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GENERALIZED SKEW DERIVATIONS ON MULTILINEAR POLYNOMIALS IN RIGHT IDEALS OF PRIME RINGS

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Abstract

Let R be a prime ring, $f(x_1, \ldots, x_n)$ a multilinear polynomial over C in n noncommuting indeterminates, I a nonzero right ideal of R, and $F : R \to R$ be a nonzero generalized skew derivation of R.
Suppose that $F(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) \in C$, for all $r_1, \ldots, r_n \in I$. If $f(x_1, \ldots, x_n)$ is not central valued on R, then either $char(R) = 2$ and R satisfies s_4 or one of the following holds:

(*i*) $f(x_1, \ldots, x_n)x_{n+1}$ is an identity for I;
(*ii*) $F(I)I = (0)$;
(*iii*) $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is an identity for I, there exist $b, c, q \in Q$ with q an invertible element such that $F(x) = bx - qxq^{-1}c$ for all $x \in R$, and $q^{-1}cI \subseteq I$. Moreover, in this case either $(b - c)I = (0)$ or $b - c \in C$ and $f(x_1, \ldots, x_n)^2$ is central valued on R.

Keywords: Identity, generalized skew derivation, automorphism, (semi-)prime ring.

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1. Introduction.

Throughout this paper, unless specially stated, K denotes a commutative ring with unit, R is always a prime K-algebra with center $Z(R)$, right Martindale quotient ring Q and extended centroid C. The definition, axiomatic formulations and properties of this quotient ring can be found in [2] (Chapter 2).

Many results in literature indicate how the global structure of a ring R is often tightly connected to the behaviour of additive mappings defined on R. A well known result of Posner [32] states that if d is a derivation of R such that $[d(x), x] \in Z(R)$, for any $x \in R$, then either $d = 0$ or R is commutative. Later in [3], Bresar proved that if d and δ are derivations of R such that $d(x)x - x\delta(x) \in Z(R)$, for all $x \in R$, then either $d = \delta = 0$ or R is commutative. In [29], Lee and Wong extended Bresar's result to the Lie case. They proved that if $d(x)x - x\delta(x) \in Z(R)$, for all x in some non-central Lie ideal L of R then either $d = \delta = 0$ or R satisfies s_4, the standard identity of degree 4.

Recently in [28], Lee and Zhou considered the case when the derivations d and δ are replaced respectively by the generalized derivations H and G, and proved that if $R \neq M_2(GF(2))$, H, G are two generalized derivations of R, and m, n are two fixed positive integers, then $H(x^m)x^n = x^n G(x^m)$ for all $x \in R$ if and only if the following two conditions hold: (1) There exists $w \in Q$ such that $H(x) = xw$ and $G(x) = wx$ for all $x \in R$; (2) either $w \in C$, or x^m and x^n are C-dependent for all $x \in R$.

More recently in [5], a similar situation is examined: more precisely it is proved that if $H(u^n)u^n + u^n G(u^n) \in C$, for all $u \in L$, a non-central Lie ideal of R, then there exists $a \in Q$ such that $H(x) = xa$, $G(x) = -ax$, or R satisfies the standard identity s_4. Moreover in this last case a complete description of H and G is given.

Finally, as a partial extension of the above results to the case of derivations and generalized derivations acting on multilinear polynomials, we have the following:

1.1. Fact. (Theorem 2 in [27]) Let R be a prime ring, $f(x_1, \ldots, x_n)$ a multilinear polynomial over C in n noncommuting indeterminates, and $d : R \to R$ a nonzero derivation of R. If $d(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) \in C$, for all $r_1, \ldots, r_n \in R$ and $f(x_1, \ldots, x_n)$ is not central valued on RC, then $char(R) = 2$ and R satisfies s_4.

1.2. Fact. (Lemma 3 in [1]) Let R be a prime ring, $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over C in n noncommuting indeterminates, and $G : R \to R$ a nonzero generalized derivation of R. If $G(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) \in C$, for all $r_1, \ldots, r_n \in R$, then either $char(R) = 2$ and R satisfies s_4 or there exists $b \in C$ such that $G(x) = bx$ for all $x \in R$ and $f(x_1, \ldots, x_n)^2$ is central valued on R.

These facts in a prime K-algebra are natural tests which evidence that, if d is a derivation of R and G is a generalized derivation of R, then the sets $\{d(x)x \mid x \in S\}$ and $\{G(x)x \mid x \in S\}$ are rather large in R, where S is either a non-central Lie ideal of R, or the set of all the evaluations of a non-central multilinear polynomial over K.

In this paper we will continue the study of the set

$$\{F(f(x_1, \ldots, x_n))f(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in R\}$$

for a generalized skew derivation F of R instead of a generalized derivation, and for a multilinear polynomial $f(x_1, \ldots, x_n)$ in n noncommuting variables over C. For the sake of clearness and completeness we now recall the definition of a generalized skew derivation of R. Let R be an associative ring and α be an automorphism of R. An additive mapping $d : R \longrightarrow R$ is called a *skew derivation* of R if

$$d(xy) = d(x)y + \alpha(x)d(y)$$

for all $x, y \in R$. The automophism α is called an *associated automorphism* of d. An additive mapping $F : R \longrightarrow R$ is said to be a *generalized skew derivation* of R if there exists a skew derivation d of R with associated automorphism α such that

$$F(xy) = F(x)y + \alpha(x)d(y)$$

for all $x, y \in R$, and d is said to be an *associated skew derivation* of F and α is called an *associated automorphism* of F. For fixed elements a and b of R, the mapping $F : R \to R$ defined as $F(x) = ax - \sigma(x)b$ for all $x \in R$ is a generalized skew derivation of R. A generalized skew derivation of this form is called an *inner* generalized skew derivation. The definition of generalized skew derivations is a unified notion of skew derivation and generalized derivation, which have been investigated by many researchers from various view points (see [8, 9, 10], [11], [26]).

The main result of this paper is the following:

1. Theorem. Let R be a prime ring, $f(x_1, \dots, x_n)$ a multilinear polynomial over C in n noncommuting indeterminates, I a nonzero right ideal of R, and $F : R \to R$ a nonzero generalized skew derivation of R.

Suppose that $F(f(r_1, \dots, r_n))f(r_1, \dots, r_n) \in C$, for all $r_1, \dots, r_n \in I$. If the polynomial $f(x_1, \dots, x_n)$ is not central valued on R, then either $char(R) = 2$ and R satisfies s_4 or one of the following holds:

(*i*) $f(x_1, \dots, x_n)x_{n+1}$ is an identity for I;

(*ii*) $F(I)I = (0)$;

(*iii*) $[f(x_1, \dots, x_n), x_{n+1}]x_{n+2}$ is an identity for I, there exist $b, c, q \in Q$ with q an invertible element such that $F(x) = bx - qxq^{-1}c$ for all $x \in R$, and $q^{-1}cI \subseteq I$. Moreover, in this case either $(b - c)I = (0)$ or $b - c \in C$ and $f(x_1, \dots, x_n)^2$ is central valued on R.

It is well known that automorphisms, derivations and skew derivations of R can be extended to Q. Chang in [8] extended the definition of a generalized skew derivation to the right Martindale quotient ring Q of R as follows: by a (right) generalized skew derivation we mean an additive mapping $F : Q \longrightarrow Q$ such that $F(xy) = F(x)y + \alpha(x)d(y)$ for all $x, y \in Q$, where d is a skew derivation of R and α is an automorphism of R. Moreover, there exists $F(1) = a \in Q$ such that $F(x) = ax + d(x)$ for all $x \in R$ (Lemma 2 in [8]).

