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ON STRONGLY AND SEPARABLY ω_1-$p^{\omega+n}$-PROJECTIVE ABELIAN p-GROUPS

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Abstract

Let $n \geq 0$ be an arbitrary integer. We prove some results for strongly n-simply presented abelian p-groups with C-decomposable property, extending classical achievements due to Keef in Commun. Algebra (1990). As applications we define the classes of strongly ω_1-$p^{\omega+n}$-projective and separably ω_1-$p^{\omega+n}$-projective abelian p-groups which are also properly contained in all ω_1-$p^{\omega+n}$-projectives, recently defined by Keef in J. Alg. Numb. Th. Acad. (2010). Moreover, some principal descriptions concerning these new objects are obtained as well.

Keywords: C-decomposable groups, $p^{\omega+n}$-projective groups, strongly n-simply presented groups, ω_1-$p^{\omega+n}$-projective groups, strongly ω_1-$p^{\omega+n}$-projective groups, bounded subgroups, countable subgroups, nice subgroups, Ulm subgroups, Ulm factors.

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1. Introduction and Terminology

Let all groups into consideration throughout the paper be abelian p-torsion groups where p is a fixed prime integer. As usual, for some ordinal $\alpha \geq 0$ and a group G, we state the *α-th Ulm subgroup* $p^\alpha G$, consisting of all elements of G with height $\geq \alpha$, inductively as follows: $p^0 G = G$, $pG = \{pg \mid g \in G\}$, $p^\alpha G = p(p^{\alpha-1}G)$ if $\alpha - 1$ exists (so α is non-limit) and $p^\alpha G = \cap_{\beta<\alpha} p^\beta G$ if $\alpha - 1$ does not exist (so α is limit). The group G is named *p^α-bounded* if $p^\alpha G = \{0\}$; note that these groups have to be reduced. We shall say that G is *Σ-cyclic* if it is a direct sum of cyclic groups, and *separable* if it is p^ω-bounded - notice that Σ-cyclic groups are separable. Most of the important unexplained here notations and notions will follow mainly those from [9].

The class of $p^{\omega+n}$-projective groups, defined originally as in [14], plays an important if not facilitating role in the theory of abelian groups whenever $n \geq 0$ is an integer. There are two similar characterizations of the $p^{\omega+n}$-projectives given in [14] and [1], respectively.

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1.1. Theorem. *The group G is $p^{\omega+n}$-projective if and only if precisely one of the following conditions holds:*

(a) there exists a p^n-bounded subgroup P of G such that G/P is Σ-cyclic.

(b) there exists a Σ-cyclic group S with a p^n-bounded subgroup B such that $G \cong S/B$.

Observe that when $n = 0$ we obtain the classical Σ-cyclic groups, i.e., the p^ω-projective groups. Moreover, note that P is of necessity nice in G because G/P is separable.

On the other hand, a few years ago, Keef established in ([12], Proposition 1.4 and Theorem 1.2 (a1)) the following intriguing generalization of $p^{\omega+n}$-projective groups:

1.2. Theorem. *The group G is ω_1-$p^{\omega+n}$-projective if and only if exactly one of the following conditions is valid:*

(i) there is a countable subgroup C of G such that $C \subseteq p^\omega G$ and G/C is $p^{\omega+n}$-projective.

(ii) there is a p^n-bounded subgroup H of G such that G/H is the direct sum of a countable group and a Σ-cyclic group.

Notice that the subgroup C of point (i) of the last theorem is necessarily nice in G satisfying the inequalities $p^{\omega+n}G \subseteq C \subseteq p^\omega G$. So, it is interesting to know whether or not the subgroup H in point (ii) of the same theorem can be chosen to be nice in G. Unfortunately or not, the answer is "no" as it will be demonstrated in the sequel.

Thus adding the niceness will be a non-trivial procedure, and thereby we come to the main concept which motivates the writing of this article.

Definition 1.1. A group G is called *strongly ω_1-$p^{\omega+n}$-projective* if it contains a p^n-bounded nice subgroup A such that G/A is a direct sum of a countable group and a Σ-cyclic group.

Each $p^{\omega+n}$-projective group is necessarily strongly ω_1-$p^{\omega+n}$-projective, while the converse is untrue provided that the group has length strictly greater $\omega + n$. However, $p^{\omega+n}$-bounded strongly ω_1-$p^{\omega+n}$-projective groups must be $p^{\omega+n}$-projective, instead of ω_1-$p^{\omega+n}$-projectives (cf. [12]) which are not.

A weaker version of the last group class is the following:

Definition 1.2. A group G is said to be *separably ω_1-$p^{\omega+n}$-projective* if it contains a p^n-bounded nice subgroup M such that $M \cap p^\omega G = \{0\}$ and G/M is a direct sum of a countable group and a Σ-cyclic group.

It is worthwhile noticing that such a subgroup M, for which $G/(M \oplus p^\omega G)$ is Σ-cyclic, must be nice in G as it will be demonstrated below. Also, Σ-cyclic groups are separably ω_1-$p^{\omega+n}$-projective and, for $n = 1$, p^ω-bounded $p^{\omega+1}$-projective groups are necessarily separably ω_1-$p^{\omega+1}$-projective, whereas in both cases the converse is not true provided that the group has length greater than ω. Even more, p^ω-bounded separably ω_1-$p^{\omega+n}$-projective groups need not be Σ-cyclic; in fact they are $p^{\omega+n}$-projective.

On the other hand, in [4] we enlarged the Keef's concept to the so-termed *weakly ω_1-$p^{\omega+n}$-projective groups* that are groups G containing countable nice subgroups $N \subseteq p^\omega G$ such that $G/N/p^{\omega+n}(G/N) \cong G/(p^{\omega+n}G + N)$ is $p^{\omega+n}$-projective. Likewise, some other improvements of ω_1-$p^{\omega+n}$-projectivity were established in [2] and [5], respectively.

On another vein, in [8] the present author along with Keef defined the class of *(strongly) n-simply presented groups* G which are groups containing a (nice) p^n-bounded subgroup P such that G/P is simply presented. Clearly, (strongly) ω_1-$p^{\omega+n}$-projective groups are (strongly) n-simply presented.

Besides, in [10], it was introduced and investigated the class of *separably n-simply presented groups* that are strongly n-simply presented groups G for which $P \cap p^\omega G = \{0\}$. Evidently, all separably ω_1-$p^{\omega+n}$-projective groups are themselves separably n-simply presented.

In some of the next sections we shall study the above stated concepts more carefully.

2. A Survey of Known Results

In this brief section, we shall list a few more useful results, needed for applicable purposes in the next sections. These results are stated here only for the sake of completeness and for the readers' convenience, and will be utilized below without some more special and concrete referring.

