Conference Proceedings of Science and Technology, 2(2), 2019, 104–109

Conference Proceeding of 17th International Geometry Symposium (2019) Erzincan/Turkey.

On Quasi-Einstein Manifolds Admitting Space-Matter Tensor

ISSN: 2651-544X

http://dergipark.gov.tr/cpost

Amit Kumar Debnath[1] Sanjib Kumar Jana[2] Fusun Nurcan[3] Joydeep Sengupta[4]*

[1] *Department of Mathematics, University of Gour Banga, West Bengal, India, ORCID:0000-0001-5398-1955.*
[2] *Department of Mathematics, Vidyasagar College, West Bengal, India, ORCID:0000-0002-4221-8249.*
[3] *Department of Mathematics, Marmara Universty, Istanbul, Turkey, ORCID:0000-0003-0146-992X.*
[4] *Department of Mathematics, Alia University, West Bengal, India, ORCID:0000-0002-1609-0798.*
** Corresponding Author E-mail: fnurcan622@gmail.com*

Abstract: The subject matter of this paper lies in the interesting domain of Differential Geometry and the Theory of General Relativity. Although the space has its motivation in Relativity, we study the geometric properties of the space, inspired by the papers on the geometry related to curvature restrictions. Such a study was joined by A. Z. Petrov to Einstein spaces. We extend the study on quasi-Einstein spaces which can be considered as a generalization of Einstein spaces. This study is supported by an example.

Keywords: Einstein's field equation, Quasi-Einstein manifold, Scalar curvature, Space-Matter tensor.

1 Introduction

In 1949, the celebrated theorem [8] showing the existence of three types of Einstein space with signature (-, -, -, +) and the corresponding three canonical forms were established. During this study, the gravitation fields are classified on the basis of the algebraic structures of the space-matter tensor. A. Z. Petrov introduced a tensor field P of type (0, 4) and defined as follows:

$$P = R + \frac{k}{2}\, g \wedge T - \sigma G, \tag{1}$$

where R is the Riemannian curvature tensor of type (0, 4), T is the energy-momentum tensor of type (0, 2), k is a cosmological constant, σ is the energy density (scalar), G is a tensor of type $(0,\ 4)$ given by

$$G(X,\ Y,\ Z,\ U) = g(X,\ U)g(Y,\ Z) - g(X,\ Z)g(Y,\ U) \tag{2}$$

for all $X,\ Y,\ Z,\ U \in \chi(M)$, $\chi(M)$ being the Lie algebra of smooth vector fields on M and the Kulkarni-Nomizu product $E \wedge F$ of two $(0,\ 2)$ tensors E and F is defined by

$$\begin{aligned}(E \wedge F)(X_1,\ X_2,\ X_3,\ X_4) &= E(X_1,\ X_4)F(X_2,\ X_3) + E(X_2,\ X_3)F(X_1,\ X_4)\\ &- E(X_1,\ X_3)F(X_2,\ X_4) - E(X_2,\ X_4)F(X_1,\ X_3),\end{aligned}$$

$X_i \in \chi(M)$, $i = 1,\ 2,\ 3,\ 4$. The tensor P is known as the space-matter tensor of type (0, 4) of the manifold M.
Einstein's field equation with cosmological constant is given by

$$kT = S + \left(\lambda - \frac{r}{2}\right) g, \tag{3}$$

where λ is a cosmological constant, r is the scalar curvature and S is the Ricci tensor of type (0, 2) and r is the scalar curvature defined by the following equations:

$$S(X,Y) = g(QX,Y) = \sum_{i=1}^{n} R(e_i,\ X,\ Y,\ e_i) \quad , \ r = \sum_{i=1}^{n} S(e_i,\ e_i) = \sum_{i=1}^{n} g((QE)_i,\ e_i),$$

where $\{e_i : i = 1,\ 2,\ \ldots,\ n\}$ be an orthonormal basis of the tangent space at any point of the manifold and Q denotes the symmetric endomorphism corresponding to the Ricci tensor S.

(1) takes the form

$$P = R + \frac{1}{2} g \wedge S - \left(\sigma - \lambda + \frac{r}{2}\right) G, \tag{4}$$

by virtue of (3).

