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# ON A REDUCTION FORMULA FOR THE KAMPÉ de FÉRIET FUNCTION

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**Abstract**

The aim of this short research note is to provide a reduction formula for the Kampé de Fériet function $F^{h:2;0}_{g:2;0}[-x, x]$ by employing a new summation formula for Clausen's series ${}_3F_2[1]$ obtained recently by the authors [Miskolc Math. Notes **10**(2), 145–153, 2009.]

**Keywords:** Clausen's series ${}_3F_2$, Euler's transformation for ${}_2F_2$, Kampé de Fériet function, Kummer–type I transformation for ${}_2F_2$, summation formula.

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## 1. Introduction and results required

Recently Paris [9] established a Kummer–type I transformation formula for the generalized hypergeoemtric function ${}_2F_2[x]$, namely

$$(1.1)\qquad {}_2F_2\left[\begin{matrix} a, & c+1 \\ b, & c \end{matrix}\; ; x\right] = \mathrm{e}^x\, {}_2F_2\left[\begin{matrix} b-a-1, & f+1 \\ b, & f \end{matrix}\; ; -x\right] \qquad x \in \mathbb{C}\,,$$

where

$$f = \frac{c(1+a-b)}{a-c}\,.$$

Equation (1.1) is seen to be analogous to the well–known and much employed Kummer's first transformation for the confluent hypergeometric function

$${}_1F_1\left[\begin{matrix} a \\ b \end{matrix}\; ; x\right] = \mathrm{e}^x\, {}_1F_1\left[\begin{matrix} b-a \\ b \end{matrix}\; ; -x\right].$$

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Paris' result (1.1) may be regarded as the generalization of the Exton's result [5], by letting $2c = a$ so that $f = 1 + a - b$, given by

$$ {}_2F_2\left[\begin{matrix} a, & 1+\frac{1}{2}a \\ & \frac{1}{2}a \end{matrix} ; x\right] = \mathrm{e}^x\, {}_2F_2\left[\begin{matrix} b-a-1, & 2+a-b \\ b, & 1+a-b \end{matrix} ; -x\right]. $$

Recently Kim *et al.* [8] have obtained a new summation formula for Clausen's ${}_3F_2[1]$ series given by

$$ {}_3F_2\left[\begin{matrix} -n, & b-a-1, & f+1 \\ & b, & f \end{matrix} ; 1\right] = \frac{(a)_n(c+1)_n}{(b)_n(c)_n}, \tag{1.2} $$

where $(a)_n = \Gamma(a+n)/\Gamma(n) = a(a+1)\cdots(a+n-1)$, $a \in \mathbb{C} \setminus Z_0^-$ stands for the Pochhammer symbol and $f$ is the same as in (1.1). We note that by convention $(a)_0 = 1$.

By utilizing (1.2), Kim *et al.* [8] have obtained the following result:

$$ (1-x)^{-h}\, {}_3F_2\left[\begin{matrix} h, & b-a-1, & f+1 \\ & b, & f \end{matrix} ; -\frac{x}{1-x}\right] = {}_3F_2\left[\begin{matrix} h, & a, & c+1 \\ & b, & c \end{matrix} ; x\right]. $$

This result is also recorded in [10], in a slightly modified form. On the other hand, this relation may be regarded as a generalization of the following result due to Exton [5]:

$$ (1-x)^{-h}\, {}_3F_2\left[\begin{matrix} h, & a, & 1+\frac{1}{2}a \\ & b, & \frac{1}{2}a \end{matrix} ; -\frac{x}{1-x}\right] = {}_3F_2\left[\begin{matrix} h, & b-a-1, & 2+a-b \\ & b, & 1+a-b \end{matrix} ; x\right]. $$

On the other hand, just as the Gauss function ${}_2F_1$ was extended to generalized hypergeometric function ${}_pF_q$ by increasing the number of parameters in the numerator as well as in the denominator, the four Appell functions were introduced and generalized by Appell and Kampé de Fériet [1] who defined a general hypergeometric function in two variables. For further details see [12]. The notation defined and introduced originally by Kampé de Fériet for this double hypergeometric function of superior order was subsequently abbreviated by Burchnall and Chaundy [3]. We, however, recall here the definition of a more general double hypergeometric function (than the one defined by Kampé de Fériet) in a sligthly modified notation given by Srivastava and Panda [14, p. 423, Eq. (26)]. For this, let $(H_h)$ denotes the sequence of parameters $(H_1, \cdots, H_h)$ and for nonnegative integers define the Pochhammer symbols $((H_h)) := (H_1)_n(H_2)_n\cdots(H_h)_n$, where when $n = 0$, the product is understood to reduce to unity. Therefore, the convenient generalization of the Kampé de Fériet function is defined as follows:

$$ F^{h:a;b}_{g:c;d}\left[\begin{matrix} (H_h): & (A_a); (B_b) & ; \\ (G_g): & (C_c); (D_d) & ; \end{matrix}\; x,\, y\right] = \sum_{m,n\geq 0} \frac{((H_h))_{m+n}((A_a))_m((B_b))_n}{((G_g))_{m+n}((C_c))_m((D_d))_n}\, \frac{x^m}{m!}\, \frac{y^n}{n!}. \tag{1.3} $$

For more details about the convergence for the function (1.3) we refer to [1]. Various authors (see e.g. [1, 4, 5, 6, 7, 11, 12]) have discussed the reducibility of the Kampé de Fériet function.