2.1. Proposition. *(*[9]*)*

(j) (Nunke's property) A group G is simply presented if and only if $p^\alpha G$ and $G/p^\alpha G$ are both simply presented for some ordinal α.

(jj) (Direct summand property) Direct summands of simply presented groups are again simply presented.

2.2. Proposition. *(*[14]*) Subgroups of $p^{\omega+n}$-projective groups are again $p^{\omega+n}$-projective.*

2.3. Proposition. *(*[12]*) Subgroups of ω_1-$p^{\omega+n}$-projective groups are again ω_1-$p^{\omega+n}$-projective.*

2.4. Proposition. *(*[8]*)*

(j) If G is a strongly n-simply presented group with $p^{\omega+n}G = \{0\}$, then G is $p^{\omega+n}$-projective.

(jj) If G is a strongly n-simply presented group, then $G/p^\alpha G$ is strongly n-simply presented for some ordinal α. In particular, $G/p^{\omega+n}G$ is $p^{\omega+n}$-projective.

Moreover, G is strongly n-simply presented if and only if $p^{\alpha+n}G$ and $G/p^{\alpha+n}G$ are both strongly n-simply presented.

3. C-Decomposable Strongly n-Simply Presented p-Groups

As mentioned in the first section, a strongly n-simply presented group is such a group G for which there is a p^n-bounded nice subgroup N with G/N being simply presented.

The next assertion strengthens ([11], Theorem 3).

3.1. Theorem. *Suppose G is a strongly n-simply presented group with $p^\omega G$ simply presented and $G \cong H \oplus K$ where K is a Σ-cyclic group whose final rank is at least $r(p^{\omega+n}G)$. Then G is a direct sum of a simply presented group and a $p^{\omega+n}$-projective group.*

Proof. Since G is strongly n-simply presented, in virtue of [8] the quotient $G/p^{\omega+n}G$ should be $p^{\omega+n}$-projective. But $H/p^{\omega+n}H$ is obviously isomorphic to a summand of $G/p^{\omega+n}G$, and hence it is $p^{\omega+n}$-projective as well. Moreover, $p^\omega H \cong p^\omega G$ is simply presented and hence so is $p^{\omega+n}H$ applying [9]. Therefore, H is strongly n-simply presented again by the utilization of [8]. It follows from Theorem 1.1 (a) that there exists a subgroup $Q \subseteq (H/p^{\omega+n}H)[p^n]$ such that $(H/p^{\omega+n}H)/Q$ is Σ-cyclic. Let P be the subgroup of H containing $p^{\omega+n}H \cong p^{\omega+n}G$ and defined by the equation $P/p^{\omega+n}H = Q$; thus $p^n P \subseteq p^{\omega+n}H$, and $H/p^{\omega+n}H/P/p^{\omega+n}H \cong H/P$ is Σ-cyclic with $p^\omega H \subseteq P$.

Using the idea behind a "standard $\omega+n$-decomposition", there is clearly a subgroup $P_1 \subseteq p^\omega H \subseteq P$ such that if L is a $p^{\omega+n}$-high subgroup of H, and thus it is $p^{\omega+n}$-bounded, then there exists a decomposition $p^\omega G \cong p^\omega H = P_1 \oplus p^\omega L$; so, in particular, $p^n P_1 = p^{\omega+n}H$ since $p^{\omega+n}L = \{0\}$. Indeed, we first claim that $p^\omega L$ is a maximal p^n-bounded summand of $p^\omega H$, so that it is pure and bounded in $p^\omega H$, whence its direct summand. To prove this, we foremost see that $p^n(p^\omega L) = p^{\omega+n}L = \{0\}$, hence $p^\omega L$ is bounded by p^n. Furthermore, because L is isotype in H and hence obviously $p^\omega L$ is pure in $p^\omega H$, we write $p^\omega H = P_1 \oplus p^\omega L$ (see, e.g., [9]). To show the maximality, also write $p^\omega H = X \oplus T$ for some $X \leq p^\omega H$ and $T \leq p^\omega H$ such that $p^n T = \{0\}$. It is apparently seen that $p^{\omega+n}H = p^n X$ and thus immediately $T \cap p^{\omega+n}H = \{0\}$. But $L \cap p^{\omega+n}H = \{0\}$ is maximal with this property, so that $T \subseteq L \cap p^\omega H = p^\omega L$ because as mentioned above L is isotype in H, as required. This gives the claim.

If now $P_2 = P \cap L$, we even have a valuated direct decomposition $P = P_1 \oplus P_2$. In fact, it is elementary to verify that $(p^{\omega+n}H)[p] = P_1[p] = (p^n P_1)[p]$. This insures at once that $P_1 \cap L = \{0\}$ and hence $P_1 \cap P_2 = \{0\}$. Next, since $H[p] = (p^{\omega+n}H)[p] \oplus L[p] = (p^n P_1)[p] \oplus L[p] = P_1[p] \oplus L[p]$ and since L is pure in H (see, cf. [9]), it easily follows that $H[p^n] = P_1[p^n] \oplus L[p^n]$. Therefore, intersecting the last equality with $P \leq H$, the modular law yields that $P[p^n] = P_1[p^n] \oplus (L \cap P)[p^n] = P_1[p^n] \oplus P_2[p^n]$. By what we have just shown above, $p^n P \subseteq p^{\omega+n}H = p^n P_1$ which, because of $P_1 \subseteq P$, is tantamount to $p^n P = p^n P_1$. The last equality directly implies that $P = P_1 + P[p^n]$, that is equivalent to $P = P_1 \oplus P_2$, as asserted. That this decomposition is valuated follows routinely, which technical details we leave to the reader. It is also worth noticing that the equality $P = P_1 \oplus P_2$ is an extension of the equality $p^\omega H = P_1 \oplus p^\omega L$; in fact the modular law ensures for $P_1 \leq p^\omega H \leq P$ that $p^\omega H = P_1 \oplus (P_2 \cap p^\omega H)$. But the latter summand is equal to $L \cap p^\omega H = p^\omega L$ because L is pure in H (e.g., [9]), and consequently we conclude that $p^\omega H = P_1 \oplus p^\omega L$ which was our initial pivotal relation.

We further observe that $L/P_2 \cong (L+P)/P \subseteq H/P$ is Σ-cyclic, and that $p^n P_2 \subseteq p^n P \cap p^n L \subseteq p^{\omega+n}H \cap L = \{0\}$, whence L is $p^{\omega+n}$-projective owing to Theorem 1.1 (a) as well.

Let us now T be a simply presented group with the following Ulm-Kaplansky function: $f_T(\alpha) = f_K(\alpha)$, when $\alpha < \omega$; $f_T(\alpha) = 0$, when $\omega \leq \alpha < \omega + n - 1$, and $f_T(\alpha) = f_G(\alpha)$, when $\omega + n - 1 \leq \alpha$. Note that the existence of such a group T is guaranteed by the fact that K has final rank no less than $r(p^{\omega+n}G)$ - see, for example, ([9], Theorem 83.6).

