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On Special Curves of General Hyperboloid in E_1^3

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Abstract:
In this work, we give the Darboux vectors $\{\gamma(s), T(s), Y(s)\}$ of a given curve using the hyperbolically motion and hyperbolically inner product defined by Simsek and Özdemir in [9]. Then, we present the variations of the geodesic curvature function $\kappa_g(s,w)$ and the speed function $v(s,w)$ of the curve γ at $w=0$. Also, we define the new type curves whose Darboux frame vectors of a given curve makes a constant angle with the constant Killing vector field and also we obtain the parametric characterizations of these curves. At the end of this article, we exemplify these curves on the general hyperboloid with their figures using the program Mathematica.

Keywords: General hyperboloid, Special curves, Lorentzian space, Darboux frame.

1 Introduction

The helix is known as a curve in DNA double, carbon nanotubes, form of plants. In the geometry, the definition of helix is a curve that its tangent vector field makes a constant angle with a fixed straight line called the axis of the helix. The study of these curves in the 3-dimensional Lorentzian space forms in [4]. On the other hand, the slant helices are studied by Izumiya and Takeuchi. If the curve is called as a slant helix, then its principal normal vector make a constant angle with a fixed direction [6]. In [8], the authors show that the path of a charged particle moves in a static magnetic field in 3D Riemannian space is the circular helix or slant helix path. Another of the important curves in geometry is the spherical curves. Thus, so many authors study in this field (see for details in [2, 3, 7]). In [10, 11], the authors give a characterization for a curve to be on a sphere. Breuer et al. obtain an explicit characterization of the spherical curve [5].

In this study, we summarize the some basic notations of general hyperbolical space which are defined by Simsek and Özdemir in [9]. In addition, we give the features of variation vector field along a curve as well as the variational formulas for its Darboux curvatures in the third section. The connection between the geometric variational formulas for curvatures and the Killing equations along a space curve according to Darboux frame. Then, we define the special curves whose Darboux frame vectors of a given curve makes a constant angle with the constant Killing vector field. Also, we generate the parametric representations of all kind of helices on the general hyperboloid $\mathcal{GH}$ and illustrate these curves on the general hyperboloid with their figures using the Mathematica program language.

2 Preliminaries

In [9], the $g-$hyperbolic $2-$space and the $g-$de Sitter $2-$space are defined as

$$\begin{aligned} H^{2,1}_{a_1,a_2,a_3} &= \left\{ u=(x,y,z)\in\mathbb{R}^3 \,\middle|\, -a_1x^2+a_2y^2+a_3z^2=-1 \right\}, \\ S^{2,1}_{a_1,a_2,a_3} &= \left\{ u=(x,y,z)\in\mathbb{R}^3 \,\middle|\, -a_1x^2+a_2y^2+a_3z^2=1 \right\}, \end{aligned}$$

respectively. The scalar product on the $g-$hyperbolic $2-$space and the $g-$de Sitter $2-$space of $\mathbb{R}^3$ with the cartesian equation $-a_1x^2+a_2y^2+a_3z^2=\pm1$, $a_1,a_2,a_3\in\mathbb{R}^+$ is defined as

$$g:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R};\ g(u,v)=-a_1x_1y_1+a_2x_2y_2+a_3x_3y_3$$

where $u=(u_1,u_2,u_3)$, $v=(v_1,v_2,v_3)\in\mathbb{R}^3$. The real vector space $\mathbb{R}^3$ equipped with the hyperbolical $g-$inner product will be represented by $\mathbb{R}^{2,1}_{a_1,a_2,a_3}$. The norm of a vector associated with the scalar product g is defined as

$$\|u\|_g=\sqrt{|g(u,u)|}.$$

Two vectors u and v are called hyperbolically orthogonal vectors if $g(u,v)=0$. If u is a hyperbolically orthonormal vector then $g(u,u)=1$. The hyperbolic angle of the between two timelike vectors u and v on the same timecone is given by

$$\cosh\theta = \frac{-g(u,v)}{\|u\|_g \|v\|_g}$$

where θ is compatible with the parameters of the angular parametric equations of pseudo-spheres.

