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On The Spectral Properties of Non- Self-Adjoint Elliptic Differential Operators in Hilbert space

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Abstract

The non-self-adjoint operators appear in many branches of science, from kinetic theory and quantum mechanics to linearizations of equations of mathematical physics. Non-self-adjoint operators are usually difficult to study because of the lack of general spectral theory. In this paper, our aim is to study the resolvent and the spectral properties of a class of non-self-adjoint differential operators.

Keywords: resolvent, asymptotic spectrum, distribution of eigenvalues, non-self-adjoint differential operator.
2010 MSC: 35P20, 35J99, 35P99.

1. Introduction

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$ (i.e., $\partial\Omega \in C^\infty$). We introduce the weighted Sobolev space $\mathcal{H} = W^2_{2,\alpha}(\Omega)$ as the space of complex value functions $u(x)$ defined on Ω with finite norm:

$$|u|_+ = \left(\sum_{i=1}^{n}\int_\Omega \sigma^{2\alpha}(x)|u'_{x_i}(x)|^2dx + \int_\Omega |u(x)|^2dx\right)^{1/2}$$

where $0 \le \alpha < 1$, and Here $\sigma(x)$ is weighted function, $\mu(x) \in C^2(\overline{\Omega})$. We denote by $\overset{\circ}{\mathcal{H}}$ the closure of $C_0^\infty(\Omega)$ in $\mathcal{H}$ with respect to the above norm. i.e., $\overset{\circ}{\mathcal{H}}$ is the closure of $C_0^\infty(\Omega)$ in $W^2_{2,\alpha}(\Omega)$. The notion $C_0^\infty(\Omega)$ stands for the space of infinitely differentiable functions with compact support in Ω. In this paper we investigate the

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spectral properties, in particular we estimate the resolvent of a non-self adjoint elliptic differential operator of type $(Au)(x) = -\sum_{i,j=1}^{n} \left(\sigma^{2\alpha}(x)a_{ij}(x)\mu(x)u'_{x_i}(x)\right)'_{x_j}$ acting in Hilbert space $H = L^2(\Omega)$. Now, we need to extend its domain as follows:

$$D(A) = \{y \in \overset{\circ}{H} \cap W^2_{2,loc}(\Omega) : \sum_{i,j=1}^{n} (\sigma^{2\alpha} a_{ij}\mu y'_{x_i})'_{x_j} \in H\}$$

(see cite2, [3]) where the local space $W^2_{2,\,loc}(\Omega)$ is the class of the functions $u(x)$ $(x \in \Omega)$ in this form

$$W^2_{2,loc}(\Omega) = \{u(x) : \ \sum_{i=0}^{2} \int_J |u^{(i)}(x)|^2 dx < \infty, \quad J \subset \Omega, \ \text{open}\}$$

Here $\sigma(x)$ is weighted function, $0 \le \alpha < 1$, $\mu(x) \in C^2(\bar{\Omega})$, $a_{ij}(x) \in C^2(\bar{\Omega})$,
$a_{ij}(x) = a_{ji}(x)$ and the functions $a_{ij}(x)$ satisfies the uniformly elliptic condition, i.e., there exists $c > 0$ such that: $c|s|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)s_i\overline{s_j}$ where $s = (s_1, \ldots, s_n) \in \mathbf{C}^n$, $x \in \Omega$. Assume that $\mu(x) \in \mathbf{C}\backslash\Phi$, $\forall x \in \overline{\Omega}$ where $\Phi = \{z \in \mathbf{C} : \ |arg\ z| \le \varphi\}$, $\varphi \in (0, \pi)$ (i.e., the value of $\mu(x)$ lie on the complex plane and outside of the closed angle Φ). To get a feeling for the history of the subject under study, refer to our earlier papers [9], [10]. Indeed this paper was written in continuing with earlier our papers, the paper is sufficiently more general than earlier our papers.