Next, consider the direct sum $A = T \oplus L$. If $B \subseteq A$ is the subgroup $p^\omega T \oplus P_2$, then apparently $A/B \cong (T/p^\omega T) \oplus (L/P_2)$ is Σ-cyclic. Moreover, $p^n P_1 = p^{\omega+n}H \cong p^{\omega+n}G$ is simply presented, hence in virtue of [9] so is P_1. But $p^\omega T$ is also simply presented (cf. [9]) and, in accordance with the preceding paragraph, it is readily checked that both $p^\omega T$ and P_1 have same Ulm-Kaplansky invariants. Thus [9] allows us to conclude that $p^\omega T \cong P_1$, and so there is an isometry $\phi : B = p^\omega T \oplus P_2 \to P_1 \oplus P_2 = P$. It is easy to check that $f_{G,P}(\alpha) = f_{A,B}(\alpha) = f_{L,P_2}(\alpha) + f_K(\alpha)$, when $\alpha < \omega$, or $f_{G,P}(\alpha) = f_{A,B}(\alpha) = 0$, when $\alpha \geq \omega$. This, however, implies in view of ([9], Theorem 83.4) that ϕ extends to an isomorphism $\Phi : A = T \oplus L \to G$, thus proving the result. □

Remark. It is worth noting that the first part of the above proof actually demonstrates that any $p^{\omega+n}$-high subgroup of a strongly n-simply presented group is $p^{\omega+n}$-projective.

As a direct consequence, we derive a generalization of Corollary 4 from [11].

3.2. Corollary. *The group G is a summand of the direct sum of a simply presented group and a $p^{\omega+n}$-projective group if and only if G is a strongly n-simply presented group such that $p^\omega G$ is simply presented.*

Proof. "$\Rightarrow$". Write $T \oplus P = G \oplus H$ where T is simply presented and P is $p^{\omega+n}$-projective. Evidently, $p^\omega G$ is a summand of $p^\omega T \oplus p^\omega P$ which is simply presented. Therefore, $p^\omega G$ is simply presented referring to [9].

On the other hand, one may observe that $(T/p^{\omega+n}T) \oplus (P/p^{\omega+n}P) \cong (G/p^{\omega+n}G) \oplus (H/p^{\omega+n}H)$. Since $T/p^{\omega+n}T$ is a direct sum of countable groups of length $\omega+n$, hence it is $p^{\omega+n}$-projective, and $P/p^{\omega+n}P$ is $p^{\omega+n}$-projective, the left hand-side is $p^{\omega+n}$-projective too, whence so is $G/p^{\omega+n}G$. Finally, [8] applies to show that G is strongly n-simply presented, as desired.

"$\Leftarrow$". Let G be strongly n-simply presented with $p^\omega G$ simply presented. Also, let C be a Σ-cyclic group whose final rank exceeds the rank of $p^{\omega+n}G$. Then $G \oplus C$ is, by Theorem 3.1, a direct sum of a simply presented group and a $p^{\omega+n}$-projective group, as required. □

Recall that a group G is C-decomposable if $G \cong H \oplus C$ where C is a Σ-cyclic group with the same final rank as that of G.

An other (second) valuable consequence of the chief result of this section is the following generalization of Corollary 5 in [11].

3.3. Corollary. *If G is a C-decomposable strongly n-simply presented group such that $p^\omega G$ is simply presented, then G is the direct sum of a simply presented group and a $p^{\omega+n}$-projective group.*

Proof. It is clear that the final rank of G must be at least as large as the rank of $p^{\omega+n}G$. Furthermore, we apply Theorem 3.1 to get the claim. □

4. Strongly ω_1-$p^{\omega+n}$-Projective p-Groups

As stated in the introductory Section 1, a group G is strongly ω_1-$p^{\omega+n}$-projective if it has a nice subgroup $N \leq G[p^n]$ such that G/N is ω_1-p^ω-projective (= ω-totally Σ-cyclic in terms of [7]), that is, the direct sum of a countable group and a Σ-cyclic group. Respectively, a group G is separably ω_1-$p^{\omega+n}$-projective if it possesses a subgroup $L \leq G[p^n]$ with $L \cap p^\omega G = \{0\}$ such that G/L is ω_1-p^ω-projective (= ω-totally Σ-cyclic), i.e., the direct sum of a countable group and a Σ-cyclic group. It is pretty easy to check that $p^{\omega+n}$-projective groups are strongly ω_1-$p^{\omega+n}$-projective (in fact, the countable group in the direct decomposition of G/N must be exactly $\{0\}$) as well as separable $p^{\omega+n}$-projective groups are separably ω_1-$p^{\omega+n}$-projective (indeed, $p^\omega G = \{0\}$ and again the countable summand from the direct decomposition of G/L has to be precisely $\{0\}$).

In [13] the following useful technicality due to B. Charles was stated explicitly:

Lemma (Charles). *Suppose A is a group with a countable subgroup B such that A/B is Σ-cyclic. Then A is the direct sum of a countable group and a Σ-cyclic group.*

In the case when A/B is $p^{\omega+n}$-projective for some $n \geq 1$, the group A is defined in [12] to be ω_1-$p^{\omega+n}$-projective (compare also with Theorem 1.2 (i) stated above in Section 1) and it is not necessarily a direct sum of a countable group and a $p^{\omega+n}$-projective group; indeed there exists a $p^{\omega+n}$-bounded ω_1-$p^{\omega+n}$-projective group which is not $p^{\omega+n}$-projective (see the comments on pp. 56 and 57 of [12]).

However, it is rather natural to ask whether the following strengthening is true: For some group A let A/B be Σ-cyclic and let B be the direct sum of a countable group and a p^n-bounded group (i.e., p^nB is countable) for some $n \geq 1$. Does it follow that A is the direct sum of a countable group and a $p^{\omega+n}$-projective group? Unfortunately or not, it is untrue, and A is in general a proper subgroup of such a direct sum being an

ω_1-$p^{\omega+n}$-projective group (see, for instance, Theorem 1.2 (b1) and Theorem 1.5 (b) of [12]); that is why an equality may not be fulfilled.

Reciprocally, if A is a group with a Σ-cyclic subgroup C such that A/C is countable, then A is again a direct sum of a countable group and a Σ-cyclic group - see, e.g., [6], or Theorem 1.5 (b) from [12] when $n = 0$.

On the other hand, Megibben proved in [13] the following statement (for some non-trivial generalizations to that fact see also [7] and [3]).

Proposition (Megibben). *Suppose G is a group. Then the following are equivalent:*
(i) $G/p^\omega G$ is Σ-cyclic with $p^\omega G$ countable;
(ii) G is the direct sum of a countable group and a Σ-cyclic group.