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be a standard unit vectors of $\mathbb{R}^3$. The $g-$vector product of the vector fields $u, v \in \mathbb{R}^3$ is described as

$$\mathcal{V}_g(u\times v) = \Delta^* \begin{vmatrix} -\mathbf{e}_1/a_1 & \mathbf{e}_2/a_2 & \mathbf{e}_3/a_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}, \tag{1}$$

where $\Delta^* = \sqrt{a_1a_2a_3}$, $a_1, a_2, a_3 \in \mathbb{R}^+$ (*see for details* in [9]).

Let's take the general hyperboloid $-a_1x^2+a_2y^2+a_3z^2=\pm1$. The sectional curvature of the hyperboloid generated by the non-degenerated plane $\{u,v\}$ is defined as

$$K(u,v) = \frac{g(R(u,v)u,v)}{g(u,u)g(v,v)-g(u,v)^2} \tag{2}$$

where R is the Riemannian curvature tensor given by

$$R(X,Y)Z = -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z \tag{3}$$

where $X, Y, Z \in \chi(H^{2,1}_{a_1,a_2,a_3})$ or $\chi(S^{2,1}_{a_1,a_2,a_3})$. The general hyperboloid has the constant sectional curvature. Therefore, the curvature tensor R written as follows

$$R(X,Y)Z = \mathcal{C}\{g(Z,X)Y - g(Z,Y)X\} \tag{4}$$

where $\mathcal{C}$ is the constant sectional curvature.

Let γ be a curve with arc length parameter s on the general hyperboloid ($\mathcal{GH}$). To calculate the Darboux frame apparatus $\{\gamma(s), T(s), Y(s), \kappa_g(s)\}$ along the curve γ on the hyperboloid surface, firstly we will calculate the unit normal vector field of the general hyperboloid. The surfaces $H^{2,1}_{a_1,a_2,a_3}$ and $S^{2,1}_{a_1,a_2,a_3}$ are Lorentzian spheres according to the hyperbolical inner product, the unit normal vector field along the general hyperboloid equal to the position vector of the curve γ. Then we found an orthonormal frame $\{\gamma(s), T(s)=\gamma'(s), Y(s)=\mathcal{V}_g(\gamma(s)\times T(s))\}$ which is called the hyperbolical Darboux frame along the curve γ. The corresponding Darboux formulas of the curve γ is written as

$$\begin{aligned} \gamma'(s) &= T(s), \\ T'(s) &= -\varepsilon_\gamma\varepsilon_T\gamma(s) + \varepsilon_Y\kappa_g(s)Y(s), \\ Y'(s) &= \varepsilon_\gamma\varepsilon_Y\kappa_g(s)T(s), \end{aligned} \tag{5}$$

$$\mathcal{V}_g(\gamma\times T) = Y,\ \ \mathcal{V}_g(\gamma\times Y) = -\varepsilon_T\varepsilon_Y T\ ,\ \mathcal{V}_g(T\times Y) = \varepsilon_\gamma\varepsilon_Y\gamma$$

where $\kappa_g(s) = g(T'(s), Y(s))$ is an *geodesic curvature function* of the curve γ on the Lorentzian spheres $H^{2,1}_{a_1,a_2,a_3}$ and $S^{2,1}_{a_1,a_2,a_3}$. Also, $\varepsilon_\gamma = g(\gamma,\gamma)$, $\varepsilon_T = g(T,T)$ and $\varepsilon_Y = g(Y,Y)$ are shown the signature functions of the hyperbolical Darboux vectors. Namely, $\varepsilon_k = 1$ $(k=\gamma,T,Y)$ when the vector k is spacelike vector and $\varepsilon_k=-1$ $(k=\gamma,T,Y)$ when the vector k is timelike vector [1].

Lemma 1. *Let $\varphi: U\subset\mathbb{E}^2\to\mathbb{E}^3$, $\varphi(U)=H^{2,1}_{a_1,a_2,a_3}$ or $S^{2,1}_{a_1,a_2,a_3}$ be a general hyperboloid and $\gamma: I\subset\mathbb{R}\to U$ be a regular curve on semi-Riemannian manifold M . Provided that V be a vector field along the curve γ then the variation of γ defined by $\Gamma: I\times(-\varepsilon,\varepsilon)\to\mathcal{GH}(C)$ with $\Gamma(s,0)=\gamma(s)$ where $\gamma(s,0)$ is the initial curve. The variations of the geodesic curvature function $\kappa_g(s,w)$ and the speed function $v(s,w)$ at $w=0$ are calculated as follows:*