If the energy momentum tensor is of Codazzi type and the energy density is constant in our manifold, then the space-matter tensor satisfies the second Bianchi identity [7] i.e.,

$$(\nabla_X P)(Y,\ Z,\ U,\ V) + (\nabla_Y P)(Z,\ X,\ U,\ V) + (\nabla_Z P)(X,\ Y,\ U,\ V) = 0. \tag{5}$$

We know that the space matter tensor comprises of two parts. First part deals with the curvature of the space and the remaining on the motion of the matter. Our investigation focused on attributing different constraints on space-matter tensor-their admissibility, and if admissible the after effects. As our intention is purely geometric, we investigate the corresponding change in its scalar curvature.

Section 2 deals with several properties of space-matter tensor P satisfying certain curvature conditions. In the last section an example proving the existence of a quasi-Einstein manifold with space-matter tensor is given.

2 Quasi-Einstein manifold with space-matter tensor

This section is concerned with a quasi-Einstein manifold with space-matter tensor satisfying certain curvature conditions. A Riemannian manifold $(M^n,\ g)$ is said to be quasi-Einstein ([1], [2]-[6]) if its Ricci tensor S is not identically zero and if there exists two real valued functions α_1 and $\alpha_2 (\neq 0)$ and a smooth unit 1-form π on M, such that the Ricci tensor S satisfies

$$S = \alpha_1 g + \alpha_2 \pi \otimes \pi. \tag{6}$$

The vector field ς, which is metrically equivalent to the unit 1-form π, is called the generator of the manifold. Such a manifold of dimension n is usually denoted by $(QE)_n$. The scalars α_1, α_2 are known as the associated scalars.

Firstly let us consider a Riemannian manifold $(M^n,\ g)$ $(n > 3)$ admitting Einstein's field equation in which the space-matter tensor P of type $(0,\ 4)$ is recurrent i.e.,it satisfies the relation

$$(\nabla_X P)(Y,\ Z,\ U,\ V) = L(X) P(Y,\ Z,\ U,\ V), \tag{7}$$

where L is the non-zero 1-form of recurrence and ρ be the vector field associated with L. In view of the above relation, (4) is reduced to the following equation

$$\begin{aligned} & 2(\nabla_X R)(Y,\ Z,\ U,\ V) + (\nabla_X S)(Y,\ V) g(Z,\ U) + (\nabla_X S)(Z,\ U) g(Y,\ V) \\ & -(\nabla_X S)(Y,\ U) g(Z,\ V) - (\nabla_X S)(Z,\ V) g(Y,\ U) - [2d\sigma(X) + dr(X)] G(Y,\ Z,\ U,\ V) \\ = \ & L(X)[2R(Y,\ Z,\ U,\ V) + (g \wedge S)(Y,\ Z,\ U,\ V) - 2(\sigma + \frac{r}{2} - \lambda) G(Y,\ Z,\ U,\ V)]. \end{aligned} \tag{8}$$

Taking contraction of (8) with respect to Y and V, we obtain

$$\begin{aligned} & n(\nabla_X S)(Z,\ U) - \{(n-2)dr(X) + 2(n-1)d\sigma(X)\} g(Z,\ U) \\ = \ & L(X)[nS(Z,\ U) - \{(n-2)r + 2(n-1)(\sigma - \lambda)\} g(Z,\ U)]. \end{aligned} \tag{9}$$

Setting $Z = U = e_i$ in (9) and taking summation over i, $1 \leq i \leq n$, we find

$$dq(X) = qL(X), \tag{10}$$

where $q = (n-3)r + 2(n-1)(\sigma - \lambda)$. If q is a constant then the only possible value of q is zero, since $L(X) \neq 0$. Hence we have

$$r = \frac{2(n-1)}{(n-3)}(\lambda - \sigma), \ \text{ since } \ L(X) \neq 0. \tag{11}$$

Putting $X = \rho$ in (10) and by the virtue of (6), we get

$$(n-3)[nd\alpha_1(\rho) + d\alpha_2(\rho)] + 2(n-1)d\sigma(\rho) = 2(n-1)(\sigma - \lambda) + (n-3)(n\alpha_1 + \alpha_2). \tag{12}$$

Hence we get the following:

Theorem 1. *The scalars ρ, σ, λ, α_1, α_2 are connected by the relation (12) in a $(QE)_n$ $(n > 3)$ admitting Einstein's field equation and recurrent space-matter tensor.*

In the view of (6), (11) is reduced to the following form

$$(n-3)(n\alpha_1 + \alpha_2) + 2(n-1)(\sigma - \lambda) = 0. \tag{13}$$

This leads to the following:

Theorem 2. *If $\alpha_1,\ \alpha_2$ and σ are constants in a $(QE)_n$ $(n > 3)$ admitting Einstein's field equation and recurrent space-matter tensor, then they are connected by the relation* (13).

Again in the view of (5), the relation (7) reduces to

$$L(X)P(Y,\ Z,\ U,\ V) + L(Y)P(Z,\ X,\ U,\ V) + L(Z)P(X,\ Y,\ U,\ V) = 0.$$

Setting $Y = V = e_i$ in the above relation and taking summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned}&\{2(n-1)(\lambda-\sigma)-(n-2)r\}[L(X)g(Z,\ U)-L(Z)g(X,\ U)]\\&+n[L(X)S(Z,\ U)-L(Z)S(X,\ U)]+2L(P(Z,\ X)U)=0,\end{aligned}$$

by the virtue of (4). Now contracting the above relation with respect to X and U we obtain

$$L(QZ) = \frac{r_0}{2n}L(Z), \text{ which yields } S(Z,\ \rho) = \frac{r_0}{2n}g(Z,\ \rho), \tag{14}$$

where $r_0 = (n-1)[2(n-2)(\lambda-\sigma)-(n-4)r]$ and $g(X,\rho) = L(X)$.
By the virtue of (6) and (14), we get

$$L(QZ) = \frac{n-1}{2n}[2(n-2)(\lambda-\sigma)-(n-4)(n\alpha_1+\alpha_2)]L(Z),$$

which yields

$$S(Z,\ \rho) = \frac{n-1}{2n}[2(n-2)(\lambda-\sigma)-(n-4)(n\alpha_1+\alpha_2)]g(Z,\ \rho).$$

This gives the following:

Theorem 3. *If the energy-momentum tensor is of Codazzi type in a $(QE)_n$ $(n > 3)$ admitting Einstein's field equation and recurrent space-matter tensor, then $\frac{n-1}{2n}[2(n-2)(\lambda-\sigma)-(n-4)(n\alpha_1+\alpha_2)]$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ, defined by $g(X,\ \rho) = L(X)$, provided that the energy density is constant.*

Now we assume that the space-matter tensor P of type (0, 4) in a Riemannian manifold $(M^n,\ g)$ $(n > 3)$ admitting Einstein's field equation, is weakly symmetric [9] in nature. Then there exist 1-forms A, B and E (not simultaneously zero) such that the following relation holds:

$$\begin{aligned}(\nabla_X P)(Y,\ Z,\ U,\ V) &= A(X)P(Y,\ Z,\ U,\ V)+B(Y)P(X,\ Z,\ U,\ V)\\&\quad+B(Z)P(Y,\ X,\ U,\ V)+E(U)P(Y,\ Z,\ X,\ V)\\&\quad+E(V)P(Y,\ Z,\ U,\ X).\end{aligned}\tag{15}$$

Here ρ_1, ρ_2, ρ_3 be the vector fields metrically equivalent to A, B, E respectively.
Contracting Y and V and using (4), the equation (15) yields

$$\begin{aligned}&\frac{n}{2}(\nabla_X S)(Z,\ U)-\left[\frac{n-2}{2}dr(X)+(n-1)d\sigma(X)\right]g(Z,\ U)\\=\ &A(X)\left[\frac{n}{2}S(Z,\ U)-\left\{\frac{n-2}{2}r+(n-1)(\sigma-\lambda)\right\}g(Z,\ U)\right]\\&+B(Z)\left[\frac{n}{2}S(X,\ U)-\left\{\frac{n-2}{2}r+(n-1)(\sigma-\lambda)\right\}g(X,\ U)\right]\\&+E(U)\left[\frac{n}{2}S(Z,X)-\left\{\frac{n-2}{2}r+(n-1)(\sigma-\lambda)\right\}g(Z,X)\right]\\&+P(X,Z,U,\rho_2)+P(X,U,Z,\rho_3).\end{aligned}\tag{16}$$

