Results in Nonlinear Analysis 2 (2019) No. 3, 143–146
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

A note on Kannan type mappings with a F-contractive iterate

Selma Gülyaz-Özyurt[a]

[a] *Department of Mathematics, Cumhuriyet University, Sivas, Turkey*

Abstract

In this paper, we revisited fixed point theorem for Kannan type mapping with a contractive iterate at a point in the setting of F-construction. The given results extend and improve the related results in the literature.

Keywords: Fixed point; b-metric space; contractive iteration at a point.
2010 MSC: 47H10, 54H25

1. Introduction and Preliminaries

One of the outstanding extension of the Banach's fixed point theorem was given by Bryant [2]. Indeed, Bryant [2] just relaxed the continuity condition. More precisely, he proved that a mapping can not forms contraction although one of its iteration satisfies being a contraction.

Theorem 1. *[2] If f is a mapping of a complete metric space into itself and if, for some positive integer k, f^k is a contraction, then f has a unique fixed point.*

The following example indicates the aspect of Bryant [2].

Example 1. *[2] Let $T : [0,2] \to [0,2]$ be defined by*

$$T(x) = \begin{cases} 0 & \text{if } x \in [0,1], \\ 1 & \text{if } x \in (1,2]. \end{cases}$$

Then 2nd iteration of T is equal to 0 for all $x \in [0,2]$ although T is not continuous.

This interesting result of Bryant [2] was improved by Sehgal [5] by proposing the idea of the "contractive iterate at each point".

The significant result of Sehgal [5] is the following:

Email addresses: selmagulyaz@gmail.com (Selma Gülyaz-Özyurt), sgulyaz@cumhuriyet.edu.tr (Selma Gülyaz-Özyurt)

Received November 12, 2019, Accepted: November 22,2019, Online: November 24,2019.

Theorem 2. *Let (X, d) be a complete metric space, $q \in [0, 1)$ and $T : X \to X$ be a continuous mapping. If for each $x \in X$ there exists a positive integer $k = k(x)$ such that*

$$d(T^{k(x)}x, T^{k(x)}y) \leq qd(x, y) \tag{1.1}$$

for all $y \in X$, then T has a unique fixed point $u \in X$. Moreover,for any $x \in X$, $u = \lim_{n\to\infty} T^n x$.

Throughout the paper, $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive integers and the set of nonnegative integers. Similarly, let $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ represent the set of reals, positive reals and the set of nonnegative reals, respectively. Throughout the paper, all consider set X is non-empty.

We start with the definition of auxiliary function that was used by Wardowski [6] to define the new type contraction.

Definition 1. *[6] Let $F : \mathbb{R}^+ \to \mathbb{R}$ and we are considering the following conditions:*

$(F1)$ *F is strictly increasing, that is, for all $\xi, \eta \in \mathbb{R}_+$ if $\xi < \eta$ then $F(\xi) < F(\eta)$.*

$(F2)$ *For every sequence $\{t_n\}_{n=1}^{\infty}$ of positive real numbers*

$$\lim_{n\to\infty} t_n = 0 \text{ if and only if } \lim_{n\to\infty} F(t_n) = -\infty.$$

$(F3)$ *There is $k \in (0, 1)$ such that $\lim_{t\to 0^+} \left(t^k F(t)\right) = 0$.*

Example 2. *[6] Let $F_i : \mathbb{R}^+ \to \mathbb{R}$, for $i = 1, 2, 3, 4$, be mappings that are defined by*

(E1) $F_1(t) = \ln t$,

(E2) $F_2(t) = t + \ln t$,

(E3) $F_3(t) = -1/\sqrt{t}$,

(E4) $F_4(t) = \ln(t^2 + t)$,

Then $F_1, F_2, F_3, F_4 \in \mathcal{F}$.