$$V(v) = \left(\frac{\partial v}{\partial t}(s,w)\right)\Big|_{w=0} = -v\wp, \tag{6}$$

$$V(\kappa_g) = \left(\frac{\partial\kappa_g}{\partial w}(s,w)\right)\Big|_{w=0} = g(Y, -R(V,T)T+\nabla_T^2V) + \frac{1}{\kappa_g}g(\gamma, -R(V,T)T+\nabla_T^2V) + 2\wp(\kappa_g - \frac{\varepsilon_T}{\kappa_g}).$$

where $\wp = g(\nabla_T V, T)$ and R stands for the curvature tensor of general hyperboloid [1].

Proposition 1. *[1] If $V(s)$ is the restriction to $\gamma(s)$ of a Killing vector field V of $\mathcal{GH}$ then the variations of the hyperbolical Darboux curvature functions and speed function of γ satisfy the following condition*

$$V(v) = 0 = V(\kappa_g). \tag{7}$$

3 Special helices on the general hyperboloid $\mathcal{GH}$

In this section, we define a new kind of slant helices called as type-1, type-2 and type-3 special helices in 3-dimensional Lorentzian space. Moreover, we obtain some characterizations using the Killing vector field and give examples of these curves. We plot the figures of these curves on the general hyperboloid by using the Mathematica.

Definition 1. *Let $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3_{a_1,a_2,a_3}$, $\varphi(U) = \mathcal{GH}$ be a general hyperboloid and $\gamma : I \subset \mathbb{R} \to U$ be a regular curve on the $\mathcal{GH}$. Then we say that γ is a type-1 special helix, type-2 special helix, or type-3 special helix if $g(V,T) = const.$, $g(V,\gamma) = const.$, and $g(V,Y) = const.$, respectively.*

Theorem 1. *[1] Let $\varphi : U \subset \mathbb{E}^2 \to \mathbb{E}^3_1$, $\varphi(U) = \mathcal{GH}$ be a general hyperboloid and $\gamma : I \subset \mathbb{R} \to U$ be a regular curve on $\mathcal{GH}$ and V be a Killing vector field along the curve γ. Then γ is a type-1 special helix with the axis V if and only if the geodesic curvature of the curve γ satisfy the following equations:*
(i) If the curve γ has the spacelike tangent vector, then the geodesic curvature is

$$\kappa_g(s) = -\varepsilon_\gamma \coth\theta(s)$$

where $\theta''(s)\sinh^2\theta(s) + \varepsilon_\gamma \omega\theta'(s)\cosh\theta(s) = 0$ and $\varepsilon_\gamma = \pm 1$,
(ii) If the curve γ has the timelike tangent vector, then the geodesic curvature is

$$\kappa_g(s) = -\cot\theta(s)$$

where $\theta''(s)\sin^2\theta(s) + \omega\theta'(s)\cos\theta(s) = 0$.

Theorem 2. *Let $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3_{a_1,a_2,a_3}$, $\varphi(U) = \mathcal{GH}$ be a general hyperboloid and $\gamma : I \subset \mathbb{R} \to U$ be a regular curve on the $\mathcal{GH}$. Then γ is a type-2 special helix with the axis V if and only if the geodesic curvature of the curve γ satisfies the following conditions which are given according to the casual character of the position vector field of the curve γ:*
(i) If $g(\gamma,\gamma) = 1$ is satisfied, then the geodesic curvature is

$$k_g(s) = -\left(\frac{c_1}{\sinh\theta(s)} + \theta'(s)\right) \tag{8}$$

with $\theta = constant$ or $\varepsilon_T(C+1)\sinh^4\theta(s) + c_1\theta'(s)\sinh\theta(s) + c_1^2 = 0$ and C is a constant.

(ii) If $g(\gamma,\gamma) = -1$ is satisfied, then the geodesic curvature is

$$k_g(s) = -\theta'(s) + \frac{c_1}{\sin\theta(s)} \tag{9}$$

with $\theta = constant$ or $(1-C)\sin^4\theta(s) + c_1\theta'(s)\sin\theta(s) - c_1^2 = 0$ and C is a constant.