Setting $Z = U = e_i$ in (16) and taking summation over i, $1 \leq i \leq n$, we have

$$\begin{aligned}&n(n-3)dr(X)+2n(n-1)d\sigma(X)+2nH_1(QX)\\=\ &n[(n-3)r+2(n-1)(\sigma-\lambda)]A(X)+2[(n-2)r+2(n-1)(\sigma-\lambda)]H_1(X)\end{aligned}\tag{17}$$

by the virtue of (4), where $H_1(X) = B(X)+E(X)$ for all vector fields X.
Contracting (16) with respect to $X,\ Z$ and replacing U by X and by the virtue of (4), we find

$$\begin{aligned}\frac{n-4}{4}dr(X)\ &+\ (n-1)d\sigma(X)+\frac{n}{2}[A(QX)+B(QX)-E(QX)]\\&=\ \left\{\frac{n-2}{2}r+(n-1)(\sigma-\lambda)\right\}[A(X)+B(X)]\\&+\ \left\{\frac{n^2-4n+2}{2}r+(n-1)^2(\sigma-\lambda)\right\}E(X).\end{aligned}\tag{18}$$

Further contracting (16) with respect to $X,\ U$ and replacing Z by X, we find

$$\begin{aligned} & \frac{n-4}{4}dr(X)+(n-1)d\sigma(X)+\frac{n}{2}[A(QX)-B(QX)+E(QX)] \\ = & \left\{\frac{n-2}{2}r+(n-1)(\sigma-\lambda)\right\}[A(X)+E(X)]+\left\{\frac{n^2-4n+2}{2}r+(n-1)^2(\sigma-\lambda)\right\}B(X) \end{aligned} \tag{19}$$

by the virtue of (4). Now (18) and (19) both yield

$$H_2(QX)=-\frac{n-1}{2n}[(n-4)r+2(n-2)(\sigma-\lambda)]H_2(X), \tag{20}$$

which gives

$$S(X,\ \tau_2)=\frac{n-1}{2n}[2(n-2)(\lambda-\sigma)-(n-4)r]g(X,\ \tau_2),$$

where $g(X,\ \tau_2)=H_2(X)=B(X)-E(X)$ for all vector fields X.

Now using (18) and (20), we obtain

$$\begin{aligned} & (n-4)dr(X)+4(n-1)d\sigma(X)+2nA(QX) \\ = & 2[(n-2)r+2(n-1)(\sigma-\lambda)]A(X)+n[(n-3)r+2(n-1)(\sigma-\lambda)]H_1(X). \end{aligned} \tag{21}$$

In the view of (17), (6) and (21), we get

$$\begin{aligned} & (n^2-2n-4)[nd\alpha_1(X)+d\alpha_2(X)]+2(n-1)(n+2)d\sigma(X) \\ = & [(n^2-n-4)(n\alpha_1+\alpha_2)+2(n-1)(n+2)(\sigma-\lambda)]H_3(X)-2nH_3(QX), \end{aligned}$$

where $H_3(X)=g(X,\tau_3)=A(X)+B(X)+E(X)$ for all vector fields X. If $\alpha_1,\ \alpha_2,\ \sigma$ are constants then we get from the above relation

$$H_3(QX)=\frac{1}{2n}[(n^2-n-4)(n\alpha_1+\alpha_2)+2(n-1)(n+2)(\sigma-\lambda)]H_3(X),$$

which yields

$$S(X,\tau_3)=\frac{1}{2n}[(n^2-n-4)(n\alpha_1+\alpha_2)+2(n-1)(n+2)(\sigma-\lambda)]g(X,\tau_3).$$

