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RESULTS ON THE COMPOSITION AND NEUTRIX COMPOSITION OF THE DELTA FUNCTION

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Abstract

The neutrix composition $F(f(x)))$ of a distribution $F(x)$ and a locally summable function $f(x)$ is said to exist and be equal to the distribution $h(x)$ if the neutrix limit of the sequence $\{F_n(f(x))\}$ is equal to $h(x)$, where $F_n(x) = F(x) * \delta_n(x)$ and $\{\delta_n(x)\}$ is a certain sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. It is proved that the neutrix composition $\delta^{(s)}\{[\exp_+(x) - 1]^r\}$ exists and

$$\delta^{(s)}\{[\exp_+(x) - 1]^r\} = \sum_{k=0}^{rs+r-1} \frac{(-1)^{s+k} s! c_{rs+r-1,k}}{2rk!} \delta^{(k)}(x),$$

for $r = 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$. Further results are also proved.

Keywords: distribution, dirac-delta function, composition of distributions, neutrix, neutrix limit.

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1. Introduction

Certain operations on smooth functions (such as addition, and multiplication by scalars) can be extended without difficulty to arbitrary distributions. Others (such as multiplication, convolution, and change of variables) can be defined only for particular distributions. Note that it is a difficult task to give a meaning to the expression $F(f(x))$, if F and f are singular distributions.

The technique of neglecting appropriately defined infinite quantities was devised by Hadamard and the resulting finite value extracted from the divergent integral is usually referred to as the Hadamard finite part. In fact, Hadamard's method can be regarded

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as a particular application of the neutrix calculus developed by van der Corput, see [1]. This is a very general principle for the discarding of unwanted infinite quantities from asymptotic expansions and has been widely exploited in the context of distributions, by Fisher in connection with the problem of compositions of distributions, see [2] or [3].

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions φ with compact support and let $\mathcal{D}[a,b]$ be the space of infinitely differentiable functions with support contained in the interval $[a,b]$. We let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$ and let $\mathcal{D}'[a,b]$ be the space of distributions defined on $\mathcal{D}[a,b]$.

Now let $\rho(x)$ be a function in $\mathcal{D}[-1,1]$ having the following properties:
(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\displaystyle\int_{-1}^{1} \rho(x)\,dx = 1.$

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. Further, if F is a distribution in $\mathcal{D}'$ and $F_n(x) = \langle F(x-t), \delta_n(x)\rangle$, then $\{F_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to $F(x)$.

There have been several attempts recently to define distributions of the form $F(f(x))$ in $\mathcal{D}'$, where F and f are distributions in $\mathcal{D}'$, see [6] and [4]. At the beginning, we look at the following definition which is a generalization of Gel'fand and Shilov's definition of the composition involving the delta function, see [10]. This definition was given in [2] by Fisher, it involves neutrix limit and was originally called the neutrix composition of distributions.

1.1. Definition. Let F be a distribution in $\mathcal{D}'$ and let f be a locally summable function. We say that the neutrix composition $F(f(x))$ exists and is equal to h on the open interval (a,b) if

$$\operatorname*{N-lim}_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all φ in $\mathcal{D}[a,b]$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \ldots$ and N is the neutrix, see [1], having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^{\lambda}\ln^{r-1} n,\ \ln^r n : \qquad \lambda > 0,\ r = 1, 2, \ldots$$

and all functions which converge to zero in the usual sense as n tends to infinity.

If f, g are two distributions then in the ordinary sense the composition $f(g)$ does not necessarily exist, but the neutrix composition can exist. Thus the definition of the neutrix composition is an extension of the regular definition of compositions of distributions. Some neutrix composition of distributions are considered in [9], [11] and [12].

Recently, Jack Ng and van Dam applied the neutrix calculus, in conjuction with the Hadamard integral, developed by van der Corput, to quantum field theories, in particular, to obtain finite results for the cofficients in the perturbation series. They also applied neutrix calculus to quantum field theory, obtaining finite renormalization in the loop calculations, see [13] and [14].

Now let $f(x)$ be an infinitely differentiable function having a single simple root at the point $x = x_0$. Gel'fand and Shilov defined the distribution $\delta^{(r)}(f(x))$ by the equation

$$\delta^{(r)}(f(x)) = \frac{1}{|f'(x_0)|}\left[\frac{1}{|f'(x)|}\frac{d}{dx}\right]^r \delta(x-x_0),$$

for $r = 0, 1, 2, \dots$, see [10].