2. Main Results

Theorem 2.1. *Let* $(Au)(x) = -\sum_{i,j=1}^{n} \left(\sigma^{2\alpha}(x)a_{ij}(x)\mu(x)u'_{x_i}(x)\right)'_{x_j}$ *acting in Hilbert space* $H = L^2(\Omega)$ *with Dirichlet-type boundary conditions. Here* $\sigma(x)$ *is weighted function,* $0 \le \alpha < 1$, $\mu(x) \in C^2(\overline{\Omega})$, *Choose a closed sector* $S \subset \Phi$ *with its vertex at zero (for more explain see [8]), such that* $S \cap R_+ = \emptyset$. *Let the complex function* $\mu(x)$ *satisfies the following conditions*

$$\mu(x) \in C^1(\overline{\Omega}),\ \mu(x) \in \mathbf{C}\backslash S,\ (\forall x \in \overline{\Omega}), \tag{2.1}$$

$$|arg\{\mu(x_1)\mu^{-1}(x_2)\}| \le \frac{\pi}{8}, \quad (\forall\ x_1, x_2 \in \overline{\Omega}). \tag{2.2}$$

Then, for sufficiently large in modulus $\lambda \in S$, *the inverse operator* $(A - \lambda I)^{-1}$ *exists and is continuous in* H, *and the following estimates are valid*

$$\|(A - \lambda I)^{-1}\| \le M_S|\lambda|^{-1}\ (\lambda \in S,\ |\lambda| > C_S), \tag{2.3}$$

$$\|\sigma^\alpha \frac{\partial}{\partial x_i}(A - \lambda I)^{-1}\| \le M'_S|\lambda|^{-\frac{1}{2}}\ (\lambda \in S,\ |\lambda| > C_S), \tag{2.4}$$

$$\textit{for}\ i = 1, \ldots, n$$

where M_S, $C_S > 0$ *are sufficiently large numbers depending on* S. *The symbol* $\|.\|$ *stands for the norm of a bounded arbitrary operator* T *in* H.

Proof. Here, to establish Theorem 2.1, we will first prove the assertion of Theorem 2.1 together with estimate (2.3). So, as in Section 1 for the closed extension the operator A (for more explain see chapter 6 of [8]), we need to extend its domain to the closed set

$$D(A) = \{y \in \overset{\circ}{H} \cap W^2_{2,loc}(\Omega) : \sum_{i,j=1}^{n} (\sigma^{2\alpha} a_{ij}\mu y'_{x_i})'_{x_j} \in H\}$$

Let the operator A, now satisfy (2.1), (2.2). Then there exists a complex number $Z \in C$ (noticed that we can take $Z = e^{i\gamma}$, for a fix real $\gamma \in (-\pi, \pi]$), then we have $|Z = e^{i\gamma}| = 1$, and so

$$c' \le Re\{Z\mu(x)\},\ \ c'|\lambda| \le -Re\{Z\lambda\}, \quad c' > 0\ (\forall\ x \in \overline{\Omega},\ \lambda \in \Phi). \tag{2.5}$$

In view of the uniformly elliptic condition, we have

$$c|s|^2 = c\sum_{i=1}^{n}|s_i|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)s_i\overline{s_j},\ (c>0,\ s=(s_1,\ldots,s_n)\in \mathbf{C}^n,\ x\in\Omega)$$

take $s_i = y'_{x_i}$ implies that $c\sum_{i=1}^{n}|y'_{x_i}(x)|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)y'_{x_i}(x)\overline{y'_{x_j}(x)}$. From this, and according to $c' \leq Re\{Z\mu(x)\}$ in (2.4), we then multiply these two positive relations with each other implies that

$$c_1\sum_{i=1}^{n}|y'_{x_i}(x)|^2 \leq ReZ\mu(x)\sum_{i,j=1}^{n} a_{ij}(x)y'_{x_i}(x)\overline{y'_{x_j}(x)}.\text{for } y\in D(A)$$