2. X-inner Generalized Skew Derivations on Prime Rings.

In this section we consider the case when F is an X-inner generalized skew derivation induced by the elements $b, c \in R$, that is, $F(x) = bx - \alpha(x)c$ for all $x \in R$, where $\alpha \in Aut(R)$ is the associated automorphism of F. Here $Aut(R)$ denotes the group of automorphisms of R.

At the outset, we will study the case when $R = M_m(K)$ is the algebra of $m \times m$ matrices over a field K. Notice that the set $f(R) = \{f(r_1, \dots, r_n) : r_1, \dots, r_n \in R\}$ is invariant under the action of all inner automorphisms of R. Hence if we denote $r = (r_1, \dots, r_n) \in R \times \ldots \times R = R^n$, then for any inner automorphism φ of $M_m(K)$, we have that $\underline{r} = (\varphi(r_1), \dots, \varphi(r_n)) \in R^n$ and $\varphi(f(r)) = f(\underline{r}) \in f(R)$.

Let us recall some results from [23] and [30]. Let T be a ring with 1 and let $e_{ij} \in M_m(T)$ be the matrix unit having 1 in the (i, j)-entry and zero elsewhere. For a sequence $u = (A_1, \dots, A_n)$ in $M_m(T)$ the value of u is defined to be the product $|u| = A_1A_2 \cdots A_n$ and u is nonvanishing if $|u| \neq 0$. For a permutation σ of $\{1, 2, \cdots, n\}$ we write $u^\sigma = (A_{\sigma(1)}, \dots, A_{\sigma(n)})$. We call u *simple* if it is of the form $u = (a_1e_{i_1j_1}, \dots, a_ne_{i_nj_n})$, where $a_i \in T$. A simple sequence u is called *even* if for some σ, $|u^\sigma| = be_{ii} \neq 0$, and *odd* if for some σ, $|u^\sigma| = be_{ij} \neq 0$, where $i \neq j$ and $b \in T$. We have:

2.1. Fact. (Lemma in [23]) Let T be a K-algebra with 1 and let $R = M_m(T)$, $m \geq 2$. Suppose that $h(x_1, \ldots, x_n)$ is a multilinear polynomial over K such that $h(u) = 0$ for all odd simple sequences u. Then $h(x_1, \ldots, x_n)$ is central valued on R.

2.2. Fact. (Lemma 2 in [30]) Let T be a K-algebra with 1 and let $R = M_m(T)$, $m \geq 2$. Suppose that $h(x_1, \ldots, x_n)$ is a multilinear polynomial over K. Let $u = (A_1, \ldots, A_n)$ be a simple sequence from R.

1. If u is even, then $h(u)$ is a diagonal matrix.
2. If u is odd, then $h(u) = ae_{pq}$ for some $a \in T$ and $p \neq q$.

2.3. Fact. Suppose that $f(x_1, \ldots, x_n)$ is a multilinear polynomial over a field K not central valued on $R = M_m(K)$. Then by Fact 2.1 there exists an odd simple sequence $r = (r_1, \ldots, r_n)$ from R such that $f(r) = f(r_1, \ldots, r_n) \neq 0$. By Fact 2.2, $f(r) = \beta e_{pq}$, where $0 \neq \beta \in K$ and $p \neq q$. Since $f(x_1, \ldots, x_n)$ is a multilinear polynomial and K is a field, we may assume that $\beta = 1$. Now, for distinct i and j, let $\sigma \in S_n$ be such that $\sigma(p) = i$ and $\sigma(q) = j$, and let ψ be the automorphism of R defined by $\psi(\sum_{s,t} \xi_{st} e_{st}) = \sum_{s,t} \xi_{st} e_{\sigma(s)\sigma(t)}$. Then $f(\psi(r)) = f(\psi(r_1), \ldots, \psi(r_n)) = \psi(f(r)) = \beta e_{ij} = e_{ij}$.

In all that follows we always assume that $f(x_1, \ldots, x_n)$ is not central valued on R.

2.4. Lemma. *Let $R = M_m(K)$ be the algebra of $m \times m$ matrices over the field K and $m \geq 2$, $f(x_1, \ldots, x_n)$ a multilinear polynomial over K, which is not central valued on R. If there exist $b, c, q \in R$ with q an invertible matrix such that*

$$\Big(bf(r_1, \ldots, r_n) - qf(r_1, \ldots, r_n)q^{-1}c\Big)f(r_1, \ldots, r_n) \in Z(R)$$

for all $r_1, \ldots, r_n \in R$, then either $char(R) = 2$ and $m = 2$, or $q^{-1}c, b - c \in Z(R)$ and $f(x_1, \ldots, x_n)^2$ is central valued on R, provided that $b \neq c$.

Proof. If $q^{-1}c \in Z(R)$ then the conclusion follows from Fact 1.2. Thus we may assume that $q^{-1}c$ is not a scalar matrix and proceed to get a contradiction. Say $q = \sum_{hl} q_{hl} e_{hl}$ and $q^{-1}c = \sum_{hl} p_{hl} e_{hl}$, for $q_{hl}, p_{hl} \in K$. By Fact 2.3, $e_{ij} \in f(R)$ for all $i \neq j$, then for any $i \neq j$

$$X = (be_{ij} - qe_{ij}q^{-1}c)e_{ij} \in Z(R).$$

By X, we have $qe_{ij}q^{-1}ce_{ij} = qp_{ji}e_{ij} \in Z(R)$. Then for any $1 \leq k \leq m$ $[qp_{ji}e_{ij}, e_{ik}] = 0$, that is $q_{ki}p_{ji} = 0$. Since q is invertible $q_{k_0 i} \neq 0$ for some k_0, we get $p_{ji} = 0$ for all $i \neq j$. Hence $q^{-1}c$ is a diagonal matrix in R. Let $i \neq j$ and $\varphi(x) = (1 + e_{ji})x(1 - e_{ji})$ be an automorphism of R. It is well known that $\varphi(f(r_i)) \in f(R)$, then

$$\Big(\varphi(b)u - \varphi(q)u\varphi(q^{-1}c)\Big)u \in Z(R)$$

for all $u \in f(R)$. By the above argument, $\varphi(q^{-1}c)$ is a diagonal matrix, that is the (j,i)-entry of $\varphi(q^{-1}c)$ is zero. By calculations it follows $p_{ii} = p_{jj}$, and we get the contradiction that $q^{-1}c$ is central in R. □

2.5. Lemma. *Let R be a prime ring, $f(x_1, \ldots, x_n)$ be a non-central multilinear polynomial over C. If there exist $b, c, q \in R$ with q an invertible element such that*

$$(bf(r_1, \ldots, r_n) - qf(r_1, \ldots, r_n)q^{-1}c)f(r_1, \ldots, r_n) \in C$$

for all $r_1, \ldots, r_n \in R$, then either $char(R) = 2$ and R satisfies s_4, or $q^{-1}c, b - c \in Z(R)$ and $f(x_1, \ldots, x_n)^2$ is central valued on R, provided that $b \neq c$.