Proof: (i) If $g(\gamma,\gamma) = 1$ is satisfied and γ is a type-2 special helix, then the Killing axis is written as

$$V = \cosh\theta(s)T(s) + c_1\gamma(s) + \varepsilon_Y \sinh\theta(s)Y(s), c_1 = const. \tag{10}$$

Differentiating eq.(10) with respect to s, we obtain the following equation

$$\begin{aligned} \nabla_T V &= ((\theta'(s)\sinh\theta(s) + k_g(s)\sinh\theta(s) + c_1)T(s) + (-\varepsilon_T\cosh\theta(s))\gamma(s) \\ &\quad +(\varepsilon_Y\cosh\theta(s)k_g(s) + \varepsilon_Y\theta'(s)\cosh\theta(s))Y(s). \end{aligned} \tag{11}$$

Using $V(v) = 0$ in Lemma 1, we calculate

$$k_g(s) = -\left(\frac{c_1}{\sinh\theta(s)} + \theta'(s)\right). \tag{12}$$

The differentiation of eq.(11) is given by

$$\begin{aligned} \nabla_T^2 V &= (-\varepsilon_T cosh\theta(s) + (\theta'(s) + k_g(s))k_g(s)cosh\theta(s))T(s) - \varepsilon_T\theta'(s)sinh\theta(s)\gamma(s) \\ &\quad +\varepsilon_Y(\theta'(s) + k_g(s))cosh\theta(s)Y(s). \end{aligned} \tag{13}$$

Also, we have the following equation

$$R(V,T(s))T(s) = C(g(T(s),V)T(s) - g(T(s),T(s))V).$$

Using the Darboux frame equations and eq.(10) we deduce

$$R(V,T(s))T(s) = -\varepsilon_T C(c_1\gamma(s) + \varepsilon_Y\sinh\theta(s)Y(s)). \tag{14}$$

Considering the eqs.(13) and (20) with the second equation in Lemma 1 and the Proposition 1, we obtain the desired differential equation for θ.

(ii) If $g(\gamma,\gamma)=-1$ is satisfied and γ is a type-2 special helix, then the Killing axis is written as

$$V=\cos\theta(s)T(s)+c_1\gamma(s)+\sin\theta(s)Y(s), c_1=const.$$

If we make similar calculations in (i), then we obtain

$$k_g(s)=-\theta'(s)+\frac{c_1}{\sin\theta(s)},$$

$$\begin{aligned}\nabla_T^2 V &= (cos\theta(s)-\frac{c_1k_g(s)cos\theta(s)}{\sin\theta(s)})T(s)-\theta'(s)sin\theta(s)\gamma(s)\\ &\quad -\left(\frac{c_1\theta'(s)}{\sin^2\theta(s)}\right)Y(s)\end{aligned}$$

and

$$R(V,T(s))T(s)=-C(c_1\gamma(s)+\sin\theta(s)Y(s)).$$

Considering the last two equalities with the second equation in Lemma 1 and the Proposition 1, the desired result is obtained. □

Corollary 1. *Let γ be a type-2 special helix on the general hyperboloid.*
(i) If the axis $V=\cosh\theta T(s)+c_1\gamma(s)+\varepsilon_Y\sinh\theta Y(s);\quad \theta=const.$ of the type-2 special helix is taken, then the curve γ has the following parametric representation

$$\gamma(s)=A_1+\frac{A_2}{\sqrt{\frac{c_1^2}{\sinh^2\theta}-\varepsilon_T}}\exp\left(\left(\sqrt{\frac{c_1^2}{\sinh^2\theta}-\varepsilon_T}\right)s\right)+\frac{A_3}{\sqrt{\frac{c_1^2}{\sinh^2\theta}-\varepsilon_T}}\exp\left(-\left(\sqrt{\frac{c_1^2}{\sinh^2\theta}-\varepsilon_T}\right)s\right)$$

where $A_1,A_2,A_3\in\mathbb{R}^3_{a_1,a_2,a_3}$ and $c_1\in\mathbb{R}$.
(ii) If the axis of the type-2 special helix is $V=\cos\theta T(s)+c_2\gamma(s)+\sin\theta Y(s);\quad \theta=const.$, then the curve γ has the following parametric representation

$$\gamma(s)=B_1+\frac{B_2}{\sqrt{\frac{c_2^2}{\sin^2\theta}+1}}\exp\left(\left(\sqrt{\frac{c_2^2}{\sin^2\theta}+1}\right)s\right)+\frac{B_3}{\sqrt{\frac{c_2^2}{\sin^2\theta}+1}}\exp\left(-\left(\sqrt{\frac{c_2^2}{\sin^2\theta}+1}\right)s\right)$$

where $B_1,B_2,B_3\in\mathbb{R}^3_{a_1,a_2,a_3}$ and $c_2\in\mathbb{R}$.