The following theorems were proved in [5], [6], [8] and [7] respectively.

1.2. Theorem. *The neutrix composition* $\delta^{(s)}(\operatorname{sgn} x|x|^\lambda)$ *exists and*

$$\delta^{(s)}(\operatorname{sgn} x|x|^\lambda) = 0$$

for $s = 0, 1, 2, \dots$ *and* $(s+1)\lambda = 1, 3, \dots$ *and*

$$\delta^{(s)}(\operatorname{sgn} x|x|^\lambda) = \frac{(-1)^{(s+1)(\lambda+1)} s!}{\lambda[(s+1)\lambda - 1]!}\delta^{((s+1)\lambda-1)}(x)$$

for $s = 0, 1, 2, \dots$ *and* $(s+1)\lambda = 2, 4, \dots$.

1.3. Theorem. *The compositions* $\delta^{(2s-1)}(\operatorname{sgn} x|x|^{1/s})$ *and* $\delta^{(s-1)}(|x|^{1/s})$ *exist and*

$$\delta^{(2s-1)}(\operatorname{sgn} x|x|^{1/s}) = \frac{(2s)!}{2}\delta'(x),$$
$$\delta^{(s-1)}(|x|^{1/s}) = (-1^s \delta(x)$$

for $s = 1, 2, \dots$.

1.4. Theorem. *The neutrix composition* $\delta^{(s)}[\ln^r(1+|x|)]$ *exists and*

$$\delta^{(s)}[\ln^r(1+|x|)] = \sum_{k=0}^{sr+r-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{s-i}[1+(-1)^k]s!(i+1)^{rs+r-1}}{2r(rs+r-1)!k!}\delta^{(k)}(x)$$

for $s = 0, 1, 2, \dots$ *and* $r = 1, 2, \dots$.

In particular, the composition $\delta[\ln(1+|x|)]$ *exists and*

$$\delta[\ln(1+|x|)] = \delta(x).$$

1.5. Theorem. *The neutrix composition* $\delta^{(s)}(\sinh^{-1} x_+)$ *exists and*

$$\delta^{(s)}(\sinh^{-1} x_+) = \sum_{k=0}^{s}\sum_{i=0}^{k}\binom{k}{i}(-1)^{s+i+k}\frac{(k-2i+1)^s + (k-2i-1)^s}{2^k k!}\delta^{(k)}(x)$$

for $s = 0, 1, 2, \dots$.

2. Main Results

In the following, the functions $\exp_+(x)$ and $\exp_-(x)$ are defined by

$$\exp_+(x) = \begin{cases} \exp(x), & x \ge 0, \\ 0, & x < 0 \end{cases} \quad \text{and} \quad \exp_-(x) = \begin{cases} \exp(x), & x \le 0, \\ 0, & x > 0. \end{cases}$$

The constants $c_{i,k}$ are defined by the expansion

$$\frac{\ln^k(1+x)}{1+x} = \sum_{i=1}^{\infty} c_{i,k}x^i \tag{2.1}$$

for $i, k = 1, 2, \dots$ and by the expansion

$$(1+x)^{-1} = \sum_{i=0}^{\infty} c_{i,0}x^i = \sum_{i=0}^{\infty}(-1)^i x^i \tag{2.2}$$

for $i = 0, 1, 2, \dots$ and $k = 0$.

We also need the following lemma, which can be easily proved by induction:

2.1. Lemma.

$$\int_{-1}^{1} t^i \rho^{(s)}(t)\,dt = \begin{cases} 0, & 0 \le i < s, \\ (-1)^s s!, & i = s \end{cases}$$

and

$$\int_{0}^{1} t^s \rho^{(s)}(t)\,dt = \tfrac{1}{2}(-1)^s s!$$

for $s = 0, 1, 2, \ldots$.

We now prove the following theorem.