Proof. Consider the generalized polynomial

$$\Phi(x_1,\ldots,x_{n+1})=\Big[\Big(bf(x_1,\ldots,x_n)-qf(x_1,\ldots,x_n)q^{-1}c\Big)f(x_1,\ldots,x_n),x_{n+1}\Big]$$

which is a generalized polynomial identity for R. If $\{1,q^{-1}c\}$ is linearly C-dependent, then $q^{-1}c\in C$. In this case R satisfies

$$\Phi(x_1,\ldots,x_{n+1})=\Big[\Big((b-c)f(x_1,\ldots,x_n)\Big)f(x_1,\ldots,x_n),x_{n+1}\Big]$$

and we are done by Fact 1.2.

Hence we here assume that $\{1,q^{-1}c\}$ is linearly C-independent. In this case $\Phi(x_1,\ldots,x_{n+1})$ is a non-trivial generalized polynomial identity for R and by [12] $\Phi(x_1,\ldots,x_{n+1})$ is a non-trivial generalized polynomial identity for Q. By Martindale's theorem in [31], Q is a primitive ring having nonzero socle with the field C as its associated division ring. By [20] (p. 75) Q is isomorphic to a dense subring of the ring of linear transformations of a vector space V over C, containing nonzero linear transformations of finite rank. Assume first that $dim_C V=k$ a finite integer. Then $Q\cong M_k(C)$ and the conclusion follows from Lemma 2.4. Therefore we may assume that $dim_C V=\infty$. As in Lemma 2 in [33], the set $f(R)=\{f(r_1,\ldots,r_n):r_i\in R\}$ is dense in R and so from $\Phi(r_1,\ldots,r_{n+1})=0$ for all $r_1,\ldots,r_{n+1}\in R$, we have that Q satisfies the generalized identity

$$\big[(bx_1-qx_1q^{-1}c)x_1,x_2\big].$$

In particular for $x_1=1$, $[b-c,x_2]$ is an identity for Q, that is $b-c\in C$, say $b=c+\lambda$ for some $\lambda\in C$. Thus Q satisfies

$$\big[((c+\lambda)x_1-qx_1q^{-1}c)x_1,x_2\big]$$

and by replacing x_1 with y_1+t_1 we have that

$$\Big[\Big((c+\lambda)y_1-qy_1q^{-1}c\Big)t_1,x_2\Big]+\Big[\Big((c+\lambda)t_1-qt_1q^{-1}c\Big)y_1,x_2\Big]$$

is an identity for Q. Once again for $y_1=1$ it follows that Q satisfies

$$\big[\lambda t_1+(c+\lambda)t_1-qt_1q^{-1}c,x_2\big]$$

and for $x_2=t_1$

$$\big[ct_1-qt_1q^{-1}c,t_1\big].$$

By Lemma 3.2 in [17] (or [18] Theorem 1) and since R cannot satisfy any polynomial identity ($dim_C V=\infty$), it follows the contradiction $q^{-1}c\in C$. □

2.6. Proposition. *Let R be a prime ring, $f(x_1,\ldots,x_n)$ a non-central multilinear polynomial over C in n non-commuting variables, $b,c\in R$ and $\alpha\in Aut(R)$ such that $F(x)=bx-\alpha(x)c$ for all $x\in R$. If $F(f(r_1,\ldots,r_n))f(r_1,\ldots,r_n)\in C$, for all $r_1,\ldots,r_n\in R$, and F is nonzero on R, then either $char(R)=2$ and R satisfies s_4, or $f(x_1,\ldots,x_n)^2$ is central valued on R and there exists $\gamma\in C$ such that $F(x)=\gamma x$, for all $x\in R$. When this last case occurs, we have:*

- *(i) if α is X-outer then $\gamma=b$ and $c=0$;*
- *(ii) if $\alpha(x)=qxq^{-1}$ for all $x\in R$ and for some invertible element $q\in Q$, then $\gamma=b-c$ and $q^{-1}c\in C$.*

Proof. In case α is an X-inner automorphism of R, there exists an invertible element $q\in Q$ such that $\alpha(x)=qxq^{-1}$ for all $x\in R$ and the conclusion follows from Lemma 2.5. So we may assume here that α is X-outer. Since by [14] R and Q satisfy the same generalized identities with automorphisms, then

$$\Phi(x_1,\ldots,x_{n+1})=\big[\big(bf(x_1,\ldots,x_n)-\alpha(f(x_1,\ldots,x_n))c\big)f(x_1,\ldots,x_n),x_{n+1}\big]$$

is satisfied by Q, moreover Q is a centrally closed prime C-algebra. Note that if $c = 0$ we are done by Fact 1.2. Thus we may assume $c \neq 0$. In this case, by [13] (main Theorem), $\Phi(x_1, \dots, x_{n+1})$ is a non-trivial generalized identity for R and for Q. By Theorem 1 in [21], RC has non-zero socle and Q is primitive. Moreover, since α is an outer automorphism and any $(x_i)^\alpha$-word degree in $\Phi(x_1, \dots, x_n)$ is equal to 1, then by Theorem 3 in [14], Q satisfies the identity

$$\big[\big(bf(x_1, \dots, x_n) - f^\alpha(y_1, \dots, y_n)c\big)f(x_1, \dots, x_n), x_{n+1}\big],$$

where $f^\alpha(X_1, \dots, X_n)$ is the polynomial obtained from f by replacing each coefficient γ of f with $\alpha(\gamma)$. By Fact 1.2 we conclude that either $char(R) = 2$ and R satisfies s_4 or $b, c \in C$ and $f(x_1, \dots, x_n)^2$ is central valued on R. Moreover, in this last case we also have that Q satisfies

$$c\big[f(y_1, \dots, y_n)f(x_1, \dots, x_n), x_{n+1}\big].$$

Since $c \neq 0$ we have $[f(y_1, \dots, y_n)f(x_1, \dots, x_n), x_{n+1}]$ is a polynomial identity for Q. Thus there exists a suitable field K such that Q and the $l \times l$ matrix ring $M_l(K)$ satisfy the same polynomial identities by Lemma 1 in [22]. In particular, $M_l(K)$ satisfies $[f(y_1, \dots, y_n)f(x_1, \dots, x_n), x_{n+1}]$. Hence, since $f(x_1, \dots, x_n)$ is not central valued on $M_l(K)$ (and hence $l \geq 2$), by Fact 2.3 we have that for all $i \neq j$ there exist $r_1, \dots, r_n, s_1, \dots, s_n \in M_l(K)$ such that $f(r_1, \dots, r_n) = e_{ij}$ and $f(s_1, \dots, s_n) = e_{ji}$. As a consequence we get $0 = [e_{ij}e_{ji}, x_{n+1}] = [e_{ii}, x_{n+1}]$, which is a contradiction for a suitable choice of $x_{n+1} \in M_l(K)$ (for example $x_{n+1} = e_{ij}$). □

2.7. Fact. (Theorem 1 in [15]) Let R be a prime ring, D be an X-outer skew derivation of R and α be an X-outer automorphism of R. If $\Phi(x_i, D(x_i), \alpha(x_i))$ is a generalized polynomial identity for R, then R also satisfies the generalized polynomial identity $\Phi(x_i, y_i, z_i)$, where x_i, y_i and z_i are distinct indeterminates.