Actually, the implication (i) $\Rightarrow$ (ii) in this assertion follows immediately from the above Lemma of Charles. Besides, a subgroup of the direct sum of a countable group and a Σ-cyclic group is again a direct sum of a countable group and a Σ-cyclic group; in fact, if $H \leq G$ where G is such a group, then $p^\omega G$ is countable and $G/p^\omega G$ is Σ-cyclic. But $H/(H \cap p^\omega G) \cong (H + p^\omega G)/p^\omega G \subseteq G/p^\omega G$ is Σ-cyclic as being a subgroup with countable intersection $H \cap p^\omega G$, so that the Lemma of Charles applies to get the assertion.

It is now quite usual to ask whether or not the following enlargement holds:

Question. Let G be a group and $n \geq 0$. Does it follow that the next two points are equivalent?
(a) $G/p^{\omega+n}G$ is $p^{\omega+n}$-projective and $p^{\omega+n}G$ is countable;
(b) G is the direct sum of a countable group and a $p^{\omega+n}$-projective group.

This is true only when $n = 1$ – see Corollary 2.11 from [7]. However, when $n = 2$, the answer is negative – see Example on p. 533 from [7]. (See also [3] for more details when $n \geq 1$.)

Reciprocally, if A is a group with a $p^{\omega+n}$-projective subgroup S such that A/S is countable, then A need not be the direct sum of a countable group and a $p^{\omega+n}$-projective group whenever $n \geq 1$. Indeed, an appeal to Theorem 1.2 (c3) from [12] gives that A is ω_1-$p^{\omega+n}$-projective, whereas Theorem 1.5 (b) of [12] insures that A is only a (proper) subgroup of such a direct sum.

We will now provide the reader with some equivalent characterizations of strongly (respectively, separably) ω_1-$p^{\omega+n}$-projectives.

4.1. Lemma. *The group G is strongly ω_1-$p^{\omega+n}$-projective if and only if there exists a nice subgroup N of G such that $p^nN = \{0\}$, $G/(N + p^\omega G)$ is Σ-cyclic and $p^\omega(G/N) \cong p^\omega G/(p^\omega G \cap N)$ is countable.*

Proof. "$\Rightarrow$". Write $G/N = (A/N) \oplus (B/N)$ where A/N is countable and B/N is Σ-cyclic for some p^n-bounded nice subgroup N of G. Therefore $p^\omega(G/N) = p^\omega(A/N)$ is countable, i.e., same is true for $(p^\omega G + N)/N \cong p^\omega G/(N \cap p^\omega G)$. On the other hand, $G/N/p^\omega(G/N) = G/N/(p^\omega G + N)/N \cong G/(p^\omega G + N)$ should be Σ-cyclic, as stated.

"$\Leftarrow$". Since $G/(N + p^\omega G) \cong G/N/(N + p^\omega G)/N$ is Σ-cyclic and $(N + p^\omega G)/N \cong p^\omega G/(N \cap p^\omega G)$ is countable, the Lemma of Charles applies to deduce that G/N is the direct sum of a countable group and a Σ-cyclic group, as required. □

4.2. Lemma. *The group G is separably ω_1-$p^{\omega+n}$-projective if and only if there exists a nice subgroup P of G such that $p^nP = \{0\}$, $P \cap p^\omega G = \{0\}$ and $G/(P \oplus p^\omega G)$ is Σ-cyclic with countable $p^\omega G$.*

Proof. Follows in the same manner as the above Lemma 4.1, taking into account Lemma 1 from [10] which says that $P \oplus p^\omega G$ is nice in G if and only if P is nice in G (see [4] too). Also, $p^\omega G/(p^\omega G \cap P) \cong p^\omega G$ is now countable. □

4.3. Corollary. *If G is strongly (respectively, separably) ω_1-$p^{\omega+n}$-projective, then so is $p^\alpha G$ for any ordinal α.*

Proof. Let N be a nice p^n-bounded subgroup of G such that G/N is ω-totally Σ-cyclic (in addition, $p^\omega G \cap N = \{0\}$). Consequently, owing to ([7], Theorem 2.6) or to the comments after the Proposition of Megibben, one can see that $(p^\alpha G + N)/N \subseteq G/N$ is also ω-totally Σ-cyclic as being a subgroup, and thus $(p^\alpha G + N)/N \cong p^\alpha G/(p^\alpha G \cap N)$ is also ω-totally Σ-cyclic, where $p^\alpha G \cap N$ is p^n-bounded and nice in $p^\alpha G$. In addition, $p^\omega(p^\alpha G) \cap (p^\alpha G \cap N) = p^{\alpha+\omega} G \cap N \subseteq p^\omega G \cap N = \{0\}$, as needed. □

4.4. Corollary. *If G is strongly (respectively, separably) ω_1-$p^{\omega+n}$-projective, then so is $G/p^\alpha G$ for each ordinal α.*

Proof. Let N be a p^n-bounded nice subgroup of G such that G/N is the direct sum of a countable group and a Σ-cyclic group . Put $N' = (N + p^\alpha G)/p^\alpha G$, and it is easily seen that N' is p^n-bounded and nice in $G/p^\alpha G$. Likewise,

$$G/p^\alpha G/(N + p^\alpha G)/p^\alpha G \cong G/(N + p^\alpha G) \cong G/N/(N + p^\alpha G)/N = G/N/p^\alpha(G/N).$$

But $p^\alpha(G/N)$ is again countable whenever $\alpha \geq \omega$, hence $G/N/p^\alpha(G/N)$ remains a direct sum of a countable group and a Σ-cyclic group. Finally, $G/p^\alpha G$ is a strongly ω_1-$p^{\omega+n}$-projective group, as expected. In addition, the modular law from [9] ensures that $N' \cap p^\omega(G/p^\alpha G) = N' \cap (p^\omega G/p^\alpha G) = [(N + p^\alpha G) \cap p^\omega G]/p^\alpha G = (p^\alpha G + N \cap p^\omega G)/p^\alpha G = \{0\}$ provided $\alpha > \omega$ and $N \cap p^\omega G = \{0\}$. For $\alpha \leq \omega$, the intersection is again clearly equal to zero. □

The next two corollaries are also consequences of results from [8].