Proof: (i) Let γ be a type-2 special helix on the general hyperboloid with the axis

$$V=\cosh\theta T(s)+c_1\gamma(s)+\varepsilon_Y\sinh\theta Y(s);\quad \theta=const.$$

then the hyperbolical curvature of γ calculated as

$$k_g=-\frac{c_1}{\sinh\theta}. \tag{15}$$

On the other hand, from the Darboux frame equations γ satisfy the following third order differential equation

$$k_g\gamma'''-k_g'\gamma''+(\varepsilon_Tk_g-k_g^3)\gamma'-\varepsilon_Tk_g'\gamma=0. \tag{16}$$

If k_g is written in the eq.(16) and the differential equation is solved then it is obtained that γ has the following parametric representation

$$\gamma(s)=A_1+\frac{A_2}{\sqrt{\frac{c_1^2}{\sinh^2\theta}-\varepsilon_T}}\exp\left(\left(\sqrt{\frac{c_1^2}{\sinh^2\theta}-\varepsilon_T}\right)s\right)+\frac{A_3}{\sqrt{\frac{c_1^2}{\sinh^2\theta}-\varepsilon_T}}\exp\left(-\left(\sqrt{\frac{c_1^2}{\sinh^2\theta}-\varepsilon_T}\right)s\right) \tag{17}$$

here $A_1,A_2,A_3\in\mathbb{R}^3_{a_1,a_2,a_3}$ and $c_1\in\mathbb{R}$.
(ii) Let γ be a type-2 special helix on the general hyperboloid with the axis

$$V=\cos\theta T(s)+c_2\gamma(s)+\sin\theta Y(s);\quad \theta=const.$$

then the hyperbolical curvature of γ calculated as

$$k_g=\frac{c_2}{\sin\theta}. \tag{18}$$

From the Darboux frame equations, the γ satisfy the following differential equation

$$k_g\gamma'''-k_g'\gamma''+(k_g^3-k_g)\gamma'+k_g'\gamma=0. \tag{19}$$

If k_g is written in the eq.(19) and the differential equation is solved then it is obtained parametric representation of the curve γ. □

Theorem 3. *Let $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3_{a_1,a_2,a_3}$, $\varphi(U) = \mathcal{GH}$ be a general hyperboloid and $\gamma : I \subset \mathbb{R} \to U$ be a regular curve on the $\mathcal{GH}$. Then γ is type-3 special helix with the axis V if and only if the geodesic curvature of the curve γ satisfy the following conditions:*
(i) If $g(Y(s), Y(s)) = 1$ is satisfied, then the geodesic curvature is

$$k_g(s) = -\frac{\varepsilon_\gamma \left(1 + \varepsilon_T \theta'(s)\right) \cosh\theta(s)}{c_3} \tag{20}$$

with $\theta'' \sinh\theta \left(-\varepsilon_\gamma \left(1 + \varepsilon_T \theta'\right) \cosh^2\theta - c_3^2\right) + \theta' \left((C + \varepsilon_\gamma) c_3^2 + \left(1 + \varepsilon_T \theta'\right)^2 (\sinh^2\theta + \cosh^2\theta)\right) \cosh\theta - c_3^2 (\theta')^2 \cosh\theta = 0$
or $\theta = constant$ and C is a constant.
(ii) If $g(Y(s), Y(s)) = -1$ is satisfied, then the geodesic curvature is

$$k_g(s) = \frac{(\theta'(s) + 1) \cos\theta(s)}{c_4} \tag{21}$$

with $\theta = constant$ or $\theta'' \sin\theta \left(\left(1 + \theta'\right) \cos^2\theta - c_4^2\right) - c_4^2 \theta' \cos\theta \left(C + \theta' + 1\right) + (\theta' + 1)^2 \theta' \cos\theta \cos 2\theta = 0$ *and C is a constant.*

Proof: (i) If γ is a type-3 special helix with the Killing axis V then V is written as

$$V = \varepsilon_T \sinh\theta T(s) + \cosh\theta \gamma(s) + c_3 Y(s). \tag{22}$$