We close this section by collecting the results we obtained so far in the following

2.8. Proposition. *Let R be a prime ring, $f(x_1, \dots, x_n)$ a non-central multilinear polynomial over C in n non-commuting variables, $F : R \to R$ a nonzero X-inner generalized skew derivation of R.*

If $F(f(r_1, \dots, r_n))f(r_1, \dots, r_n) \in C$, for all $r_1, \dots, r_n \in R$, then either $char(R) = 2$ and R satisfies s_4, or $f(x_1, \dots, x_n)^2$ is central valued on R and there exists $\gamma \in C$ such that $F(x) = \gamma x$, for all $x \in R$.

Proof. We can write $F(x) = bx + d(x)$ for all $x \in R$ where $b \in Q$ and d is a skew derivation of R (see [8]). We denote $f(x_1, \dots, x_n) = \sum_{\sigma \in S_n} \gamma_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$ with $\gamma_\sigma \in C$. By Theorem 2 in [15] R and Q satisfy the same generalized polynomial identities with a single skew derivation, then Q satisfies

$$\left[\left(bf(x_1, \dots, x_n) + d(f(x_1, \dots, x_n))\right)f(x_1, \dots, x_n), x_{n+1}\right]. \tag{2.1}$$

Since F is X-inner then d is X-inner, that is there exist $c \in Q$ and $\alpha \in Aut(Q)$ such that $d(x) = cx - \alpha(x)c$, for all $x \in R$. Hence $F(x) = (b + c)x - \alpha(x)c$ and we conclude by Proposition 2.6. □

2.9. Corollary. *Let R be a prime ring, $f(x_1, \dots, x_n)$ a non-vanishing multilinear polynomial over C in n non-commuting variables, $F : R \to R$ a non-zero X-inner generalized skew derivation of R. If $F(f(r_1, \dots, r_n))f(r_1, \dots, r_n) = 0$, for all $r_1, \dots, r_n \in R$, then $char(R) = 2$ and R satisfies s_4.*

3. Generalized Skew Derivations on Right Ideals.

We premit the following:

3.1. Fact. (Main Theorem in [1]) Let R be a prime ring, I a nonzero right ideal of R, $f(x_1,\dots,x_n)$ a multilinear polynomial over C in n non-commuting indeterminates, which is not an identity for R, and $g : R \to R$ a nonzero generalized derivation of R with the associated derivation $d : R \to R$, that is $g(x) = ax + d(x)$, for all $x \in R$ and a fixed $a \in Q$.

Suppose that $g(f(r_1,\dots,r_n))f(r_1,\dots,r_n) \in C$, for all $r_1,\dots,r_n \in I$. Then either $char(R) = 2$ and R satisfies s_4 or $f(x_1,\dots,x_n)x_{n+1}$ is an identity for I, or there exist $b, c \in Q$ such that $g(x) = bx + xc$ for all $x \in R$ and one of the following holds:

(i) $b, c \in C$ and $f(x_1,\dots,x_n)^2$ is central valued on R;
(ii) there exists $\lambda \in C$ such that $b = \lambda - c$ and $f(x_1,\dots,x_n)$ is central valued on R;
(iii) $(b + c)I = (0)$ and I satisfies the identity $[f(x_1,\dots,x_n), x_{n+1}]x_{n+2}$;
(iv) $(b + c)I = (0)$ and there exists $\gamma \in C$ such that $(c - \gamma)I = (0)$.

3.2. Fact. (Theorem 1 in [1]) Under the same situation as in above Fact, we notice that in case $g(f(r_1,\dots,r_n))f(r_1,\dots,r_n) = 0$, for all $r_1,\dots,r_n \in I$, the conclusions (i) and (ii) cannot occur. Hence we have that either $char(R) = 2$ and R satisfies s_4 or $f(x_1,\dots,x_n)x_{n+1}$ is an identity for I, or there exist $b, c \in Q$ such that $g(x) = bx + xc$ for all $x \in R$ and one of the following holds:

(i) $(b + c)I = (0)$ and I satisfies the identity $[f(x_1,\dots,x_n), x_{n+1}]x_{n+2}$;
(ii) $(b + c)I = (0)$ and there exists $\gamma \in C$ such that $(c - \gamma)I = (0)$.

3.3. Proposition. *Let R be a prime ring, $f(x_1,\dots,x_n)$ a non-central multilinear polynomial over C in n non-commuting indeterminates, I a nonzero right ideal of R, $F : R \to R$ an X-outer generalized skew derivation of R. If*

$$F(f(r_1,\dots,r_n))f(r_1,\dots,r_n) \in C, \tag{3.1}$$

for all $r_1,\dots,r_n \in I$, then either $char(R) = 2$ and R satisfies $s_4(x_1,\dots,x_4)$, or $f(x_1,\dots,x_n)x_{n+1}$ is an identity for I.

Proof. As above we write $F(x) = bx + d(x)$ for all $x \in R$, $b \in Q$ and d is an X-outer skew derivation of R. Let $\alpha \in Aut(Q)$ be the automorphism which is associated with d. Notice that in case α is the identity map on R, then d is a usual derivation of R and so F is a generalized derivation of R. Therefore by Fact 3.1 we obtain the required conclusions. Hence in what follows we always assume that $\alpha \neq 1 \in Aut(R)$.

We denote by $f^d(x_1,\dots,x_n)$ the polynomial obtained from $f(x_1,\dots,x_n)$ by replacing each coefficient γ_σ with $d(\gamma_\sigma)$. Notice that

$$\begin{aligned} d\big(\gamma_\sigma x_{\sigma(1)}\cdots x_{\sigma(n)}\big) &= d(\gamma_\sigma)x_{\sigma(1)}\cdots x_{\sigma(n)} \\ &\quad + \alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\cdots x_{\sigma(n)} \end{aligned}$$

so that

$$\begin{aligned} d(f(x_1,\dots,x_n)) &= f^d(x_1,\dots,x_n) \\ &\quad + \sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\cdots x_{\sigma(n)}. \end{aligned}$$

Since IQ satisfies (3.1), then for all $0 \neq u \in I$, Q satisfies

$$\Big[\Big(bf(ux_1,\ldots,ux_n)+f^d(ux_1,\ldots,ux_n)\Big)f(ux_1,\ldots,ux_n),x_{n+1}\Big]$$
$$+\Big[\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(ux_{\sigma(1)}\ldots ux_{\sigma(j)})d(ux_{\sigma(j+1)})ux_{\sigma(j+2)}\ldots ux_{\sigma(n)}\Big)f(ux_1,\ldots,ux_n),x_{n+1}\Big].$$

By Theorem 1 in [15], Q satisfies

$$\Big[\big(bf(ux_1,\ldots,ux_n)+f^d(ux_1,\ldots,ux_n)\big)f(ux_1,\ldots,ux_n),x_{n+1}\Big]$$
$$+\Big[(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(ux_{\sigma(1)}\ldots ux_{\sigma(j)})d(u)x_{\sigma(j+1)}\ldots ux_{\sigma(n)}\Big)f(ux_1,\ldots,ux_n),x_{n+1}\Big]$$
$$+\Big[\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(ux_{\sigma(1)})\ldots ux_{\sigma(j)})\alpha(u)y_{\sigma(j+1)}ux_{\sigma(j+2)}\ldots ux_{\sigma(n)}\Big)f(ux_1,\ldots,ux_n),x_{n+1}\Big].$$

In particular Q satisfies

(3.2)
$$\Big[\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(ux_{\sigma(1)}\ldots ux_{\sigma(j)})\alpha(u)y_{\sigma(j+1)}ux_{\sigma(j+2)}\ldots ux_{\sigma(n)}\Big)f(ux_1,\ldots,ux_n),x_{n+1}\Big].$$

Here we suppose that either $char(R)\neq 2$ or R does not satisfy s_4, moreover $f(x_1,\ldots,x_n)x_{n+1}$ is not an identity for I, if not we are done. Hence suppose there exist $a_1,\ldots,a_{n+1}\in I$ such that $f(a_1,\ldots,a_n)a_{n+1}\neq 0$. We proceed to get a number of contradictions.