The main objective of this short research note is to establish a reduction formula for the Kampé de Fériet function $F^{h:2;0}_{g:2;0}[-x, x]$ by employing the summation formula (1.2).

## 2. Main result

**2.1. Theorem.** *There holds true*

$$ F^{h:2;0}_{g:2;0}\left[\begin{matrix} (H_h): & b-a-1, & f+1 & ; - ; \\ (G_g): & b, & f & ; - ; \end{matrix}\; -x,\, x\right] = {}_{h+2}F_{g+2}\left[\begin{matrix} (H_h), & a, & c+1 \\ (G_g), & b, & c \end{matrix} ; x\right], \tag{2.1} $$

*where $f$ is given in (1.1). Here the series (2.1) converges either for all $x \in \mathbb{C}$ for $g \geq h$; or inside the unit circle $|x| < 1$ when $g = h - 1$; or on the unit circle $|x| = 1$ when*

$$\Re\left\{\sum_{j=1}^{h-1} G_j - \sum_{j=1}^{h} H_j + b - a\right\} > 1\,.$$

*Proof.* In order to derive (2.1), we proceed as follows. Denoting the left–hand side of (2.1) by $S$ and expressing the Kampé de Fériet function as a double series, we have

$$S = \sum_{m,n\geq 0} \frac{((H_h))_{m+n}\,(b-a-1)_m\,(f+1)_m}{((G_g))_{m+n}\,(b)_m\,(f)_m}\,\frac{(-1)^m x^{n+m}}{m!\,n!}\,.$$

Making use of the well–known Bailey–transform technique in summing up double infinite series [2]

$$\sum_{n\geq 0}\sum_{k\geq 0} A(k,n) = \sum_{n\geq 0}\sum_{k=0}^{n} A(k, n-k)\,,$$

we have, after some little algebra, using

$$(n-m)! = \frac{(-1)^m\,n!}{(-n)_m}\,,$$

that

$$S = \sum_{n\geq 0} \frac{((H_h))_n}{((G_g))_n}\,\frac{x^n}{n!}\,\sum_{m=0}^{n} \frac{(-n)_m (b-a-1)_m\,(f+1)_m}{(b)_m\,(f)_m\,m!}\,.$$

The inner–most finite series we recognize as a ${}_3F_2[1]$ expression, that is

$$S = \sum_{n\geq 0} \frac{((H_h))_n}{((G_g))_n}\,\frac{x^n}{n!}\,{}_3F_2\left[\begin{array}{c} -n,\ b-a-1,\ f+1 \\ b,\quad f \end{array};1\right].$$

Using (1.2) we have

$$S = \sum_{n\geq 0} \frac{((H_h))_n}{((G_g))_n}\cdot\frac{(a)_n\,(c+1)_n}{(b)_n\,(c)_n}\cdot\frac{x^n}{n!}\,,$$

which gives in fact the right–hand side of the series (2.1).

By conditions that hold for the generalized hypergeometric function we easily conclude the stated convergence constraints. □

## 3. Special cases

**3.1.** In (2.1), if we take $2c = a$, so that $f = 1 + a - b$, we get the following result due to Exton [5]:

$$F^{h:2;0}_{g:2;0}\left[\begin{array}{lccc} (H_h): & b-a-1, & 2+a-b & ;\,-\,; \\ (G_g): & b, & 1+a-b & ;\,-\,; \end{array} -x,\,x\right] = {}_{h+2}F_{g+2}\left[\begin{array}{l} (H_h),\ a,\ \tfrac{1}{2}a+1 \\ (G_g),\ \tfrac{1}{2}a,\quad b \end{array};x\right],$$

where the series converges under the same conditions which hold for (2.1).

**3.2.** If we take $b = c+1$, so that $f = c$, we arrive at the following result:

$$F^{h:1;0}_{g:1;0}\left[\begin{array}{lcc} (H_h): & c-a & ;-; \\ (G_g): & c & ;-; \end{array} -x,\, x\right] = {}_{h+1}F_{g+1}\left[\begin{array}{l} (H_h),\ a \\ (G_g),\ c \end{array} ;\, x\right],$$

where the series converges under the same conditions which hold for (2.1), exception is the convergence for $g = h-1$ on the unit circle $|x| = 1$ which follows for

$$\Re\left\{\sum_{j=1}^{h-1} G_j - \sum_{j=1}^{h} H_j + c - a\right\} > 0\,.$$

**3.3.** Finally, if we take $(H) = (G)$ and $h = g = 0$, we arrive at Paris' result (1.1). In this case, the formula is valid in the whole complex plane $\mathbb{C}$.

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