4.5. Corollary. *If G is a group such that $p^{\omega+n}G = \{0\}$, then G is strongly ω_1-$p^{\omega+n}$-projective if and only if G is $p^{\omega+n}$-projective.*

Proof. In accordance with Proposition 4.1, the quotient $G/(N + p^\omega G)$ is Σ-cyclic for some $N \leq G[p^n]$. Thus $p^n(N + p^\omega G) = \{0\}$ and Theorem 1.1 is manifestly applicable to obtain the claim. □

4.6. Corollary. *If G is strongly ω_1-$p^{\omega+n}$-projective, then $G/p^{\omega+n}G$ is $p^{\omega+n}$-projective.*

Proof. Follows directly from the combination of Corollaries 4.4 and 4.5. □

Remark. In ([12], Example 2.3) was constructed an example of an ω_1-$p^{\omega+n}$-projective group of length $\omega + n$ which is not $p^{\omega+n}$-projective; thereby in view of Corollary 4.5 it is not strongly ω_1-$p^{\omega+n}$-projective as well. Invoking [8], it is not even strongly n-simply presented.

Moreover, the following inclusions hold:

{separable $p^{\omega+n}$-projective groups} $\subseteq$ {$p^{\omega+n}$-projective groups} $\cap$ {separably ω_1-$p^{\omega+n}$-projective groups} $\subseteq$ {strongly ω_1-$p^{\omega+n}$-projective groups} $\subseteq$ {ω_1-$p^{\omega+n}$-projective groups} $\cap$ {strongly n-simply presented groups}.

Below we shall demonstrate that the last containment is actually tantamount to an equality - see Corollary 4.16.

On the other hand, Keef also showed in [12] that for any $n \geq 2$ there is a $p^{\omega+n}$-projective group G with the property that G is not separably ω_1-$p^{\omega+n}$-projective (see too the Example on p. 4382 of [11] where a $p^{\omega+n}$-projective group was exhibited which is not separably n-simply presented and thus not separably ω_1-$p^{\omega+n}$-projective; however every $p^{\omega+1}$-projective group is separably 1-simply presented). That is why there exists an example of a strongly ω_1-$p^{\omega+n}$-projective group that is not separably ω_1-$p^{\omega+n}$-projective (and even not separably n-simply presented) whenever $n > 1$. For $n = 1$ this will be illustrated below as well.

As a matter of fact, we begin with the following affirmation that restricts strong (separable) ω_1-$p^{\omega+1}$-projectivity to Ulm subgroups and Ulm factors.

4.7. Proposition. *The group G is strongly ω_1-$p^{\omega+1}$-projective if and only if*
(i) $p^{\omega+1}G$ is countable;
(ii) $G/p^{\omega+1}G$ is $p^{\omega+1}$-projective.

Proof. The necessity being already established in the series of our previous assertions, we concentrate now on the sufficiency.

And so, using ([7], Corollary 2.11), the decomposition $G = K \oplus S$ holds, where K is countable and S is $p^{\omega+1}$-projective. Thus, by Theorem 1.1 (a), there is $T \leq S[p]$ with S/T being Σ-cyclic. Hence T is nice in S and so in G. Finally, $G/T \cong K \oplus (S/T)$ is the direct sum of a countable group and a Σ-cyclic group, as required in Definition 1.1. Besides, even $T \cap p^{\omega+1}G = T \cap p^{\omega+1}K \subseteq S \cap K = \{0\}$ is fulfilled. □

4.8. Proposition. *The group G is separably ω_1-$p^{\omega+1}$-projective if and only if*
(i) $p^{\omega}G$ is countable;
(ii) $G/p^{\omega+1}G$ is $p^{\omega+1}$-projective.

Proof. The necessity being already obtained in the series of our preceding statements, we deal now with the sufficiency. And so, utilizing ([7], Corollary 2.11), one may decompose $G = L \oplus R$, where L is countable and R is separable $p^{\omega+1}$-projective. Thus, again an appeal to Theorem 1.1 (a), leads to the existence of $M \leq R[p]$ such that R/M is Σ-cyclic. Hence M is nice in R and so it is nice in G. Furthermore, $G/M \cong L \oplus (R/M)$ is the direct sum of a countable group and a Σ-cyclic group. Moreover, $M \cap p^{\omega}G = M \cap p^{\omega}L \subseteq R \cap L = \{0\}$, as required in Definition 1.2. □

As promised above, the wanted example of a strongly ω_1-$p^{\omega+1}$-projective non separably ω_1-$p^{\omega+1}$-projective group can be produced by choosing a group G whose subgroup $p^{\omega+1}G$ is countable but such that $p^{\omega}G$ is uncountable, and $G/p^{\omega+1}G$ is $p^{\omega+1}$-projective. There exists an abundance of such groups; in fact, any $p^{\omega+1}$-projective group G with uncountable $p^{\omega}G$ may be applied in this situation. Nevertheless, each $p^{\omega+1}$-projective group G with countable $p^{\omega}G$ (in particular, each separable $p^{\omega+1}$-projective group) is separably ω_1-$p^{\omega+1}$-projective, as it will be seen below. This crucial property is due to the fact that $p^{\omega+1}$-projectives are C-decomposable (for more details see, for instance, [11] and [12]).

4.9. Proposition. *Suppose that G is a group whose $p^{\omega}G$ is countable. Then G is separably n-simply presented if and only if G is separably ω_1-$p^{\omega+n}$-projective.*

Proof. The sufficiency being trivial, we are now attack the necessity. Thus the application of [10] guarantees that $G/(M \oplus p^{\omega}G)$ is Σ-cyclic for some p^n-bounded nice subgroup M of G such that $M \cap p^{\omega}G = \{0\}$. But $G/(M \oplus p^{\omega}G) \cong G/M/(M \oplus p^{\omega}G)/M$ and since $(M \oplus p^{\omega}G)/M \cong p^{\omega}G$ is countable, the Lemma of Charles listed above applies to show that G/M is the direct sum of a countable group and a Σ-cyclic group. So, by Definition 1.2, the group G has to be separably ω_1-$p^{\omega+n}$-projective, as desired. $\square$

4.10. Proposition. *Let G be a group such that $p^\omega G$ is countable. Then G is both separably ω_1-$p^{\omega+1}$-projective and $p^{\omega+1}$-bounded if and only if G is $p^{\omega+1}$-projective.*

Proof. The necessity follows immediately from Corollary 4.5.

Concerning the sufficiency, it was proved in [11] that any $p^{\omega+1}$-projective groups belongs to the class of separably 1-simply presented groups. We now employ the preceding Proposition 4.9 to get the claim. $\square$

4.11. Corollary. *Suppose G is a group with countable $p^\omega G$. Then $G/p^{\omega+1}G$ is separably ω_1-$p^{\omega+1}$-projective if and only if $G/p^{\omega+1}G$ is $p^{\omega+1}$-projective.*

Proof. Observe that $p^\omega(G/p^{\omega+1}G) = p^\omega G/p^{\omega+1}G$ is countable and we next apply Proposition 4.10. $\square$

4.12. Corollary. *If G is a separably ω_1-$p^{\omega+1}$-projective group and H is a subgroup such that $H \cap p^{\omega+1}G = p^{\omega+1}H$, then H is separably ω_1-$p^{\omega+1}$-projective.*

In particular, isotype subgroups of separably ω_1-$p^{\omega+1}$-projectives are separably ω_1-$p^{\omega+1}$-projective.