Since $0\neq\alpha(u)$ is a fixed element of Q, we notice that (3.2) is a non-trivial generalized polynomial identity for Q, then Q has nonzero socle H which satisfies the same generalized polynomial identities of Q (see [12]). In order to prove our result, we may replace Q by H, and by Lemma 1 in [19], we may assume that Q is a regular ring. Thus there exists $0\neq e=e^2\in IQ$ such that $\sum_{i=1}^{n+1}a_iQ=eQ$, and $a_i=ea_i$ for each $i=1,\ldots,n+1$. Notice that eQ satisfies the same generalized identities with skew derivations and automorphisms of I. So that we may assume $e\neq 1$, if not $eQ=Q$ and the conclusion follows from Proposition 2.6.

Assume that α is X-outer. Thus, by Fact 2.7 and (3.2), Q satisfies

(3.3)
$$\Big[\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(e)t_{\sigma(1)}\cdots\alpha(e)t_{\sigma(j)}\alpha(e)y_{\sigma(j+1)}ex_{\sigma(j+2)}\cdots ex_{\sigma(n)}\Big)f(ex_1,\ldots,ex_n),x_{n+1}\Big]$$

and in particular

$$\Big[\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\alpha(e)y_{\sigma(1)}\cdots\alpha(e)y_{\sigma(n)}\Big)f(ex_1,\ldots,ex_n),x_{n+1}\Big]. \tag{3.4}$$

We also denote by $f^\alpha(x_1,\ldots,x_n)$ the polynomial obtained from $f(x_1,\ldots,x_n)$ by replacing each coefficient γ_σ with $\alpha(\gamma_\sigma)$. Therefore we may rewrite (3.4) as follows:

$$\Big[f^\alpha(\alpha(e)r_1,\ldots,\alpha(e)r_n)f(es_1,\ldots,es_n),X\Big]=0 \tag{3.5}$$

for all $r_1,\ldots,r_n,s_1,\ldots,s_n,X\in Q$. Choose in (3.5) $X=Y(1-\alpha(e))$, then we get

$$f^\alpha(\alpha(e)r_1,\ldots,\alpha(e)r_n)f(es_1,\ldots,es_n)Y(1-\alpha(e))=0$$

and by the primeness of Q and since $e \neq 1$, it follows that Q satisfies

$$f^{\alpha}(\alpha(e)y_1, \dots, \alpha(e)y_n) f(ex_1, \dots, ex_n)$$

that is $f^{\alpha}(\alpha(e)Q)f(eQ) = (0)$, where $\alpha(e)Q$ and eQ are both right ideals of Q and f^{α} and f are distinct polynomials over C (since $\alpha \neq 1$). In this situation, applying the result in [16] (see the proof of Lemma 3, pp. 181), it follows that either $f^{\alpha}(\alpha(e)Q)\alpha(e) = (0)$ or $f(eQ) = (0)$. Since this last case cannot occur, we have that $f^{\alpha}(\alpha(e)r_1, \dots, \alpha(e)r_n)\alpha(e) = 0$ for all $r_1, \dots, r_n \in Q$. Hence

$$0 = \alpha^{-1}\Big(f^{\alpha}(\alpha(e)r_1, \dots, \alpha(e)r_n)\alpha(e)\Big) = f(e\alpha^{-1}(r_1), \dots, e\alpha^{-1}(r_n))e$$

and since α^{-1} is an automorphism of Q, it follows that $f(es_1, \dots, es_n)e = 0$, for all $s_1, \dots, s_n \in Q$, which is again a contradiction.

Finally consider the case when there exists an invertible element $q \in Q$ such that $\alpha(x) = qxq^{-1}$, for all $x \in Q$. Thus from (3.2) we have that Q satisfies

(3.6)
$$\left[\Big(\sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} q(ex_{\sigma(1)} \cdots ex_{\sigma(j)})eq^{-1} y_{\sigma(j+1)} ex_{\sigma(j+2)} \cdots ex_{\sigma(n)}\Big) f(ex_1, \dots, ex_n), x_{n+1}\right].$$

Since $\alpha(\gamma_\sigma) = \gamma_\sigma$ and by replacing $y_{\sigma(i)}$ with $qx_{\sigma(i)}$, for all $\sigma \in S_n$ and for all $i = 1, \dots, n$, it follows that Q satisfies

$$\left[\Big(\sum_{\sigma \in S_n} \gamma_\sigma q e x_{\sigma(1)} \cdots ex_{\sigma(j)} ex_{\sigma(j+1)} ex_{\sigma(j+2)} \cdots ex_{\sigma(n)}\Big) f(ex_1, \dots, ex_n), x_{n+1}\right] \tag{3.7}$$

that is

$$\left[\Big(qf(ex_1, \dots, ex_n)\Big) f(ex_1, \dots, ex_n), x_{n+1}\right]. \tag{3.8}$$

By Fact 3.1 it follows that one of the following holds:

1. $char(Q) = 2$ and Q satisfies s_4;
2. $f(x_1, \dots, x_n)x_{n+1}$ is an identity for eQ;
3. $q \in C$;
4. $qeQ = (0)$.

Since in any case we get a contradiction, we are done. □

3.4. Lemma. *Let R be a prime ring, $f(x_1, \dots, x_n)$ a non-central multilinear polynomial over C in n non-commuting indeterminates, I a nonzero right ideal of R, $b, c \in Q$ and $\alpha \in Aut(R)$ be an automorphism of R such that $F(x) = bx - \alpha(x)c$, for all $x \in R$. Assume that $F(f(r_1, \dots, r_n))f(r_1, \dots, r_n) \in C$, for all $r_1, \dots, r_n \in I$. If R does not satisfy any non-trivial generalized polynomial identity then $F(I)I = (0)$.*

Proof. Let u be any nonzero element of I. By the hypothesis R satisfies the following:

$$\left[\Big(b(f(ux_1, \dots, ux_n)) - \alpha(f(ux_1, \dots, ux_n))c\Big) f(ux_1, \dots, ux_n), x_{n+1}\right].$$

Also here we denote by $f^{\alpha}(x_1, \dots, x_n)$ the polynomial obtained from $f(x_1, \dots, x_n)$ by replacing each coefficient γ_σ of $f(x_1, \dots, x_n)$ with $\alpha(\gamma_\sigma)$. Thus R satisfies

$$\left[\Big(bf(ux_1, \dots, ux_n) - f^{\alpha}(\alpha(u)\alpha(x_1), \dots, \alpha(u)\alpha(x_n))c\Big) f(ux_1, \dots, ux_n), x_{n+1}\right]. \tag{3.9}$$

In case α is X-outer, by Theorem 3 in [14] and (3.9) we have that R satisfies

$$\left[\left(b(f(ux_1,\ldots,ux_n))-f^{\alpha}(\alpha(u)y_1,\ldots,\alpha(u)y_n)c\right)f(ux_1,\ldots,ux_n),x_{n+1}\right]$$

and in particular R satisfies both

$$\left[bf(ux_1,\ldots,ux_n)^2,x_{n+1}\right] \tag{3.10}$$

and

$$\left[f^{\alpha}(\alpha(u)y_1,\ldots,\alpha(u)y_n)cf(ux_1,\ldots,ux_n),x_{n+1}\right]. \tag{3.11}$$

Since (3.10) and (3.11) must be trivial generalized polynomial identities for R, by [12] it follows that $bu=0$ and $cu=0$ that is $F(I)I=(0)$.