By differentiating eq.(22) we get

$$\nabla_T V = (\cosh\theta + \varepsilon_T \theta' \cosh\theta + \varepsilon_\gamma c_3 k_g) T(s) + (\theta' \sinh\theta - \varepsilon_\gamma \sinh\theta) \gamma(s) + \varepsilon_T k_g \sinh\theta Y(s). \tag{23}$$

By using the equation $V(v) = 0$ in Lemma 1 we reach

$$k_g = -\frac{\varepsilon_\gamma (1 + \varepsilon_T \theta') \cosh\theta}{c_3}. \tag{24}$$

If we take the differentiation of eq.(23) we obtain

$$\begin{aligned} \nabla_T^2 V = {} & ((\theta' - \varepsilon_\gamma) \sinh\theta - k_g^2 \sinh\theta) T(s) + (\theta'' \sinh\theta + (\theta' - \varepsilon_\gamma) \theta' \cosh\theta) \gamma(s) \\ & + (\varepsilon_T k_g' \sinh\theta + \varepsilon_T k_g \theta' \cosh\theta) Y(s). \end{aligned} \tag{25}$$

Furthermore, we have the following equation

$$R(V, T(s))T(s) = C(g(T(s), V) T(s) - g(T(s), T(s)) V). \tag{26}$$

By using the Darboux frame equations and eq.(22) we obtain

$$R(V, T(s))T(s) = -C \varepsilon_T (\cosh\theta \gamma(s) - c_3 Y(s)). \tag{27}$$

If we consider the eq.(25) and eq.(27) with the second equation in Lemma 1 and the Proposition 1, we deduce

$$\theta = const. \tag{28}$$

or satisfy the following equation

$$\theta'' \sinh\theta \left(-\varepsilon_\gamma \left(1 + \varepsilon_T \theta'\right) \cosh^2\theta - c_3^2\right) + \theta' \left((C + \varepsilon_\gamma) c_3^2 + \left(1 + \varepsilon_T \theta'\right)^2 (\sinh^2\theta + \cosh^2\theta)\right) \cosh\theta - c_3^2 (\theta')^2 \cosh\theta = 0. \tag{29}$$

(ii) If γ is a type-3 special helix with the Killing axis V then V is written as

$$V = \sin\theta T(s) + \cos\theta \gamma(s) + c_4 Y(s). \tag{30}$$

If we make similar calculations in (i), then we obtain

$$k_g(s) = \frac{(\theta'(s) + 1) \cos\theta(s)}{c_4},$$

$$\begin{aligned} \nabla_T^2 V \quad = \quad & (-(\theta'(s) + 1) \sin\theta(s) + k_g^2(s) \sin\theta(s)) T(s) - (\theta''(s) \sin\theta(s) + (\theta'(s) + 1) \theta'(s) \cos\theta(s)) \gamma(s) \\ & - \left(k_g'(s) \sin\theta(s) + k_g(s) \theta'(s) \cos\theta(s)\right) Y(s) \end{aligned}$$

and

$$R(V, T(s))T(s) = -C(\cos\theta(s) \gamma(s) + c_4 Y(s)).$$

Considering the last two equalities with the second equation in Lemma 1 and the Proposition 1, the desired result is obtained. □

Corollary 2. *Let γ be a type-3 special helix on the general hyperboloid.*
(i) If the axis $V=\varepsilon_T \sinh\theta T(s)+\cosh\theta\gamma(s)+c_3Y(s); \quad \theta=const.$ of the type-3 special helix is taken, then the curve γ has the following parametric representation

$$\gamma(s)=C_1+\frac{C_2}{\sqrt{\frac{\varepsilon_\gamma\cosh^2\theta}{c_3^2}+1}}\exp\left(\left(\sqrt{\frac{\varepsilon_\gamma\cosh^2\theta}{c_3^2}}+1\right)s\right)+\frac{C_3}{\sqrt{\frac{\varepsilon_\gamma\cosh^2\theta}{c_3^2}+1}}\exp\left(-\left(\sqrt{\frac{\varepsilon_\gamma\cosh^2\theta}{c_3^2}}+1\right)s\right) \tag{31}$$