Definition 2. *[6] Let (X, d) be a metric space. A map $T : X \to X$ is said to be an $F-$contraction on (X, d) if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$*

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)) \tag{1.2}$$

From $(F1)$ and $(F2)$ easily conclude that every F-contraction is a contractive mapping, that is, for all $x, y \in X$ with $Tx \neq Ty$, we have

$$d(Tx, Ty) < d(x, y)$$

Theorem 3. *[6] Let T be a self-mapping on a complete metric space (X, d). If T forms an $F-$contraction, then it possesses a unique fixed point u. Moreover, for any $x \in X$ the sequence $\{T^n x\}$ is convergent to u.*

Remark 1. *From $(F1)$ and (1.2) it follows that*

$$\begin{aligned} F(d(Tx, Ty)) &\leq F(d(x, y)) - \tau < F(d(x, y)) \Rightarrow \\ &\Rightarrow d(Tx, Ty) < d(x, y) \end{aligned}$$

for all $x, y \in X$ such that $Tx \neq Ty$. Also, T is a continuous operator.

In this paper, we combine the notions of F-contraction, Kannan type mapping in the setting of a complete metric space. This paper can be considered as a continuation of the recent result [4].

2. Main results

Theorem 4. *Let (X, d) be a complete metric space and $T : X \to X$ a mapping which satisfies the condition: If there exists $F \in \mathcal{F}$ and $\tau > 0$ such that for each $x \in X$ there is a positive integer $n(x)$ such that for all $y \in X$*

$$d(T^{n(x)}(x), T^{n(x)}(y)) > 0 \Rightarrow \tau + F(d(T^{n(x)}(x), T^{n(x)}(y))) \leq F(K(x, y)), \tag{2.1}$$

where $K(x, y) := \max\{d(T^{n(x)}(x), x), d(y, T^{n(x)}(y))\}$. Then, T has a unique fixed point $z \in X$ and $T^n(x_0) \to z$ for each $x_0 \in X$, as $n \to \infty$.

Proof. We shall construct a sequence $\{x_k\}$ in the following way. For the chooses arbitrary point $x_0 \in X$ with $n_0 = n(x_0)$, we set $x_1 = T^{n_0}x_0$ and inductively we get

$$x_{i+1} = T^{n_i}x_i \text{ with } n_i = n(x_i).$$

We assert that $x_i \neq x_{i+1}$ for all $i \in \mathbb{N}_0$. Suppose, on the contrary, there exists $i_0 \in \mathbb{N}_0$ such that $x_{i_0} = x_{i_0+1} = T^{n_{i_0}}x_{i_0}$. Then, x_{i_0} turns to be a fixed point of $T^{n_{i_0}}$. On the other hand,

$$Tx_{i_0} = T(T^{n_{i_0}}x_{i_0}) = T^{n_{i_0}}(Tx_{i_0}).$$

Thus, Tx_{i_0} form a fixed point of $T^{n_{i_0}}$. If $Tx_{i_0} \neq x_{i_0}$, then we conclude that T has a fixed point and that terminate the proof. Suppose, on the contrary, that $Tx_{i_0} \neq x_{i_0}$ and hence $d(T^{n_{i_0}}(Tx_{i_0}), T^{n_{i_0}}(x_{i_0})) > 0$. Then, by (2.13) we have

$$\tau + F(d(x_{i_0}, Tx_{i_0})) = \tau + F(d(T^{n_{i_0}}x_{i_0}, T^{n_{i_0}}Tx_{i_0})) \leq F(K(x_{i_0}, Tx_{i_0})), \tag{2.2}$$

with $K(x, y) := \max\{d(T^{n_{i_0}}(x_{i_0}), x_{i_0}), d(Tx_{i_0}, T^{n_{i_0}}(Tx_{i_0}))\}$=0, a contradiction. Consequently, we deduce that

$$x_i \neq x_{i+1} \text{ for all } i \in \mathbb{N}_0. \tag{2.3}$$

Taking the expression (2.3) into account (2.13) implies that

$$d(x_{i+1}, x_i) > 0 \Rightarrow \tau + F(d(x_{i+1}, x_i)) \leq F(K(x_i, x_{i-1})), \tag{2.4}$$

where $K(x_i, x_{i-1}) = \max\{d(x_i, x_{i-1}), d(x_{i+1}, x_i)\}$. It is clear that if $K(x_i, x_{i-1}) = d(x_{i+1}, x_i)$ yields a contradiction. Thus, $K(x_i, x_{i-1}) = d(x_i, x_{i-1})$ and (2.4) implies that

$$F(\delta_i) \leq F(\delta_{i-1}) - \tau \leq F(\delta_{i-1}) - 2\tau \leq ... \leq F(\delta_0) - i\tau, \tag{2.5}$$

where $\delta_j = d(T^{n_i}x_j, x_j)$ for all $j \in \mathbb{N}_0$.