Consider now the case $\alpha(x)=qxq^{-1}$ for all $x\in R$, for some invertible element $q\in Q$. Since by (3.9)

$$\left[\left(bf(ux_1,\ldots,ux_n)-qf(ux_1,\ldots,ux_n)q^{-1}c\right)f(ux_1,\ldots,ux_n),x_{n+1}\right] \tag{3.12}$$

is a trivial generalized polynomial identity for R, again by [12] we have that $bu=\lambda qu$, for some $\lambda\in C$. Thus we may write (3.12) as follows

$$\left[qf(ux_1,\ldots,ux_n)(\lambda-q^{-1}c)f(ux_1,\ldots,ux_n),x_{n+1}\right]. \tag{3.13}$$

Once again (3.13) is a trivial identity for R, moreover $qu\neq 0$. This implies that $(\lambda-q^{-1}c)u=0$ and hence $(\lambda_u-q^{-1}c)u=0$ for all $u\in I$ and for some $\lambda_u\in C$. Then u and $q^{-1}cu$ are C-dependent for all $u\in I$. By a standard argument we conclude that $(\lambda-q^{-1}c)I=(0)$ for some $\lambda\in C$, and thus $F(I)I=(0)$. □

3.5. Lemma. *Let R be a prime ring, $f(x_1,\ldots,x_n)$ a non-central multilinear polynomial over C in n non-commuting indeterminates, I a nonzero right ideal of R, $b,c\in Q$ and $\alpha\in Aut(R)$ be an X-outer automorphism of R such that $F(x)=bx-\alpha(x)c$, for all $x\in R$. If $F(f(r_1,\ldots,r_n))f(r_1,\ldots,r_n)\in C$, for all $r_1,\ldots,r_n\in I$, then either $char(R)=2$ and R satisfies s_4 or one of the following holds:*

- (*i*) *$f(x_1,\ldots,x_n)x_{n+1}$ is an identity for I;*
- (*ii*) *$F(I)I=(0)$;*
- (*iii*) *$cI=(0)$, $b\in C$ and $f(x_1,\ldots,x_n)^2$ is central valued on R.*

Proof. Firstly we notice that in case $cI=(0)$, then $bf(r_1,\ldots,r_n)^2\in C$, for all $r_1,\ldots,r_n\in I$. Thus by Fact 3.1 it follows that either $cI=(0)$, $b\in C$ and $f(x_1,\ldots,x_n)^2$ is central valued on R, or $cI=bI=(0)$ that is $F(I)I=(0)$. Hence in the following we assume $cI\neq(0)$. By previous Lemma we may assume that R satisfies some non-trivial generalized polynomial identity. As above let u be any nonzero element of I. By the hypothesis R satisfies the following:

$$\left[\left(bf(ux_1,\ldots,ux_n)-f^{\alpha}(\alpha(u)\alpha(x_1),\ldots,\alpha(u)\alpha(x_n))c\right)f(ux_1,\ldots,ux_n),x_{n+1}\right]. \tag{3.14}$$

Since α is X-outer, by Theorem 3 in [14], R satisfies

$$\left[\left(bf(ux_1,\ldots,ux_n)-f^{\alpha}(\alpha(u)y_1,\ldots,\alpha(u)y_n)c\right)f(ux_1,\ldots,ux_n),x_{n+1}\right] \tag{3.15}$$

and in particular R as well as Q satisfy the component

$$\left[f^{\alpha}(\alpha(u)y_1,\ldots,\alpha(u)y_n)cf(ux_1,\ldots,ux_n),x_{n+1}\right]. \tag{3.16}$$

By [31] Q is a primitive ring having nonzero socle H with the field C as its associated division ring. Moreover H and Q satisfy the same generalized polynomial identities with automorphisms (Theorem 1 in [14]). Therefore H satisfies (3.14) and so we may replace Q by H. Suppose there exist $a_1, \ldots, a_{n+2} \in I$ such that $f(a_1, \ldots, a_n)a_{n+1} \neq 0$ and $ca_{n+2} \neq 0$. Since Q is a regular GPI-ring, there exists an idempotent element $e \in IQ$ such that $eQ = \sum_{i=1}^{n+2} a_iQ$ and $a_i = ea_i$, for any $i = 1, \ldots, n+2$. Therefore, by (3.14), Q satisfies

$$\left[\left(bf(ex_1, \ldots, ex_n) - f^{\alpha}(\alpha(e)\alpha(x_1), \ldots, \alpha(e)\alpha(x_n))c\right)f(ex_1, \ldots, ex_n), x_{n+1}\right]. \tag{3.17}$$

Moreover assume $e \neq 1$, if not $eQ = Q$ and by Proposition 2.6 we get $b \in C$, $c = 0$ and $f(x_1, \ldots, x_n)^2$ is central valued on R. Since α is X-outer, as above by (3.17) Q satisfies

$$\left[\left(bf(ex_1, \ldots, ex_n) - f^{\alpha}(\alpha(e)y_1, \ldots, \alpha(e)y_n)c\right)f(ex_1, \ldots, ex_n), x_{n+1}\right].$$

In particular Q satisfies

$$\left[f^{\alpha}(\alpha(e)y_1, \ldots, \alpha(e)y_n)cf(ex_1, \ldots, ex_n), x_{n+1}(1 - \alpha(e))\right]$$

that is Q satisfies

$$f^{\alpha}(\alpha(e)y_1, \ldots, \alpha(e)y_n)cf(ex_1, \ldots, ex_n)x_{n+1}(1 - \alpha(e))$$

and since Q is prime and $e \neq 0, 1$, it follows $f^{\alpha}(\alpha(e)r_1, \ldots, \alpha(e)r_n)cf(es_1, \ldots, es_n) = 0$, for all $r_1, \ldots, r_n, s_1, \ldots, s_n \in Q$. Since $f(ea_1, \ldots, ea_n)ea_{n+1} \neq 0$ and $cea_{n+2} \neq 0$ and by using the result in [16], it follows that $f^{\alpha}(\alpha(e)y_1, \ldots, \alpha(e)y_n)$ is an identity for Q. This implies that $f(e\alpha^{-1}(y_1), \ldots, e\alpha^{-1}(y_n))$ is also an identity for Q. Moreover it is clear that α^{-1} is X-outer, therefore $f(ex_1, \ldots, ex_n)$ is an identity for Q, a contradiction. □