Multiply both sides of the latter relation by the positive term $\sigma^{2\alpha}(x)$, and then integrate from both sides, we will have

$$c_1\sum_{i=1}^{n}\int_\Omega \sigma^{2\alpha}(x)|y'_{x_i}(x)|^2dx \leq ReZ\sum_{i,j=1}^{n}\int_\Omega \sigma^{2\alpha}(x)a_{ij}(x)\mu(x)y'_{x_i}(x)\overline{y'_{x_j}(x)}dx.$$

Now by applying the integration by parts, and using Dirichlet-type condition, then the right sides of the latter relation without multiple ReZ becomes:

$$\begin{aligned}
&\sum_{i,j=1}^{n}\int_\Omega \sigma^{2\alpha}(x)a_{ij}(x)\mu(x)y'_{x_i}(x)\overline{y'_{x_j}(x)}dx\\
&= -\sum_{i,j=1}^{n}\int_\Omega (\sigma^{2\alpha}(x)a_{ij}(x)\mu(x)y'_{x_i}(x))'_{x_j}\overline{y}(x)dx\\
&= (-\sum_{i,j=1}^{n}(\sigma^{2\alpha}(x)a_{ij}(x)\sigma(x)y'_{x_i}(x))'_{x_j}\ ,\ y(x)) = (Ay,y). \qquad (2.6)
\end{aligned}$$

Since $(Ay)(x) = -\sum_{i,j=1}^{n}\left(\sigma^{2\alpha}(x)a_{ij}(x)\mu(x)u'_{x_i}(x)\right)'_{x_j}$. Here, the the symbol (,) denotes the inner product in H. Notice that the above equality in (2.6) obtains by the well known theorem of the m-sectorial operators which are closed by extending its domain to the closed domain in $\mathcal{H}$. These operators are associated with the closed sectorial bilinear forms that are densely defined in $\mathcal{H}$ (for more explanation see the well known Theorem 2.1, chapter 6 of [8]). The reason we extend the domain of the operator A to the closed domain in space $\mathcal{H}$, above is now specified.
therefore

$$c_1\sum_{i=1}^{n}\int_\Omega \sigma^{2\alpha}(x)|y'_{x_i}(x)|^2dx \leq ReZ(Ay\ ,\ y)$$

from (2.4) we have: $c'|\lambda| \leq -Re\{Z\lambda\},\ c'>0,\ \forall\lambda\in\Phi$. Multiply this inequality by $\int_\Omega |y(x)|^2\,dx = (y\ ,\ y) = \|y\|^2 > 0$. It follows that $c'|\lambda|\int_\Omega |y(x)|^2\,dt \leq -Re\{Z\lambda\}(y\ ,\ y)$. From this and the above inequality we will have

$$\begin{aligned}
c_1\sum_{i=1}^{n}\int_\Omega \sigma^{2\alpha}(x)|y'_{x_i}(x)|^2dx + c'|\lambda|\int_\Omega |y(x)|^2dx &\leq Re\{Z(Ay,y) - Z\lambda(y,y)\}\\
&= Re\{Z((A-\lambda I)y,y)\}\\
&\leq \|Z\|\|y\|\|(A-\lambda I)y\|\\
&= \|y\|\|(A-\lambda I)y\|; \quad (2.7)
\end{aligned}$$

i.e.,

$$c_1 \sum_{i=1}^{n} \int_\Omega \sigma^{2\alpha}(x)|y'_{x_i}(x)|^2 dx + c'|\lambda| \int_\Omega |y(x)|^2 \, dx \le \|y\| \, \|(A \ - \ \lambda \, I)y\|.$$