As $i \to \infty$ the inequality above yields that $\lim_{i\to\infty} F(d(x_{i+1}, x_i)) = -\infty$. On account of axiom $(F2)$, we conclude that

$$\lim_{n\to\infty} d(x_{i+1}, x_i) = 0. \tag{2.6}$$

Taking the axiom $(F3)$ into the account, we find a $k \in (0, 1)$ such that

$$\lim_{i\to\infty} (d(x_{i+1}, x_i))^k F(d(x_{i+1}, x_i)) = 0. \tag{2.7}$$

On the other hand, by regarding (2.5), we find that

$$\begin{aligned} &(d(x_{i+1}, x_i))^k F(d(x_{i+1}, x_i)) - (d(x_{i+1}, x_i))^k F(\delta_0) \\ &\leq (d(x_{i+1}, x_i))^k (F(\delta_0) - i\tau) - (d(x_{i+1}, x_i))^k F(\delta_0) \\ &= -(d(x_{i+1}, x_i))^k i\tau \leq 0. \end{aligned} \tag{2.8}$$

Taking, (2.6) and (2.7), into account and by letting $n \to \infty$ in (2.8), we find

$$\lim_{i\to\infty} i(d(x_{i+1}, x_i))^k = 0. \tag{2.9}$$

Here, (2.9) implies that there exists $n_1 \in \mathbb{N}$ such that $i\delta_i^k \leq 1$ for all $i \geq n_1$. As a result, for all $i \geq n_1$, we have

$$(d(x_{i+1}, x_i))^k \leq \frac{1}{i^{1/k}}. \tag{2.10}$$

On account of (2.10), we shall show that the recursive sequence $\{x_i\}$ is Cauchy. Consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Due to the estimation (2.10) together with the triangle inequality, we get that

$$d(x_m, x_n) \leq \delta_{m-1} + \delta_{m-2} + ... + \delta_n < \sum_{j=n}^{\infty} \delta_j \leq \sum_{j=n}^{\infty} \frac{1}{j^{1/k}}. \tag{2.11}$$

It is evident that the series $\sum_{j=n}^{\infty} \frac{1}{j^{1/k}}$ converges. Thus, $\{x_i\}$ is a Cauchy sequence. Owing to the completeness of (X, d), there exists $u \in X$ such that $\lim_{i\to\infty} x_i = x^*$.

in what follows, we show that x^* is a fixed point of $T^{n(x^*)}$. Indeed, due to the continuity of T, we have

$$d(Tx^*, x^*) = \lim_{i\to\infty} d(Tx_i, x_i) = \lim_{n\to\infty} d(x_{i+1}, x_i) = 0,$$

For the proving the uniqueness of the fixed point let us consider x^* and y^* be two distinct fixed point and $n = (x^*)$. So, we have $d(x^*, y^*) > 0$ and hence we get that

$$d(Tx^*, Ty^*) > 0 \Rightarrow \tau + F(d(Tx^*, Ty^*)) \leq F(K(x^*, y^*)), \tag{2.12}$$

where $K(x, y) = \max\{d(Tx^*, x^*), d(Ty^*, y^*)\} = 0$, a contradiction. □

Theorem 5. *Let (X, d) be a complete metric space and $T : X \to X$ a mapping which satisfies the condition: If there exists $F \in \mathcal{F}$ and $\tau > 0$ such that for each $x \in X$ there is a positive integer $n(x)$ such that for all $y \in X$*

$$d(T^{n(x)}(x), T^{n(x)}(y)) > 0 \Rightarrow \tau + F(d(T^{n(x)}(x), T^{n(x)}(y))) \leq F(N(x, y)), \tag{2.13}$$

where $N(x, y) := \alpha d(T^{n(x)}(x), x) + \beta d(y, T^{n(x)}(y))$ and α, β are non-negative number with $0 \leq \alpha + \beta < 1$. Then, T has a unique fixed point $z \in X$ and $T^n(x_0) \to z$ for each $x_0 \in X$, as $n \to \infty$.

Sketch of the proof. Note that $N(x, y) \leq K(x, y)$ for all x, y and F is strictly increasing. Thus, by Theorem , we conclude the desired result.

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