Proof. With the help of Proposition 4.8 write that $p^\omega G$ is countable and $G/p^{\omega+1}G$ is $p^{\omega+1}$-projective. Hence $p^\omega H$ is countable, and $H/p^{\omega+1}H = H/(H \cap p^{\omega+1}G) \cong (H + p^{\omega+1}G)/p^{\omega+1}G \subseteq G/p^{\omega+1}G$ is $p^{\omega+1}$-projective. Consequently, again Proposition 4.8 works to get the assertion. The second half is immediate. $\square$

The above two reduction statements suggest the following stronger consideration. So we will now somewhat enlarge Propositions 4.7 and 4.8 to an arbitrary natural number $n \geq 1$ in an identical way, noticing also that Corollary 4.11 can be eventually derived from the next Theorem 4.13. In this aspect, Keef showed in [11] that a group G is separably n-simply presented if and only if $p^{\omega+n}G$ is simply presented and $G/p^{\omega+n}G$ is separably n-simply presented, while in [8] it was established that G is strongly n-simply presented if and only if $p^{\omega+n}G$ is strongly n-simply presented and $G/p^{\omega+n}G$ is $p^{\omega+n}$-projective. Moreover, Keef proved in [12] that G is ω_1-$p^{\omega+n}$-projective if and only if $p^{\omega+n}G$ is countable and $G/p^{\omega+n}G$ is ω_1-$p^{\omega+n}$-projective.

So, keeping the similarity of the formulation, we are now able to formulate and prove our first central result of the present section.

4.13. Theorem. *(First Reduction Criterion). For every $n \geq 1$ the group G is strongly ω_1-$p^{\omega+n}$-projective if and only if*

(1) $p^{\omega+n}G$ is countable;

(2) $G/p^{\omega+n}G$ is $p^{\omega+n}$-projective.

Proof. "$\Rightarrow$". According to Lemma 4.1, one may write that $p^\omega G/(p^\omega G \cap N)$ is countable for some p^n-bounded nice subgroup N of G. Thus $p^\omega G = p^\omega G \cap N + C$ where $C \leq p^\omega G$ is countable. Furthermore, $p^{\omega+n}G = p^n C$ is countable, so that clause (1) follows.

Next, point (2) follows directly from Corollary 4.6.

"$\Leftarrow$". Suppose that $P \leq G$ such that $p^{\omega+n}G \subseteq P$, $p^n P \subseteq p^{\omega+n}G$ (thereby $P/p^{\omega+n}G$ is p^n-bounded) and G/P is Σ-cyclic. Let Y be a maximal p^n-bounded summand of $p^\omega G$; so there is a decomposition $p^\omega G = X \oplus Y$ and thus the inclusions $X \subseteq p^\omega G \subseteq P$ hold. We may assume without loss of generality that X is countable; in fact, $p^{\omega+n}G = p^n X$ is countable and so we can decompose $X = K \oplus T$ where K is countable and T is p^n-bounded (whence T is a p^n-bounded summand of $p^\omega G$ and thereby $T \subseteq Y$; then even $T = T \cap Y \subseteq X \cap Y = \{0\}$ and $X = K$ - in any case $p^\omega G = K \oplus (T \oplus Y)$ where $T \oplus Y$ is

p^n-bounded). That is why $p^\omega G = K \oplus Y$ with a countable summand K, as desired. Another verification of this fact is like this: Note that $X[p] = (p^{\omega+n}G)[p] = (p^n X)[p]$, and hence $X[p]$ is countable. So X will be countable, provided that it is reduced.

Let us now H be a $p^{\omega+n}$-high subgroup of G containing Y (thus H is maximal with respect to $H \cap p^{\omega+n}G = \{0\}$). We next assert that $(G/p^{\omega+n}G)[p^n] = (X \oplus H[p^n])/p^{\omega+n}G$. To this aim, given $v \in G$ with $p^n v \in p^{\omega+n}G$, it suffices to prove that $v \in X \oplus H[p^n]$. If $x \in X$ is chosen such that $p^n x = p^n v$, then replacing v by $v - x$, we may assume that $p^n v = 0$. Since $G[p] = (p^{\omega+n}G)[p] \oplus H[p] = X[p] \oplus H[p]$ and H is pure in G, it easily follows that $G[p^n] = X[p^n] \oplus H[p^n]$. Therefore, $v = x' + h$ where $x' \in X[p^n]$ and $h \in H[p^n]$ as required. Moreover, $X \cap H = \{0\}$ because as noted above $X[p] = (p^{\omega+n}G)[p]$, which substantiates our assertion. Furthermore, by what we have just shown above, $P/p^{\omega+n}G \subseteq (G/p^{\omega+n}G)[p^n]$ implies that $P \subseteq X \oplus H[p^n]$. Note also the fact from above that $X \leq P$. Let $L = P \cap H[p^n] \subseteq H[p^n] \subseteq G[p^n]$; so $p^n L = \{0\}$. Clearly, the inclusion $L \subseteq H$ forces that $L \cap p^{\omega+n}G = \{0\}$. Likewise, $P \subseteq X \oplus H[p^n]$ yields that $P = X + (P \cap H[p^n]) = X + L$; indeed the modular law applies to get that $P = (X \oplus H[p^n]) \cap P = X + P \cap H[p^n]$ as stated. Consequently, we conclude that $P = p^\omega G + P = p^\omega G + L$. Thus $G/P = G/(p^\omega G + L)$ is Σ-cyclic.

We next will show that L is nice in G. Since $L \cap p^{\omega+n}G = \{0\}$, it readily follows via some technical efforts that $L \cap p^\omega G$ is nice in $p^\omega G$ and so nice in G. But $L + p^\omega G = P$ is also nice in G because $G/(p^\omega G + L)$ is separable, and these two conditions together imply that L is nice in G, as wanted (see, e.g., Section 79, Exercise 10 of [9]).

Furthermore, we claim that $p^\omega(G/L) = (p^\omega G + L)/L = P/L$ is countable. In fact, $P/L = P/(P \cap H[p^n]) \cong (P + H[p^n])/H[p^n] = (p^\omega G + H[p^n])/H[p^n] \cong p^\omega G/(p^\omega G \cap H[p^n])$. But $p^\omega G = X \oplus Y$ and since $Y \subseteq H$, one may have in view of the modular law that $p^\omega G \cap H = (X \oplus Y) \cap H = (X \cap H) \oplus Y = Y$. We therefore establish that $P/L \cong (X \oplus Y)/Y[p^n] \cong X \oplus (Y/Y[p^n]) \cong X \oplus p^n Y = X$, because $p^n Y = \{0\}$. As noticed above, X is countable, so that $p^\omega(G/L)$ is really countable as claimed. Finally, Lemma 4.1 allows us to infer that G is strongly ω_1-$p^{\omega+n}$-projective, as required. □

An immediate consequence is this one:

4.14. Proposition. *Suppose that G is a group whose $p^{\omega+n}G$ is countable. Then the following are equivalent:*

(a) G is strongly ω_1-$p^{\omega+n}$-projective;
(b) $G/p^{\omega+n}G$ is strongly ω_1-$p^{\omega+n}$-projective;
(c) $G/p^{\omega+n}G$ is $p^{\omega+n}$-projective.