2.2. Theorem. *The neutrix composition* $\delta^{(s)}\{[\exp_+(x) - 1]^r\}$ *exists and*

$$\delta^{(s)}\{[\exp_+(x) - 1]^r\} = \sum_{k=0}^{rs+r-1} \frac{(-1)^{s+k} s! c_{rs+r-1,k}}{2rk!} \delta^{(k)}(x), \tag{2.3}$$

for $r = 1, 2, \ldots$ *and* $s = 0, 1, 2, \ldots$, *where the constants* $c_{rs+r-1,k}$ *are defined with relations (2.1) and (2.2).*

In particular

$$\delta[\exp_+(x) - 1] = \frac{1}{2}\delta(x), \tag{2.4}$$

$$\delta\{[\exp_+(x) - 1]^2\} = -\frac{1}{4}\delta(x) + \frac{1}{4}\delta'(x), \tag{2.5}$$

$$\delta'\{[\exp_+(x) - 1]^2\} = \frac{1}{2}\delta(x) - \frac{1}{2}\delta'(x). \tag{2.6}$$

Proof. We will first of all prove equation (2.3) on the interval $[-1, 1]$. To do this, we need to evaluate

$$\begin{aligned} \int_{-1}^{1} x^k \delta_n^{(s)}\{[\exp_+(x) - 1]^r\}\,dx &= \\ &= \int_{0}^{1} x^k \delta_n^{(s)}\{[\exp(x) - 1]^r\}\,dx + \int_{-1}^{0} x^k \delta_n^{(s)}[(-1)^r]\,dx \\ &= n^{s+1}\int_{0}^{1} x^k \rho^{(s)}\{n[\exp(x) - 1]^r\}\,dx + 0 \\ &= I. \end{aligned} \tag{2.7}$$

Making the substitution $n[\exp(x) - 1]^r = t$ or

$$x = \ln[1 + (t/n)^{1/r}],$$

we have

$$dx = \frac{t^{1/r-1}\,dt}{rn^{1/r}[1 + (t/n)^{1/r}]}.$$

Then for for $n > 1$, we have

$$\begin{aligned} I &= \frac{n^{s+1}}{rn^{1/r}} \int_{0}^{1} \frac{\ln^k[1 + (t/n)^{1/r}]t^{1/r-1}}{1 + (t/n)^{1/r}} \rho^{(s)}(t)\,dt \\ &= \sum_{i=0}^{\infty} \frac{c_{i,k}}{r} \int_{0}^{1} \frac{t^{(i+1)/r-1}}{n^{(i+1)/r-s-1}} \rho^{(s)}(t)\,dt. \end{aligned}$$

It follows that

$$\begin{aligned} \mathop{\text{N}-\lim}_{n\to\infty} I &= \mathop{\text{N}-\lim}_{n\to\infty} \int_0^1 x^k \delta_n^{(s)}\{[\exp(x)-1]^r\}\,dx \\ &= \frac{(-1)^s s! c_{rs+r-1,k}}{2r}, \end{aligned} \tag{2.8}$$

on using the lemma 2.1, for $k = 0,1,2,\ldots, rs+r-1$, $r = 1,2,\ldots$ and $s = 0,1,2,\ldots$.

Next, when $k = rs+r$, we have

$$\begin{aligned} \int_0^1 \left|x^{rs+r}\delta_n^{(s)}\{[\exp(x)-1]^r\}\right| dx &\le \frac{n^{s+1}}{rn^{1/r}} \int_0^1 \left|\frac{\ln^{rs+r}[1+(t/n)^{1/r}]t^{1/r-1}}{1+(t/n)^{1/r}}\rho^{(s)}(t)\right| dt \\ &= O(n^{-1/r}), \end{aligned}$$

since $|\ln^{rs+r}[1+(t/n)^{1/r}]| = O(n^{-s-1})$. Hence, if $\psi(x)$ is an arbitrary continuous function, then

$$\lim_{n\to\infty} \int_0^1 x^{rs+r}\delta_n^{(s)}\{[\exp(x)-1]^r\}\psi(x)\,dx = 0, \tag{2.9}$$

for $r = 1,2,\ldots$ and $s = 0,1,2,\ldots$.

Further,

$$\begin{aligned} \mathop{\text{N}-\lim}_{n\to\infty} \int_{-1}^0 x^{rs+r}\delta_n^{(s)}(0)\psi(x)\,dx &= \mathop{\text{N}-\lim}_{n\to\infty} n^{s+1}\int_{-1}^0 x^{rs+r)}\rho^{(s)}(0)\psi(x)\,dx \\ &= 0, \end{aligned} \tag{2.10}$$

for $r = 1,2,\ldots$ and $s = 0,1,2,\ldots$.

