lemma 4. *If the functions $f : [1,T] \times \mathbb{R}^2 \to \mathbb{R}$ and $g : [1,T] \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$ are continuous, then the initial value problem (1.1) is equivalent to nonlinear fractional Volterra integro-differential equation*

$$\begin{aligned} x(t) &= \eta g(t, x(t)) \\ &\quad + \frac{g(t,x(t))}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f\left(s, x(s), \mathfrak{D}_1^\alpha \left(\frac{x(s)}{g(s,x(s))}\right)\right) \frac{ds}{s}, \end{aligned}$$

for $t \in [1,T]$.

Theorem 1. *Let $T > 1$. Assume that the continuous functions $f : [1,T] \times \mathbb{R}^2 \to \mathbb{R}$ and $g : [1,T] \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$ satisfy the following condition*

(H1) There exists $M_f \in \mathbb{R}^+$ such that

$$|f(t,u,v)| \leq M_f,$$

for all $u, v \in \mathbb{R}$ and $t \in [1,T]$.

(H2) There exists $M_g \in \mathbb{R}^+$ such that

$$|g(t,u)| \leq M_g,$$

for all $u \in \mathbb{R}$ and $t \in [1,T]$.

(H3) There exist $K_1, K_3 \in \mathbb{R}^+$, $K_2 \in (0,1)$ with $K_3 |\eta| \in (0,1)$ such that

$$|f(t,u,v) - f(t,\tilde{u},\tilde{v})| \leq K_1 |u-\tilde{u}| + K_2 |v-\tilde{v}|,$$

and

$$|g(t,u) - g(t,\tilde{u})| \leq K_3 |u-\tilde{u}|,$$

for all $u, v, \tilde{u}, \tilde{v} \in \mathbb{R}$ and $t \in [1,T]$.
Let

$$1 < b < \min\left\{ T, \exp\left(\frac{(1-K_3|\eta|)(1-K_2)\Gamma(\alpha+1)}{K_3(1-K_2)M_f + K_1 M_g} \right)^{\frac{1}{\alpha}} \right\},$$

then (1.1) has a unique solution $x \in C([1,b],\mathbb{R})$.

Proof. Let

$$\mathfrak{D}_1^\alpha \left(\frac{x(t)}{g(t,x(t))} \right) = z_x(t),\ x(1) = \eta g(1,x(1)),$$

then by Lemma 4,

$$x(t) = \eta g(t,x(t)) + \frac{g(t,x(t))}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} z_x(s) \frac{ds}{s},\ t \in [1,T],$$

where

$$z_x(t) = f\left(t, \eta g(t,x(t)) + g(t,x(t)) \mathfrak{I}_1^\alpha z_x(t), z_x(t)\right).$$

That is $x(t) = \eta g(t,x(t)) + g(t,x(t)) \mathfrak{I}_1^\alpha z_x(t)$. Define the mapping $P : C([1,b],\mathbb{R}) \to C([1,b],\mathbb{R})$ as follows

$$(Px)(t) = \eta g(t,x(t)) + \frac{g(t,x(t))}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} z_x(s) \frac{ds}{s}.$$

It is clear that the fixed points of P are solutions of (1.1). Let $x, y \in C([1,b],\mathbb{R})$, then we have

$$\begin{aligned}
&|(Px)(t) - (Py)(t)| \\
&= \Bigg| \eta g(t,x(t)) + \frac{g(t,x(t))}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} z_x(s) \frac{ds}{s} \\
&- \eta g(t,y(t)) - \frac{g(t,y(t))}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} z_y(s) \frac{ds}{s} \Bigg| \\
&\leq |\eta| |g(t,x(t)) - g(t,y(t))| \\
&+ |g(t,x(t)) - g(t,y(t))| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} |z_x(s) - z_y(s)| \frac{ds}{s} \\
&+ |g(t,y(t))| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} |z_x(s) - z_y(s)| \frac{ds}{s} \\
&\leq K_3 |\eta| |x(t) - y(t)| + K_3 |x(t) - y(t)| \frac{M_f}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \frac{ds}{s} \\
&+ \frac{M_g}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} |z_x(s) - z_y(s)| \frac{ds}{s},
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
|z_x(t) - z_y(t)| &\leq |f(t,x(t),z_x(t)) - f(t,x(t),z_y(t))| \\
&\leq K_1 |x(t) - y(t)| + K_2 |z_x(t) - z_y(t)| \\
&\leq \frac{K_1}{1-K_2} |x(t) - y(t)|.
\end{aligned} \tag{3.2}$$

By replacing (3.2) in the inequality (3.1), we get

$$\begin{aligned}
&|(Px)(t)-(Py)(t)| \\
&\leq K_3|\eta||x(t)-y(t)|+K_3|x(t)-y(t)|\frac{M_f}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s} \\
&+\frac{M_g}{\Gamma(\alpha)}\frac{K_1}{1-K_2}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|x(s)-y(s)|\frac{ds}{s} \\
&\leq K_3\left(|\eta|+\frac{M_f(\log t)^{\alpha}}{\Gamma(\alpha+1)}\right)\|x-y\| \\
&+\frac{M_g}{\Gamma(\alpha)}\frac{K_1}{1-K_2}\left(\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}\right)\|x-y\| \\
&\leq\left(K_3|\eta|+\left(K_3M_f+\frac{K_1M_g}{1-K_2}\right)\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}\right)\|x-y\|.
\end{aligned}$$