Proof. Follows by a direct application of Corollaries 4.4 and 4.5 as well as of Theorem 4.13. □

As a new valuable consequence of the First Reduction Criterion, we obtain an analog of Proposition 4.9 (see also Corollary 3.2):

4.15. Corollary. *Suppose $p^{\omega+n}G$ is countable. Then G is strongly n-simply presented if and only if G is strongly ω_1-$p^{\omega+n}$-projective.*

Proof. One direction "⇐" being trivial, we observe for the another one "⇒" that, appealing to [8], the quotient $G/p^{\omega+n}G$ is $p^{\omega+n}$-projective. Next, the First Reduction Criterion can be applied to derive that G is strongly ω_1-$p^{\omega+n}$-projective, as formulated. □

An interesting consequence to the last statement is the following.

4.16. Corollary. *Strongly n-simply presented ω_1-$p^{\omega+n}$-projective groups are strongly ω_1-$p^{\omega+n}$-projective, and vice versa.*

Proof. The sufficiency being elementary, we will attack the necessity. Since by Theorem 1.2 (i) for each ω_1-$p^{\omega+n}$-projective group G we have that $p^{\omega+n}G$ is countable, Corollary 4.15 applies to infer that G is, in fact, strongly ω_1-$p^{\omega+n}$-projective. □

4.17. Corollary. *Suppose G is a group such that $p^\omega G$ is countable. Then the following are equivalent:*

(1) G is strongly ω_1-$p^{\omega+1}$-projective;
(2) G is separably 1-simply presented;
(3) G is separably ω_1-$p^{\omega+1}$-projective.

Proof. The equivalence (1) $\iff$ (3) follows from directly Propositions 4.7 and 4.8. On the other hand the equivalence (2) $\iff$ (3) was proved in Proposition 4.9. □

For $n = 1$ the alluded to above Corollary 4.15 can be slightly extended in the following way:

4.18. Corollary. *Suppose that G is a group with countable $p^{\omega+1}G$. Then the following three conditions are equivalent:*

(1) G is strongly 1-simply presented;
(2) G is separably 1-simply presented;
(3) G is strongly ω_1-$p^{\omega+1}$-projective.

Proof. For the fact that (1) is tantamount to (3) we employ Corollary 4.15.

To prove that (2) and (3) are equal, we first observe that separably 1-simply presented groups are strongly 1-simply presented and thus by what we have just shown, they are strongly ω_1-$p^{\omega+1}$-projective. So (2) implies (3). In order to verify the converse, we next apply the First Reduction Criterion to deduce that $G/p^{\omega+1}G$ is $p^{\omega+1}$-projective, whence in view of [11] this quotient must be separably 1-simply presented. Finally, again an appeal to [11] insures that G has to be separably 1-simply presented, as wanted. □

Note that the last two corollaries fail for $n \geq 2$.

4.19. Corollary. *Let H be a subgroup of the strongly ω_1-$p^{\omega+n}$-projective group such that $H \cap p^{\omega+n}G = p^{\omega+n}H$. Then H is strongly ω_1-$p^{\omega+n}$-projective.*

In particular, isotype subgroups of strongly ω_1-$p^{\omega+n}$-projectives are strongly ω_1-$p^{\omega+n}$-projective.

Proof. Employing Theorem 4.13 we can write that $p^{\omega+n}G$ is countable and $G/p^{\omega+n}G$ is $p^{\omega+n}$-projective. Thus $p^{\omega+n}H$ is countable as being a subgroup of $p^{\omega+n}G$. Moreover, $H/p^{\omega+n}H = H/(H \cap p^{\omega+n}G) \cong (H + p^{\omega+n}G)/p^{\omega+n}G \subseteq G/p^{\omega+n}G$ is $p^{\omega+n}$-projective as well. So, again the utilization of the First Reduction Criterion guarantees that H is strongly ω_1-$p^{\omega+n}$-projective, as expected. The final part is immediate. □

We are now in a position and state and prove the second major result of this section.

4.20. Theorem. *(Second Reduction Criterion). For every $n \geq 1$ the group G is separably ω_1-$p^{\omega+n}$-projective if and only if*

(1) $p^\omega G$ is countable;
(2) $G/p^{\omega+n}G$ is separably ω_1-$p^{\omega+n}$-projective.

Proof. "$\Rightarrow$". That $p^\omega G$ is countable is evident in virtue of Lemma 4.2. With the aid of the same lemma write that $G/(P \oplus p^\omega G)$ is Σ-cyclic for some $P \leq G[p^n]$ with $P \cap p^\omega G = \{0\}$. But by the modular law we have

$$[(P + p^{\omega+n}G)/p^{\omega+n}G] \cap p^\omega(G/p^{\omega+n}G) = [(P + p^{\omega+n}G) \cap p^\omega G]/p^{\omega+n}G =$$

$$= [(P \cap p^\omega G) + p^{\omega+n}G]/p^{\omega+n}G = \{0\}.$$

Furthermore,

$$G/(P \oplus p^\omega G) \cong G/p^{\omega+n}G/(P \oplus p^\omega G)/p^{\omega+n}G =$$

$$= G/p^{\omega+n}G/[((P + p^{\omega+n}G)/p^{\omega+n}G) \oplus p^\omega(G/p^{\omega+n}G)]$$

is Σ-cyclic with $p^n[(P + p^{\omega+n}G)/p^{\omega+n}G] = \{0\}$ and $p^\omega(G/p^{\omega+n}G)$ countable. This verifies the necessity.