Now let φ be an arbitrary function in $\mathcal{D}[-1,1]$. By Taylor's Theorem we have

$$\varphi(x) = \sum_{k=0}^{rs+r-1} \frac{x^k\varphi^{(k)}(0)}{k!} + \frac{x^{rs+r}\varphi^{(rs+r)}(\xi x)}{s!},$$

where $0 < \xi < 1$. Then

$$\begin{aligned}
&\mathop{\mathrm{N-lim}}_{n\to\infty}\langle\delta_n^{(s)}\{[\exp(x)-1]^r\},\varphi(x)\rangle = \\
&\qquad = \mathop{\mathrm{N-lim}}_{n\to\infty}\sum_{k=0}^{rs+r-1}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^{1}x^k\delta_n^{(s)}\{[\exp(x)-1]^r\}\,dx \\
&\qquad\quad + \mathop{\mathrm{N-lim}}_{n\to\infty}\int_{-1}^{1}\frac{x^{rs+r}}{(rs+r)!}\delta_n^{(s)}\{[\exp(x)-1]^r\}\varphi^{(rs+s)}(\xi x)\,dx \\
&\qquad = \mathop{\mathrm{N-lim}}_{n\to\infty}\sum_{k=0}^{rs+r-1}\frac{\varphi^{(k)}(0)}{k!}\int_{0}^{1}x^k\delta_n^{(s)}\{[\exp(x)-1]^r\}\,dx \\
&\qquad\quad + \mathop{\mathrm{N-lim}}_{n\to\infty}\sum_{k=0}^{rs+r-1}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^{0}x^k\delta_n^{(s)}(0)\,dx \\
&\qquad\quad + \mathop{\mathrm{N-lim}}_{n\to\infty}\int_{0}^{1}\frac{x^{rs+r}}{(rs+r)!}\delta_n^{(s)}\{[\exp(x)-1]^r\}\varphi^{(s)}(\xi x)\,dx \\
&\qquad\quad + \mathop{\mathrm{N-lim}}_{n\to\infty}\int_{-1}^{0}\frac{x^{rs+r-1}}{(rs+r-1)!}\delta_n^{(s)}(0)\varphi^{(s)}(\xi x)\,dx \\
&\qquad = \sum_{k=0}^{rs+r-1}\frac{(-1)^s s!c_{rs+r-1,k}}{2rk!}\varphi^{(k)}(0) \\
&\qquad = \sum_{k=0}^{rs+r-1}\frac{(-1)^{s+k}s!c_{rs+r-1,k}}{2rk!}\langle\delta^{(k)}(x),\varphi(x)\rangle,
\end{aligned}$$

on using equations (2.7), (2.8), (2.9) and (2.10), for $r = 2, 3, \ldots$ and $s = 1, 2, \ldots$.

This proves that the neutrix composition $\delta^{(s)}\{[\exp_+(x)-1]^r\}$ exists and

$$\delta^{(s)}\{[\exp_+(x)-1]^r\} = \sum_{k=0}^{rs+r-1}\frac{(-1)^{s+k}s!c_{rs+r-1,k}}{2rk!}\delta^{(k)}(x),$$

on the interval $[-1, 1]$ for $r = 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$.

It is obvious that $\delta^{(s)}\{[\exp_+(x)-1]^r\} = 0$, if $x \neq 0$ and so the neutrix composition $\delta^{(s)}\{[\exp_+(x)-1]^r\}$ exists on the real line.

Equations (2.4), (2.5) and (2.6) follow on noting that $c_{0,0} = 1$, $c_{1,0} = -1$ and $c_{1,1} = -1$.

Finally note that when $r = 1$ and $s = 0$, the normal limits exist and so the composition $\delta[\exp_+(x)-1]$ exists. This completes the proof of the theorem 2.2.

2.3. Corollary. *The neutrix composition* $\delta^{(s)}\{[1-\exp_-(x)]^r\}$ *exists and*

$$\delta^{(s)}\{[1-\exp_-(x)]^r\} = \sum_{k=0}^{rs+r-1}\frac{(-1)^{rs+r+s+k-1}s!c_{rs+r-1,k}}{2rk!}\delta^{(k)}(x), \tag{2.11}$$

for $r = 1, 2, \ldots$ *and* $s = 0, 1, 2, \ldots$.