Since $t\in[1,b]$, then

$$\|Px-Py\|\leq\beta\|x-y\|,\ 0<\beta<1,$$

where

$$\beta=K_3|\eta|+\frac{K_3(1-K_2)M_f+K_1M_g}{1-K_2}\frac{(\log b)^{\alpha}}{\Gamma(\alpha+1)}.$$

That is to say the mapping P is a contraction in $C([1,b],\mathbb{R})$. Hence P has a unique fixed point $x\in C([1,b],\mathbb{R})$. Therefore, (1.1) has a unique solution. □

Theorem 2. *Assume that $f:[1,T]\times\mathbb{R}^2\to\mathbb{R}$ and $g:[1,T]\times\mathbb{R}\to\mathbb{R}\backslash\{0\}$ satisfy (H2) and (H3). If x is a solution of (1.1), then*

$$\begin{aligned}
|x(t)|\leq&\frac{(1-K_2)(1-K_3|\eta|)\Gamma(\alpha+1)+K_1KM_g(\log T)^{\alpha}}{(1-K_2)(1-K_3|\eta|)^2\Gamma(\alpha+1)} \\
&\times\left(|\eta|Q_1+\frac{M_gQ_2(\log T)^{\alpha}}{(1-K_2)\Gamma(\alpha+1)}\right),
\end{aligned}$$

where $Q_1=\sup_{t\in[1,T]}|g(t,0)|$, $Q_2=\sup_{t\in[1,T]}|f(t,0,0)|$ and $K\in\mathbb{R}^+$ is a constant.

Proof. Let

$$\mathfrak{D}_1^{\alpha}\left(\frac{x(t)}{g(t,x(t))}\right)=z_x(t),\ x(1)=\eta g(1,x(1)).$$

By Lemma 4, $x(t)=\eta g(t,x(t))+g(t,x(t))\mathfrak{I}_1^{\alpha}z_x(t)$. Then by (H2) and (H3), for any $t\in[1,T]$ we have

$$\begin{aligned}
|x(t)| &\leq |\eta||g(t,x(t))|+|g(t,x(t))|\mathfrak{I}_1^{\alpha}|z_x(t)| \\
&\leq |\eta|(|g(t,x(t))-g(t,0)|+|g(t,0)|)+M_g\mathfrak{I}_1^{\alpha}|z_x(t)| \\
&\leq |\eta|(Q_1+K_3|x(t)|)+M_g\mathfrak{I}_1^{\alpha}|z_x(t)|,
\end{aligned}$$

where $Q_1=\sup_{t\in[1,T]}|g(t,0)|$. On the other hand, for any $t\in[1,T]$ we get

$$\begin{aligned}
|z_x(t)| &= |f(t,x(t),z_x(t))| \\
&\leq |f(t,x(t),z_x(t))-f(t,0,0)|+|f(t,0,0)| \\
&\leq K_1|x(t)|+K_2|z_x(t)|+|f(t,0,0)| \\
&\leq \frac{K_1}{1-K_2}\|x\|+\frac{Q_2}{1-K_2},
\end{aligned}$$

where $Q_2 = \sup_{t\in[1,T]} |f(t,0,0)|$. Therefore

$$|x(t)| \le |\eta| (Q_1 + K_3 |x(t)|) + M_g \mathfrak{I}_1^\alpha \left(\frac{Q_2}{1-K_2} + \frac{K_1}{1-K_2} |x(t)| \right).$$

Thus

$$\begin{aligned}(1-K_3|\eta|)|x(t)| &\le |\eta| Q_1 + \frac{M_g Q_2 (\log T)^\alpha}{(1-K_2)\Gamma(\alpha+1)} \\ &\quad + \frac{K_1 M_g}{(1-K_2)(1-K_3|\eta|)} \mathfrak{I}_1^\alpha \{(1-K_3|\eta|)|x(t)|\}.\end{aligned}$$

By Lemma 3, there is a constant $K = K(\alpha)$ such that

$$\begin{aligned}&(1-K_3|\eta|)|x(t)| \\ &\le |\eta| Q_1 + \frac{M_g Q_2 (\log T)^\alpha}{(1-K_2)\Gamma(\alpha+1)} \\ &+ \frac{K_1 K M_g (\log T)^\alpha}{(1-K_2)(1-K_3|\eta|)\Gamma(\alpha+1)} \left(|\eta| Q_1 + \frac{M_g Q_2 (\log T)^\alpha}{(1-K_2)\Gamma(\alpha+1)} \right) \\ &\le \frac{(1-K_2)(1-K_3|\eta|)\Gamma(\alpha+1) + K_1 K M_g (\log T)^\alpha}{(1-K_2)(1-K_3|\eta|)\Gamma(\alpha+1)} \\ &\times \left(|\eta| Q_1 + \frac{M_g Q_2 (\log T)^\alpha}{(1-K_2)\Gamma(\alpha+1)} \right).\end{aligned}$$