Since $c_1 \sum_{i=1}^{n} \int_\Omega \sigma^{2\alpha}(x)|y'_{x_i}(x)|^2 dx$ is positive. We will have either $c'|\lambda| \|y(x)\|^2 = |\lambda| \int_\Omega |y(x)|^2 \, dx \le \|y\| \|(A \ - \lambda \, I)y\|$. Or

$$|\lambda| \|y(x)\| \le M_S \|(A \ - \lambda \, I)y\|. \tag{2.8}$$

This inequity ensures that the operator $(A \ - \ \lambda \, I)$ is one to one, which implies that $ker(A \ - \ \lambda \, I) = 0$. Therefore the inverse operator $(A - \ \lambda I)^{-1}$ exists, and its continuity follows from the proof of the estimate (2.3) of Theorem 2.1. To prove (2.3), we set $v = (A - \lambda I)^{-1} f, \ f \in H$ in (2.7) implies that

$$|\lambda| \int_\Omega |(A - \lambda I)^{-1} f|^2 \, dx \le M_S \|(A - \ \lambda I)^{-1} f\| \|(A \ - \lambda I)(A - \lambda I)^{-1} f\|.$$

Since $(A \ - \ \lambda \, I)(A - \lambda I)^{-1} f = I(f) = f$. Then

$$|\lambda| \int_\Omega |(A \ - \lambda I)^{-1} f|^2 \, dx \le M_S \|(A \ - \lambda I)^{-1} f\| |f|.$$

So

$$|\lambda| \|(A \ - \lambda I)^{-1}(f)\|^2 \le M_S \|(A \ - \lambda I)^{-1}(f)\| |f|.$$

Which this implies that $|\lambda| \|(A \ - \lambda I)^{-1}(f)\| \le M_S |f|$. Since $\lambda \ne 0$. Then $\|(A - \lambda I)^{-1}(f)\| \le M_S |\lambda|^{-1} |f|$; i.e., $\|(A - \lambda I)^{-1}\| \le M_S |\lambda|^{-1}$. This estimate completes the proof of the assertion of Theorem 2.1 together with the estimate (2.3). Now, we start to prove the estimate (2.4) of Theorem 2.1 As in the above argument, we drop the positive term $c'|\lambda| \int_\Omega |y(x)|^2 \, dx$ from

$$c_1 \sum_{i=1}^{n} \int_\Omega \sigma^{2\alpha}(x)|y'_{x_i}(x)|^2 dx + c'|\lambda| \int_\Omega |y(x)|^2 \, dx \le \|y\| \, \|(A \ - \ \lambda \, I)y\|.$$

It follows that

$$c_1 \sum_{i=1}^{n} \int_\Omega \sigma^{2\alpha}(x)|y'_{x_i}(x)|^2 dx \le \|y\| \, \|(A \ - \ \lambda \, I)y\|.$$

Eminently,

$$c_1 \|\sigma^\alpha \frac{\partial}{\partial x_i}(A - \lambda I)^{-1} f\|^2 \le \|y\| \, \|(A \ - \ \lambda \, I)y\|.$$

Set $y = (A - \lambda I)^{-1} f, \ f \in H$ in the latter relation, and proceeding by similar calculation as in the proof (2.3) we then obtain:

$$c_1 \|\sigma^\alpha \frac{\partial}{\partial x_i}(A - \lambda I)^{-1} f\|^2 \le \|(A - \lambda I)^{-1} f\| \, \|(A \ - \ \lambda \, I)(A - \lambda I)^{-1} f\|.$$

Since $(A \ - \ \lambda \, I)(A - \lambda I)^{-1} f = I(f) = f$, then

$$c_1 \|\sigma^\alpha \frac{\partial}{\partial x_i}(A - \lambda I)^{-1} f\|^2 \le \|(A - \lambda I)^{-1}\| \|f\|^2,$$

consequently, by (2.3) this implies that

$$c_1 \|\sigma^\alpha \frac{\partial}{\partial x_i}(A - \lambda I)^{-1} f\|^2 \le M_S |\lambda|^{-1} \|f\|^2$$

to this end we will have

$$\|\sigma^\alpha \frac{\partial}{\partial x_i}(A - \lambda I)^{-1}\| \le M'_S |\lambda|^{-\frac{1}{2}}. \tag{2.7}$$

Thus, here the proof of the estimate (2.4) is finished; i.e., this completes the proof of Theorem 2.1 □

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