3.6. Lemma. *Let R be a prime ring, $f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over C in n non-commuting indeterminates, I a nonzero right ideal of R, $b, c, q \in Q$ such that $F(x) = bx - qxq^{-1}c$, for all $x \in R$. If*

$$F(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) = 0,$$

for all $r_1, \ldots, r_n \in I$, then either $char R = 2$ and R satisfies s_4 or one of the following holds:

(*i*) *$f(x_1, \ldots, x_n)x_{n+1}$ is an identity for I;*
(*ii*) *$[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is an identity for I, $(b - c)I = (0)$ and $q^{-1}cI \subseteq I$;*
(*iii*) *$F(I)I = (0)$.*

Proof. Here I satisfies

$$\left(bf(x_1, \ldots, x_n) - qf(x_1, \ldots, x_n)q^{-1}c\right)f(x_1, \ldots, x_n) \tag{3.18}$$

and left multiplying by q^{-1}, I satisfies

$$\left(q^{-1}b(f(x_1, \ldots, x_n)) - (f(x_1, \ldots, x_n)q^{-1}c\right)f(x_1, \ldots, x_n). \tag{3.19}$$

Since we assume $f(x_1, \ldots, x_n)$ is not central valued on R, by Fact 3.2 we have that either $char R = 2$ and R satisfies the standard identity s_4, or $f(x_1, \ldots, x_n)x_{n+1}$ is an identity for I, or one of the following holds:

1. there exists $\gamma \in C$ such that $q^{-1}bx = \gamma x = q^{-1}cx$, for all $x \in I$ (this is the case $F(I)I = (0)$).
2. $q^{-1}(b - c)I = (0)$, that is $(b - c)I = (0)$, moreover $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is an identity for I.

In this last case, by (3.19) it follows that I satisfies

$$\Big(bf(ux_1,\ldots,ux_n)-qf(ux_1,\ldots,ux_n)q^{-1}b\Big)f(ux_1,\ldots,ux_n) \tag{3.20}$$

and moreover, since I satisfies the polynomial identity $[f(x_1,\ldots,x_n),x_{n+1}]x_{n+2}$, in view of Proposition in [25], $I=eQ$ for some idempotent e in the socle of Q. Here we write $f(x_1,\ldots,x_n)=\sum t_i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)x_i$, where any t_i is a multilinear polynomial in $n-1$ variables and x_i never appears in t_i. Of course, if $t_i(ex_1,\ldots,ex_{i-1},ex_{i+1},\ldots,ex_n)e$ is an identity for Q, then $f(x_1,\ldots,x_n)x_{n+1}$ is an identity for I and we are done. Thus assume there exists $i\in\{1,\ldots,n\}$ such that $t_i(er_1,\ldots,er_{i-1},er_{i+1},\ldots,er_n)e\neq 0$ for some $r_1,\ldots,r_n\in I$. In particular,

$$f(ex_1,\ldots,ex_{i-1},ex_i(1-e),ex_{i+1},\ldots,ex_n)=t_i(ex_1,\ldots,ex_n)ex_i(1-e)$$

and by (3.20) Q satisfies

$$bt_i(ex_1,\ldots,ex_n)ex_i(1-e)t_i(ex_1,\ldots,ex_n)ex_i(1-e)$$
$$-\,qt_i(ex_1,\ldots,ex_n)ex_i(1-e)q^{-1}bt_i(ex_1,\ldots,ex_n)ex_i(1-e)$$

that is Q satisfies

$$\Big(-qt_i(ex_1,\ldots,ex_n)ex_i(1-e)q^{-1}b\Big)t_i(ex_1,\ldots,ex_n)ex_i(1-e) \tag{3.21}$$

and left multiplying by $(1-e)q^{-1}bq^{-1}$, we easily have that Q satisfies

$$(1-e)q^{-1}bt_i(ex_1,\ldots,ex_n)eX(1-e)q^{-1}bt_i(ex_1,\ldots,ex_n)eX(1-e). \tag{3.22}$$

By Lemma 2 in [32] and since $e\neq 1$, it follows that

$$(1-e)q^{-1}bt_i(ex_1,\ldots,ex_{i-1},ex_{i+1},\ldots,ex_n)e$$

is an identity for Q, that is $(1-e)q^{-1}bet_i(x_1e,\ldots,x_{i-1}e,x_{i+1}e,\ldots,x_ne)$ is an identity for Q. In this case, since $t_i(x_1e,\ldots,x_{i-1}e,x_{i+1}e,\ldots,x_ne)$ is not an identity for Q, we get in view of the result in [16], $(1-e)q^{-1}be=0$, that is $q^{-1}bI\subseteq I$ and also $q^{-1}cI\subseteq I$. □

3.7. Theorem. *Let R be a prime ring, $f(x_1,\ldots,x_n)$ a multilinear polynomial over C in n non-commuting variables, I a non-zero right ideal of R, $F:R\to R$ be a non-zero generalized skew derivation of R. Suppose that*

$$F(f(r_1,\ldots,r_n))f(r_1,\ldots,r_n)\in C,$$

for all $r_1,\ldots,r_n\in I$. If $f(x_1,\ldots,x_n)$ is not central valued on R, then either $char(R)=2$ and R satisfies s_4 or one of the following holds:

- *(i) $f(x_1,\ldots,x_n)x_{n+1}$ is an identity for I;*
- *(ii) $F(I)I=(0)$;*
- *(iii) $[f(x_1,\ldots,x_n),x_{n+1}]x_{n+2}$ is an identity for I, there exist $b,c,q\in Q$ with q invertible such that $F(x)=bx-qxq^{-1}c$ for all $x\in R$, and $q^{-1}cI\subseteq I$; moreover in this case either $(b-c)I=(0)$ or $b-c\in C$ and $f(x_1,\ldots,x_n)^2$ is central valued on R provided that $b\neq c$.*

Proof. In view of all previous Lemmas and Propositions, we may assume $I\neq R$ and $F(x)=bx-qxq^{-1}c$, for all $x\in R$. Moreover we may assume that there exist $s_1,\ldots,s_n\in I$ such that $F(f(s_1,\ldots,s_n))f(s_1,\ldots,s_n)\neq 0$. Therefore

$$(bf(x_1,\ldots,x_n)-qf(x_1,\ldots,x_n)q^{-1}c)f(x_1,\ldots,x_n)$$

is a central generalized polynomial identity for I. Thus R is a PI-ring and so RC is a finite dimensional central simple C-algebra (the proof of this fact is the same of Theorem

1 in [7]). By Wedderburn-Artin theorem, $RC \cong M_k(D)$ for some $k \geq 1$ and D a finite-dimensional central division C-algebra. By Theorem 2 in [24]

$$(bf(x_1, \ldots, x_n) - qf(x_1, \ldots, x_n)q^{-1}c)f(x_1, \ldots, x_n) \in C$$

for all $x_1, \ldots, x_n \in IC$. Without loss of generality we may replace R with RC and assume that $R = M_k(D)$. Let E be a maximal subfield of D, so that $M_k(D) \otimes_C E \cong M_t(E)$ where $t = k \cdot [E : C]$. Hence $(bf(r_1, \ldots, r_n) - qf(r_1, \ldots, r_n)q^{-1}c)f(r_1, \ldots, r_n) \in C$, for any $r_1, \ldots, r_n \in I \otimes E$ (Lemma 2 in [24] and Proposition in [29]). Therefore we may assume that $R \cong M_t(E)$ and $I = eR = (e_{11}R + \cdots + e_{ll}R)$, where $t \geq 2$ and $l \leq t$.