In particular

$$\delta\{[1-\exp_-(x)]\} = \frac{1}{2}\delta(x), \tag{2.12}$$

$$\delta\{[1-\exp_-(x)]^2\} = \frac{1}{4}\delta(x) - \frac{1}{4}\delta'(x), \tag{2.13}$$

$$\delta'\{[1-\exp_-(x)]^2\} = \frac{1}{2}\delta(x) - \frac{1}{2}\delta'(x). \tag{2.14}$$

Proof. To prove equation (2.11) on the interval $[-1,1]$, we need to evaluate

$$\begin{aligned}\int_{-1}^{1} x^k \delta_n^{(s)}\{[1-\exp_-(x)]^r\}\,dx &= \\ &= \int_{-1}^{0} x^k \delta_n^{(s)}\{[1-\exp(x)]^r\}\,dx + \int_0^1 x^k \delta_n^{(s)}(1)\,dx \\ &= n^{s+1}\int_{-1}^{0} x^k \rho^{(s)}\{n[1-\exp(x)]^r\}\,dx + 0 \\ &= I. \qquad (2.15)\end{aligned}$$

Making the substitution $n[1-\exp(x)]^r = t$ or

$$x = \ln[1-(t/n)^{1/r}],$$

we have

$$dx = -\frac{t^{1/r-1}\,dt}{rn^{1/r}[1-(t/n)^{1/r}]}.$$

Then for for $n > 1$, we have

$$\begin{aligned}I &= \frac{n^{s+1}}{rn^{1/r}}\int_0^1 \frac{\ln^k[1-(t/n)^{1/r}]t^{1/r-1}}{1-(t/n)^{1/r}}\rho^{(s)}(t)\,dt \\ &= \sum_{i=0}^{\infty}\frac{(-1)^i c_{i,k}}{r}\int_0^1 \frac{t^{(i+1)/r-1}}{n^{(i+1)/r-s-1}}\rho^{(s)}(t)\,dt.\end{aligned}$$

It follows that

$$\begin{aligned}\operatorname*{N-lim}_{n\to\infty} I &= \operatorname*{N-lim}_{n\to\infty}\int_0^1 x^k\delta_n^{(s)}\{[1-\exp(x)]^r\}\,dx \\ &= \frac{(-1)^{rs+r+s-1}s!c_{rs+r-1,k}}{2r}, \qquad (2.16)\end{aligned}$$

on using the lemma 2.1, for $k = 0,1,2,\ldots,rs+r-1$, $r = 1,2,\ldots$ and $s = 0,1,2,\ldots$.

Next, when $k = rs+r$, we have

$$\begin{aligned}\int_{-1}^{0}\big|x^{rs+r}\delta_n^{(s)}\{[1-\exp(x)]^r\}\big|\,dx &\le \\ &\le \frac{n^{s+1}}{rn^{1/r}}\int_{-1}^{0}\left|\frac{\ln^{rs+r}[1-(t/n)^{1/r}]t^{1/r-1}}{1-(t/n)^{1/r}}\rho^{(s)}(t)\right|\,dt \\ &= O(n^{-1/r}).\end{aligned}$$

Hence, if $\psi(x)$ is an arbitrary continuous function, then

$$\lim_{n\to\infty}\int_0^1 x^{rs+r}\delta_n^{(s)}\{[1-\exp(x)]^r\}\psi(x)\,dx = 0, \qquad (2.17)$$

for $r = 1,2,\ldots$ and $s = 0,1,2,\ldots$.

The proof of the corollary now follows as in the proof of Theorem 2.2, using (2.15), (2.16) and (2.17). Equations (2.12), (2.13) and (2.14) follows immediately.

2.4. Corollary. *The neutrix composition $\delta^{(s)}[|\exp(x)-1|^r]$ exists and*

$$\delta^{(s)}[|\exp(x)-1|^r] = \begin{cases}\displaystyle\sum_{k=0}^{rs+r-1}\frac{(-1)^k s!c_{rs+r-1,k}}{rk!}\delta^{(k)}(x), & r \text{ odd } s \text{ even},\\ 0, & r \text{ even},\\ 0, & r, s \text{ odd}\end{cases} \qquad (2.18)$$

for $r = 1,2,\ldots$ and $s = 0,1,2,\ldots$.

Proof. Equation (2.18) follows on noting that we have

$$\delta^{(s)}[|\exp(x)-1|^r] = \delta^{(s)}[|\exp_+(x)-1|^r] + \delta^{(s)}[|\exp_-(x)-1|^r]$$

and

$$\delta^{(s)}[|\exp_-(x)-1|^r] = \begin{cases} \delta^{(s)}\{[\exp(x)-1]^r\} & r \text{ odd}, \ s \text{ even}, \\ \delta^{(s)}\{[1-\exp(x)]^r\}, & r \text{ even}, \\ -\delta^{(s)}\{[1-\exp(x)]^r\}, & r, s \text{ odd}. \end{cases}$$

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