where $C_1,C_2,C_3\in\mathbb{R}^3_{a_1,a_2,a_3}$ and $c_3\in\mathbb{R}$.
(ii) If the axis of the type-3 special helix is $V=\sin\theta T(s)+\cos\theta\gamma(s)+c_4Y(s); \quad \theta=const.$, then the curve γ has the following parametric representation

$$\gamma(s)=D_1+\frac{D_2}{\sqrt{\frac{\cos^2\theta}{c_4^2}-1}}\exp\left(\left(\sqrt{\frac{\cos^2\theta}{c_4^2}}-1\right)s\right)+\frac{D_3}{\sqrt{\frac{\cos^2\theta}{c_4^2}-1}}\exp\left(-\left(\sqrt{\frac{\cos^2\theta}{c_4^2}}-1\right)s\right) \tag{32}$$

where $D_1,D_2,D_3\in\mathbb{R}^3_{a_1,a_2,a_3}$ and $c_4\in\mathbb{R}$.

Proof: (i) Let γ be a type-3 special helix on the general hyperboloid and the hyperbolical curvature of γ is calculated as

$$k_g=-\frac{\varepsilon_\gamma\cosh\theta}{c_3} \tag{33}$$

where θ is constant. From the Darboux frame equations, it is calculated

$$k_g\gamma'''-k_g'\gamma''-(\varepsilon_\gamma k_g^3+k_g)\gamma'+k_g'\gamma=0. \tag{34}$$

If k_g is written in the eq.(34) and solution of the differential equation is eq. (31).
(ii) Let γ be a type-3 special helix on the general hyperboloid with the axis

$$V=\sin\theta T(s)+\cos\theta\gamma(s)+c_4Y(s); \quad \theta=const.$$

then the hyperbolical curvature of γ calculated as

$$k_g=\frac{\cos\theta}{c_4}. \tag{35}$$

From the Darboux frame equations, the γ satisfy the following differential equation

$$k_g\gamma'''-k_g'\gamma''-(k_g^3-k_g)\gamma'-k_g'\gamma=0. \tag{36}$$

If k_g is written in the eq.(36) and the differential equation is solved then it is obtained parametric representation of the curve γ. □

Example 1. *Let us take the timelike curve parameterized as*

$$\gamma(s)=\left(\frac{15}{16}\cos 17s,\frac{9}{64}\cos 25s+\frac{25}{64}\cos 9s,\frac{1}{16}\sin 25s-\frac{25}{144}\sin 9s\right) \tag{37}$$

on the hyperboloid $-4x^2+16y^2+81z^2=1$. The hyperbolical curvature of the curve γ calculated as

$$k_g(s)=-\cot(17s). \tag{38}$$

Thus we can easily see that γ is a type-1 special helix. It is illustrated in Figure 1.

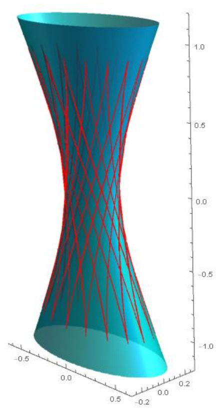

Figure 1. Type-1 special helix on the one sheet hyperboloid $S^{2,1}_{4,16,81}$.

Example 2. *Type-2 (type-3) special helices corresponding to different values of the A_i, C_i, $i = 1, 2, 3$. are illustrated in Figure 2.*

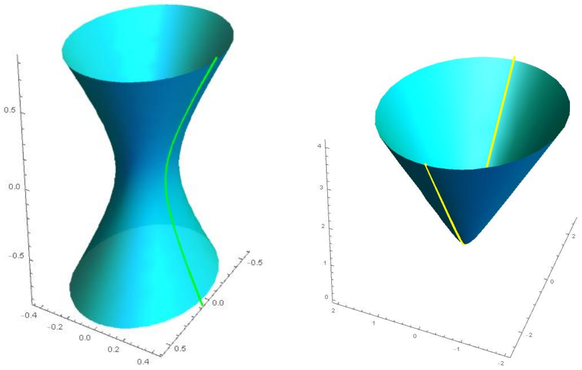

Figure 2. Type-2 (type-3) special helices on the hyperboloid $S^{2,1}_{2\sqrt{2},3\sqrt{2},4\sqrt{2}}$ and $H^{2,1}_{2\sqrt{2},3\sqrt{2},4\sqrt{2}}$.

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