Suppose that $t \geq 2$, otherwise we are done and denote $q = \sum_{r,s} q_{rs}e_{rs}$ and $q^{-1}c = \sum_{r,s} c_{rs}e_{rs}$, for $q_{rs}, c_{rs} \in E$. As in Lemma 3.6 we write

$$f(x_1, \ldots, x_n) = \sum t_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)x_i$$

and there exists some $t_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)x_i$ which is not an identity for I. In particular $qt_i(ex_1, \ldots, ex_{i-1}, ex_{i+1}, \ldots, ex_n)ex_i$ is not an identity for R, because q is invertible. Hence, again for

$$f(ex_1, , \ldots, ex_{i-1}, ex_i(1-e), ex_{i+1}, \ldots, ex_n) = t_i(ex_1, \ldots, ex_{i-1}, ex_{i+1}, \ldots, ex_n)ex_i(1-e)$$

and by our hypothesis, we have that

$$qt_i(ex_1, \ldots, ex_{i-1}, ex_{i+1}, \ldots, ex_n)ex_i(1-e)q^{-1}ct_i(ex_1, \ldots, ex_{i-1}, ex_{i+1}, \ldots, ex_n)ex_i(1-e)$$

is an identity for R, and by the primeness of R it follows that

$$(1 - e)q^{-1}ct_i(ex_1, \ldots, ex_{i-1}, ex_{i+1}, \ldots, ex_n)e$$

is an identity for R. By [16] and since $t_i(ex_1, \ldots, ex_{i-1}, ex_{i+1}, \ldots, ex_n)ex_i$ is not an identity for R, the previous identity says that $(1 - e)q^{-1}ce = 0$. Thus $q^{-1}cI \subseteq I$.
In case $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is an identity for I, then by our assumption we get $(b - c)f(r_1, \ldots, r_n)^2 \in C$ for all $r_1, \ldots, r_n \in I$. In view of Fact 3.1, either $(b - c)I = (0)$ and we are done, or $b - c \in C$ and $f(x_1, \ldots, x_n)^2$ is central valued on R, provided that $b \neq c$.

Consider finally the case $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is not an identity for I. By Lemma 3 in [6], for any $i \leq l$, $j \neq i$, the element e_{ij} falls in the additive subgroup of RC generated by all valuations of $f(x_1, \ldots, x_n)$ in I. Since the matrix $(be_{ij} - qe_{ij}q^{-1}c)e_{ij}$ has rank at most 1, then it is not central. Therefore $qe_{ij}q^{-1}ce_{ij} = 0$, i.e. $q_{ki}(q^{-1}c)_{ji} = 0$ for all k and for all $j \neq i$. Since q is invertible, there exists some $q_{ki} \neq 0$, therefore $(q^{-1}c)_{ji} = 0$ for all $j \neq i$.

Consider the following automorphism of R:

$$\lambda(x) = (1 + e_{ij})x(1 - e_{ij}) = x + e_{ij}x - xe_{ij} - e_{ij}xe_{ij}$$

for any $i, j \leq l$, and note that $\lambda(I) \subseteq I$ is a right ideal of R satisfying

$$\Big[\Big(\lambda(b)f(x_1, \ldots, x_n) - \lambda(q)f(x_1, \ldots, x_n)\lambda(q^{-1}c)\Big)f(x_1, \ldots, x_n), x_{n+1}\Big].$$

If we denote $\lambda(q^{-1}c) = \sum_{rs} c'_{rs}e_{rs}$, the above argument says that $c'_{rs} = 0$ for all $s \leq l$ and $r \neq s$. In particular the (i, j)-entry of $\lambda(q^{-1}c)$ is zero. This implies that $c_{ii} = c_{jj} = \alpha$, for all $i, j \leq l$. Therefore $q^{-1}cx = \alpha x$ for all $x \in I$. This leads to $(b - c)f(r_1, \ldots, r_n)^2 \in C$ for all $r_1, \ldots, r_n \in I$ and we conclude by the same argument above. □

For the sake of completeness, we would like to conclude this paper by showing the explicit meaning of the conclusion $F(I)I = (0)$, more precisely we state the following:

3.8. Remark. Let R be a prime ring, I be a non-zero right ideal of R and $F : R \to R$ be a non-zero generalized skew derivation of R. If $F(I)I = (0)$ then there exist $a, b \in Q$ and $\alpha \in Aut(R)$ such that $F(x) = (a + b)x - \alpha(x)b$ for all $x \in R$, $aI = (0)$ and one of the following holds:

(i) $bI = (0)$;

(ii) there exist $\lambda \in C$ and an invertible element $q \in Q$ such that $\alpha(x) = qxq^{-1}$, for all $x \in R$, and $q^{-1}by = \lambda y$, for all $y \in I$.

Proof. As previously remarked we can write $F(x) = ax + d(x)$ for all $x \in R$, where $a \in Q$ and d is a skew derivation of R (see [8]). Let $\alpha \in Aut(R)$ be the automorphism associated with d, in the sense that $d(xy) = d(x)y + \alpha(x)d(y)$, for all $x, y \in R$. Thus, by the hypothesis, for all $x, y \in I$,

(3.23) $\quad (ax + d(x))y = 0.$

For all $x, y, z \in I$ we have:

$$0 = F(xz)y = (ax + d(x))zy + \alpha(x)d(z)y$$

and by (3.23) we obtain $\alpha(x)d(z)y = 0$ for all $x, y, z \in I$. Moreover $\alpha(I)$ is a non-zero right ideal of R, so that it follows

(3.24) $\quad d(z)y = 0$

for all $y, z \in I$. Once again by (3.23) we get $azy = 0$ for all $z, y \in I$, that is $aI = (0)$.

Finally in (3.24) replace z with xs, for any $x \in I$ and $s \in R$, then:

(3.25) $\quad 0 = d(xs)y = d(x)sy + \alpha(x)d(s)y$

for all $x, y \in I$, $s \in R$. In case d is X-outer, it follows that $d(x)sy + \alpha(x)ty = 0$, for all $x, y \in I$ and $s, t \in R$ (Theorem 1 in [15]). In particular $\alpha(x)ty = 0$, which implies the contradiction $\alpha(x) = 0$ for all $x \in I$. Therefore we may assume that d is X-inner, that is there exists $b \in Q$ such that $d(r) = br - \alpha(r)b$, for all $r \in R$ and by (3.24)

(3.26) $\quad (bx - \alpha(x)b)y = 0$

for all $x, y \in I$. Consider first the case α is X-outer and replace x with xr, for any $r \in R$. Then $(bxr - \alpha(x)\alpha(r)b)y = 0$ and, by Theorem 3 in [14], $(bxr - \alpha(x)sb)y = 0$ for all $x, y \in I$ and $r, s \in R$. In particular $bIRI = (0)$, which implies $bI = (0)$ and we are done.

On the other hand, if there exists an invertible element $q \in Q$ such that $\alpha(r) = qrq^{-1}$, for all $r \in R$, from (3.26) we have $(bx - qxq^{-1}b)y = 0$, for all $x, y \in I$. Left multiplying by q^{-1}, it follows $[q^{-1}b, x]y = 0$, and by Lemma in [4] there exists $\lambda \in C$ such that $q^{-1}bx = \lambda x$ for all $x \in I$. □

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