Hence

$$\begin{aligned}|x(t)| &\le \frac{(1-K_2)(1-K_3|\eta|)\Gamma(\alpha+1) + K_1 K M_g (\log T)^\alpha}{(1-K_2)(1-K_3|\eta|)^2\Gamma(\alpha+1)} \\ &\quad \times \left(|\eta| Q_1 + \frac{M_g Q_2 (\log T)^\alpha}{(1-K_2)\Gamma(\alpha+1)} \right).\end{aligned}$$

This completes the proof. □

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References

[1] Y. Adjabi, F. Jarad, D. Baleanu, T. Abdeljawad, On Cauchy problems with Caputo Hadamard fractional derivatives, J. Comput. Anal. Appl 21(4) (2016) 661-681.

[2] R.P. Agarwal, Y. Zhou, Y. He, Existence of fractional functional differential equations, Computers and Mathematics with Applications 59 (2010) 1095-1100.

[3] B. Ahmad, S.K. NTouyas, Initial-value problems for fractional differential equations, Electronic Journal of Differential Equations 2014(161) (2014) 1-8.

[4] B. Ahmad, S. Sivasundaram, Some existence results for fractional integro-differential equations with nonlocal conditions, Communications in Applied Analysis 12 (2008) 107-112.

[5] A. Ali, K. Shah, F. Jarad, V. Gupta, T. Abdeljawad, Existence and stability analysis to a coupled system of implicit type impulsive boundary value problems of fractional-order differential equations, Advances in Difference Equations 2019(101) (2019) 1-21.

[6] H. Boulares, A. Ardjouni, Y. Laskri, Positive solutions for nonlinear fractional differential equations, Positivity 21 (2017) 1201-1212.

[7] H. Boulares, A. Ardjouni, Y. Laskri, Stability in delay nonlinear fractional differential equations, Rend. Circ. Mat. Palermo 65 (2016) 243-253.

[8] D.B. Dhaigude, S.P. Bhairat, On Ulam type stability for nonlinear implicit fractional differential equations, arXiv: 1707.07597v1, [math.CA] 24 Jul 2017.

[9] K. Diethelm, The analysis of fractional differential equations, Lecture Notes in Mathematics, Springer-verlag, Berlin, Heidelberg, (2010).

[10] J. Dong, Y. Feng, J. Jiang, A note on implicit fractional differential equations, Mathematica Aeterna 7(3) (2017) 261-267.

[11] Y.Y. Gambo, F. Jarad, D. Baleanu, T. Abdeljawad, On Caputo modification of the Hadamard fractional derivatives, Advances in Difference Equations 2014(10) (2014) 1-12.

[12] M. Haoues, A. Ardjouni, A. Djoudi, Existence, interval of existence and uniqueness of solutions for nonlinear implicit Caputo fractional differential equations, TJMM 10(1) (2018) 09-13.

[13] D. Henry, Geometric theory of semi linear parabolic equations, Springer -Verlag, Berlin, Heidelberge, New York, (1981).

[14] F. Jarad, T. Abdeljawad, D. Baleanu, Caputo-type modification of the Hadamard fractional derivatives, Advances in Difference Equations 2012(142) (2012) 1-8.

[15] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and applications of fractional differential equations, North-Holland Mathematics Studies 204, Editor: Jan Van Mill, Elsevier, Amsterdam, The Netherlands, (2006).

[16] K.D. Kucche, J.J. Nieto, V. Venktesh, Theory of nonlinear implicit fractional differential equations, Differ. Equ. Dyn. Syst., DOI 10.1007/s12591-016-0297-7.

[17] K.D. Kucche, S.T. Sutar, On existence and stability results for nonlinear fractional delay differential equations, Bol. Soc. Paran. Mat. (3s.) v. 36(4) (2018) 55-75.

[18] K.D. Kucche, S.S. Sutar, Stability via successive approximation for nonlinear implicit fractional differential equations, Moroccan J. Pure Appl. Anal. 3(1) (2017) 36-55.

[19] I. Podlubny, Fractional differential equations, Academic Press, San Diego, (1999).

[20] S.T. Sutar, K.D. Kucche, Global existence and uniqueness for implicit differential equations of arbitrary order, Fractional Differential Calculus 5(2) (2015) 199-208.

[21] J. Wang, L. Lv, Y. Zhou, New concepts and results in stability of fractional differential equations, Commun Nonlinear Sci Numer Simulat 17 (2012) 2530-2538.

[22] J.R. Wang, K. Shah, A. Ali, Existence and Hyers-Ulam stability of fractional nonlinear impulsive switched coupled evolution equations, Math. Meth. Applied Sciences 41(6) (2018) 2392-2402.