Again by (17), (6) and (21), we also get

$$\begin{aligned} & (n^2-4n+4)[nd\alpha_1(X)+d\alpha_2(X)]+2(n-1)(n-2)d\sigma(X) \\ = & 2nH_4(QX)+(n-1)[(n-4)(n\alpha_1+\alpha_2)+2(n-2)(\sigma-\lambda)]H_4(X), \end{aligned}$$

where $H_4(X)=g(X,\tau_4)=A(X)-B(X)-E(X)$ for all vector fields X. If $\alpha_1,\ \alpha_2,\ \sigma$ are constants, then the above relation gives us

$$H_4(QX)=\frac{1}{2n}(n-1)[2(n-2)(\lambda-\sigma)-(n-4)(n\alpha_1+\alpha_2)]H_4(X),$$

which yields

$$S(X,\tau_4)=\frac{n-1}{2n}[2(n-2)(\lambda-\sigma)-(n-4)(n\alpha_1+\alpha_2)]g(X,\tau_4).$$

Thus we have the following:

Theorem 4. *$\frac{1}{2n}[(n^2-n-4)(n\alpha_1+\alpha_2)+2(n-1)(n+2)(\sigma-\lambda)]$ and $\frac{n-1}{2n}[2(n-2)(\lambda-\sigma)-(n-4)(n\alpha_1+\alpha_2)]$ are the eigenvalues of the Ricci tensor S corresponding to the eigenvectors τ_3 and τ_4, respectively, in a $(QE)_n$ $(n>3)$ admitting Einstein's field equation and weakly symmetric space-matter tensor, provided $\alpha_1,\ \alpha_2,\ \sigma$ are constants.*

3 An example proving the existence of a Quasi-Einstein Manifold with Space-Matter Tensor

This section deals with an example of a quasi-Einstein manifold admitting Einstein's field equation and space-matter tensor satisfying certain curvature restrictions.

Example 1: Let $M^4 = \{(x^1,\ x^2,\ x^3,\ x^4) \in \mathbb{R}^4\}$ be an open subset of $\mathbb{R}^4$ endowed with the metric

$$ds^2 = g_{ij}dx^i dx^j = e^{x^1+1}(dx^1)^2 + e^{x^1}[(dx^2)^2 + (dx^3)^2 + (dx^4)^2], \tag{22}$$

where i, j run from 1 to 4. Then the only non-vanishing components of the curvature tensor, the Ricci tensor and the scalar curvature are given by

$$R_{2323} = -\frac{1}{4}e^{x^1-1} = R_{2424} = R_{3434};\ S_{22} = \frac{1}{2e} = S_{33} = S_{44};\ r = \frac{3}{2}e^{-(x^1+1)}. \tag{23}$$

We shall now verify that our M^4 is a quasi-Einstein manifold. To verify that the manifold M^4 is a $(QE)_4$ let us consider the 1-form π and associated scalars α_1, α_2 as follows:

$$\pi(\frac{\partial}{\partial x^i}) \quad = \quad \pi_i = \begin{cases} e^{\frac{x^1+1}{2}} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{24}$$

$$\alpha_1 = \frac{1}{2}e^{-(x^1+1)};\ \alpha_2 = -\frac{1}{2}e^{-(x^1+1)}. \tag{25}$$

According to our M^4, (6) is reduced to following equations

$$S_{ii} = \alpha_1 g_{ii} + \alpha_2 \pi_i \pi_i, \qquad \text{for } i = 1,\ 2,\ 3,\ 4, \tag{26}$$

since the components of (6) vanishes identically for the cases other than (26) and the relation (6) holds trivially. By the virtue of (22), (23), (24), (25), it follows that
r. h. s. of (26) $= \frac{1}{2}e^{-(x^1+1)}g_{ii} - \frac{1}{2}e^{-(x^1+1)}\pi_i\pi_i = 0 =$ l. h. s. of (26) for $i = 1$. By a similar argument, it can be easily shown that the relation (26) holds for $i = 2,\ 3,\ 4$. Therefore our $(M^4,\ g)$ is a $(QE)_4$. Now, considering σ as a constant, we calculate the non-vanishing components of the space-matter tensor and its covariant derivatives as follows:

$$\left\{ \begin{array}{c} P_{1212} = -\frac{1}{2}e^{x^1}[-1 + 2e^{x^1+1}(\lambda - \sigma)] = P_{1313} = P_{1414}, \\ P_{2323} = -e^{2x^1}(\lambda - \sigma) = P_{2424} = P_{3434}; \\ P_{1212,1} = e^{x^1}[-1 + 2e^{x^1+1}(\lambda - \sigma)] = P_{1313,1} = P_{1414,1}, \\ P_{2323,1} = 2e^{2x^1}(\lambda - \sigma) = P_{2424,1} = P_{3434,1}, \\ P_{1323,2} = -\frac{1}{4}e^{x^1-1} = P_{1424,2} = P_{1232,3} = P_{1434,3} = P_{1242,4} = P_{1343,4}; \end{array} \right. \tag{27}$$

where ‘,’ denotes the covariant differentiation with respect to the coordinates. Let us consider the 1-form L as follows:

$$L(\frac{\partial}{\partial x^i}) = L_i = \begin{cases} -2 & \text{for } i = 1 \\ 0 & \text{otherwise} \end{cases} \tag{28}$$

at any point of M. One can easily check the validity of the following relations with these 1-forms defined above:

$$P_{1j1j,1} = L_1 P_{1j1j}, \tag{29}$$
$$P_{jkjk,1} = L_1 P_{jkjk}, \tag{30}$$
$$P_{1jkj,k} = L_k P_{1jkj}, \tag{31}$$

where j and k run from 2 to 4 and $j \neq k$. We simply show one of the above here:

$$\text{r. h. s. of (29)} = -2P_{1j1j} = e^{x^1}[-1 + 2e^{x^1+1}(\lambda - \sigma)] = \text{l. h. s. of (29) for } j = 2.$$

In other possible cases either the result is trivial or both sides vanish identically. Here it can also be easily shown that our considered manifold is a recurrent manifold. Hence we can state the following theorem:

Theorem 5. *Let (M^4, g) be a Riemannian manifold endowed with the metric*

$$ds^2 = g_{ij}dx^i dx^j = e^{x^1+1}(dx^1)^2 + e^{x^1}[(dx^2)^2 + (dx^3)^2 + (dx^4)^2] \text{ where } (i,\ j = 1,\ 2,\ 3,\ 4).$$

Then $(M^4,\ g)$ is a $(QE)_4$ admitting Einstein's field equation and recurrent space-matter tensor with non-vanishing scalar curvature such that σ is constant. It is also a recurrent manifold.

4 Conclusion

In this article we provide several properties of space-matter tensor P on quasi-Einstein manifolds. Our future research will be focused on the study of space-matter tensor P on some other kind of manifolds.

Acknowledgement

The authors are thankful to the referee for his/her valuable comments and suggestions towards the improvement of the paper.

5 References

[1] M. C. Chaki, R. K. Maity, *On quasi-Einstein manifolds*, Publ. Math. Debrecen, **57** (2000), 297-306.
[2] F. Defever, R. Deszcz, M. Hotlos, M. Kucharski, Z. Senturk, *Generalisations of Robertson-Walker spaces*, Annales Univ. Sci. Budapest. Eötvös Sect. Math., **43** (2000), 13-24.
[3] R. Deszcz, M. Hotlos, *On some pseudosymmetry type curvature condition*, Tsukuba J. Math., **27** (2003), 13-30.
[4] R. Deszcz, M. Hotlos, Z. Senturk, *Quasi-Einstein hypersurfaces in semi-Riemannian space forms*, Colloq. Math., **81** (2001), 81-97.
[5] R. Deszcz, M. Hotlos, Z. Senturk, *On curvature properties of quasi-Einstein hypersurfaces in semi-Euclidean spaces*, Soochow J. Math., **27**(4) (2001), 375-389.
[6] R. DESZCZ, VERHEYEN, P. and VERSTRAELEN, L.: *On some generalized Einstein metric conditions*, Publ. Inst. Math. (Beograd), **60:74** (1996), pp. 108-120.
[7] S. K. Jana, A. K. Debnath, J. Sengupta: *On Riemannian manifolds satisfying certain curvature conditions*, Bulletin of Natural and Mathematical Sciences, Russia, **30**(2) (2013), 40-61.
[8] A. Z. Petrov, *Einstein Spaces*, Pergamon Press, Oxford, 1949.
[9] L. Tamassy, T. Q. Binh, *On weakly symmetric and weakly projective symmetric Rimannian manifolds*, Coll. Math. Soc., J. Bolyai **50** (1989), 663-670 .