"$\Leftarrow$". Let Q be a subgroup of G containing $p^{\omega+n}G$ such that $Q/p^{\omega+n}G$ is p^n-bounded (i.e., $p^nQ \subseteq p^{\omega+n}G$), $Q \cap p^\omega G \subseteq p^{\omega+n}G$ and $G/p^{\omega+n}G/[(Q/p^{\omega+n}G) \oplus p^\omega(G/p^{\omega+n}G)] = G/p^{\omega+n}G/(Q + p^\omega G)/p^{\omega+n}G \cong G/(Q + p^\omega G)$ is Σ-cyclic. Suppose

$$Q/p^{\omega+n}G = \oplus_{i\in I}\langle x_i + p^{\omega+n}G\rangle$$

where $x_i \in Q$ and order$(x_i + p^{\omega+n}G) = p^{t_i} \leq p^n$ in $G/p^{\omega+n}G$, which is equivalent to $p^{t_i}(x_i + p^{\omega+n}G) = p^{\omega+n}G$, i.e. to $p^{t_i}x_i \in p^{\omega+n}G$, and t_i is the minimal natural number with this property. Now, for each $i \in I$, $p^{t_i}x_i \in p^{\omega+n}G = p^n(p^\omega G) = p^{t_i}(p^{\omega+n-t_i}G)$ whence $p^{t_i}x_i = p^{t_i}g_i$ for some $g_i \in p^\omega G$. Put

$$P = \oplus_{i\in I}\langle x_i - g_i\rangle$$

observing that $P \subseteq G$. Clearly, $p^nP = \{0\}$ because $p^nx_i = p^ng_i$. If now $y \in P \cap p^\omega G$, then one may write $y = a_1(x_{i_1} - g_{i_1}) + \cdots + a_k(x_{i_k} - g_{i_k})$ for some collection of indexes i_j and integers a_j, where $j = 1, \cdots, k$. This forces that

$$a_1x_{i_1} + \cdots + a_kx_{i_k} + p^{\omega+n}G = y + a_1g_{i_1} + \cdots + a_kg_{i_k} + p^{\omega+n}G \in$$

$$(Q/p^{\omega+n}G) \cap p^\omega(G/^{\omega+n}G) = (Q/p^{\omega+n}G) \cap (p^\omega G/p^{\omega+n}G) = (Q \cap p^\omega G)/p^{\omega+n}G = \{0\},$$

hence we have $a_1x_{i_1} + \cdots + a_kx_{i_k} \in p^{\omega+n}G$ which ensures that $a_1(x_{i_1} + p^{\omega+n}G) = \cdots = a_k(x_{i_k} + p^{\omega+n}G) = p^{\omega+n}G$. Consequently, $p^{t_{i_j}}/a_j$ for every $j = 1, \cdots, k$ and hence

$$y = s_1p^{t_1}(x_{i_1} - g_{i_1}) + \cdots + s_kp^{t_k}(x_{i_k} - g_{i_k}) =$$

$$s_1(p^{t_1}x_{i_1} - p^{t_1}g_{i_1}) + \cdots + s_k(p^{t_k}x_{i_k} - p^{t_k}g_{i_k}) = 0.$$

That is why $P \cap p^\omega G = \{0\}$ as expected. Finally, since $Q = \sum_{i\in I}\langle x_i + p^{\omega+n}G\rangle = \sum_{i\in I}\langle x_i\rangle + p^{\omega+n}G$, we infer that $P + p^\omega G = Q + p^\omega G$. But $G/(P \oplus p^\omega G) = G/(Q + p^\omega G)$ is Σ-cyclic and this substantiates the sufficiency in accordance with Lemma 4.2. □

Remark. It is worthwhile noticing that, unfortunately, the Second Reduction Criterion does not directly lead to the aforementioned fact from Proposition 4.9 that separably n-simply presented groups with countable first Ulm subgroup are themselves separably ω_1-$p^{\omega+n}$-projective. The reason for this contrast with the First Reduction Criterion is that separably n-simply presented groups of length $\leq \omega + n$ need not be separably ω_1-$p^{\omega+n}$-projective for any $n \geq 1$; they are just $p^{\omega+n}$-projective.

The following example illustrates that in point (2) of Theorem 4.20 the factor-group $G/p^{\omega+n}G$ cannot be replaced to be $p^{\omega+n}$-projective when $n \geq 2$ (compare also the difference with Proposition 4.8 when $n = 1$). This is so because separably ω_1-$p^{\omega+n}$-projectives of length $\leq \omega + n$ are necessarily $p^{\omega+n}$-projective but the converse fails whenever $n \geq 2$ and even for $n = 1$ provided the first Ulm subgroup is uncountable (see Proposition 4.10 too).

Example. Let A be the $p^{\omega+n}$-projective group which is not separably ω_1-$p^{\omega+n}$-projective for some $n \geq 2$, as constructed in [11], and let G be a group such that $G/p^{\omega+n}G \cong A$. We claim that G is not separably ω_1-$p^{\omega+n}$-projective because, otherwise, Corollary 4.4 would imply that so is $G/p^{\omega+n}G$ that is against our construction. The example is sustained.

However, since $G/p^{\omega+n}G$ is $p^{\omega+n}$-projective, the First Reduction Criterion, that is Theorem 4.13, assures that G is necessarily strongly ω_1-$p^{\omega+n}$-projective.

In [10] was appeared that summands of separably n-simply presented groups are again separably n-simply presented. The same idea works and for separably ω_1-$p^{\omega+n}$-projective groups, so that one may formulate without a proof the following.

4.21. Proposition. *A summand of a separably ω_1-$p^{\omega+n}$-projective group is also separably ω_1-$p^{\omega+n}$-projective.*

5. Concluding Discussion

Certainly, the major concept of strong ω_1-$p^{\omega+n}$-projectivity can be extended as follows:

Definition 5.1. A group G is called *weakly n-ω_1-$p^{\omega+n}$-projective* if there exists a subgroup $R \leq G[p^n]$ which is nice in G such that G/R is a subgroup of the direct sum of a countable group and a $p^{\omega+n}$-projective group.

It is worth noticing that, in view of Theorem 1.5 (a) from [12], G/R must be ω_1-$p^{\omega+n}$-projective. Also, the subgroup $p^{\omega+2n}G$ must be countable.

Besides, strongly n-simply presented groups of length $\leq \omega+2n$ and n-simply presented groups of length $\leq \omega + n$ are both strongly n-ω_1-$p^{\omega+n}$-projective by taking $R = p^{\omega+n}G$ or $R = p^{\omega}G$, respectively.

Another interesting variation in a more weak form of ω_1-$p^{\omega+n}$-projectivity is given in the following new concept:

Definition 5.2. A group G is said to be *nicely ω_1-$p^{\omega+n}$-projective* if it has a nice $p^{\omega+n}$-projective subgroup X such that G/X is countable.

Apparently, owing to ([12], Theorem 1.2 (c3)), nicely ω_1-$p^{\omega+n}$-projectives are themselves ω_1-$p^{\omega+n}$-projective.

The class of nicely ω_1-$p^{\omega+n}$-projectives is also worthy of investigation, which will be done in a subsequent article.

Corrigendum. In the proof of Proposition 2.3 from [7] there is a typo, namely the subgroup P of H should satisfy $p^{n+1}P = \{0\}$ instead of the written there equality $p^{\omega+n+1}P = \{0\}$.

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