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GENERALIZED NOTION OF WEAK MODULE AMENABILITY

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Abstract

In the present paper, we introduce a new notion of weak module amenability for Banach algebras which is related to module homomorphisms. Among other results, we investigate the relationship between this concept for a Banach algebra $\mathcal{A}$ which is a Banach $\mathfrak{A}$-bimodule with compatible actions, and the quotient Banach algebra $\mathcal{A}/J$ where J is the closed ideal of $\mathcal{A}$ generated by elements of the form $(a\cdot\alpha)b-a(\alpha\cdot b)$ for $a\in\mathcal{A}$ and $\alpha\in\mathfrak{A}$. We then study this concept for an inverse semigroup S, where some examples on $\ell^1(S)$ and $C^*(S)$ are given.

Keywords: Banach modules; Module derivation; Weak amenability; Weak module amenability; Inverse semigroup.

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1. Introduction

Let S be a (discrete) semigroup. The semigroup algebra $\ell^1(S)$ is the Banach algebra consisting of all absolutely summable complex-valued functions on S, with the convolution product and the ℓ^1-norm; $\|f\|_1=\sum_{s\in S}|f(s)|$ $(f\in\ell^1(S))$. We will use δ_s to denote the point mass function at s; $\delta_s(t)=1$ if $t=s$ and $=0$ elsewhere. Using point masses we may represent a function f on S as $f=\sum_{s\in S}f(s)\delta_s$. Here we recall that an *inverse semigroup* is a discrete semigroup S such that for each $s\in S$, there is a unique element $s^*\in S$ with $ss^*s=s$ and $s^*ss^*=s^*$. The set of elements of the form s^*s are called *idempotents* of S and denoted by E.

The concept of amenability for a Banach algebra $\mathcal{A}$ was introduced by B. E. Johnson in [18]. A Banach algebra $\mathcal{A}$ is *amenable* if every bounded derivation from $\mathcal{A}$ into any dual Banach $\mathcal{A}$-module is inner, equivalently if $H^1(\mathcal{A},X^*)=\{0\}$ for every Banach $\mathcal{A}$-module X, where $H^1(\mathcal{A},X^*)$ is the *first Hochschild cohomology group* of $\mathcal{A}$ with coefficients in X^*, the first dual space of X. Also, a Banach algebra $\mathcal{A}$ is *weakly amenable* if $H^1(\mathcal{A},\mathcal{A}^*)=\{0\}$. Bade, Curtis and Dales introduced the notion of weak amenability in [5]. They considered this concept only for commutative Banach algebras. After that

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Johnson defined the weak amenability for arbitrary Banach algebras [19] and showed that for a locally compact group G, $L^1(G)$ is weakly amenable [20]. This fact fails for semigroups though. For example, if S is the bicyclic inverse semigroup, then $\ell^1(S)$ is not weakly amenable [9].

Homomorphisms on Banach algebras play an important role in Functional Analysis. Papers [8] and [21] defined and investigated two concepts of the amenability for Banach algebras by using homomorphisms which are different from weak amenability and amenability. In [1], Amini introduced the concept of module amenability of a Banach algebra $\mathcal{A}$ which is a Banach module over another Banach algebra $\mathfrak{A}$ with compatible actions. Later this notion of amenability is generalized by the author in [7]. The notion of weak module amenability of Banach algebras is defined in [4] and studied in [2]. In fact, the author and Amini investigated the concept of weak module amenability in [2] and obtained some results on the seond dual of a Banach algebra. In [6], the author showed that for an arbitrary inverse semigroup S with a set of idempotents E, the semigroup algebra $\ell^1(S)$ as an $\ell^1(E)$-module with trivial left action is always weakly module amenable. The abelian case for S was proved earlier in [4]. These papers motivated us to generalize of the concept of weak module amenability by homomorphisms.

Let $\mathcal{A}$ and $\mathfrak{A}$ be Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with compatible actions. Then every $\mathfrak{A}$-module homomorphism σ (not necessarily $\mathbb{C}$-linear) on $\mathcal{A}$ induces a linear continuous homomorphism $\widehat{\sigma}$ on $\mathcal{A}/J$, where J is a closed ideal of $\mathcal{A}$. In section three, we generalize the concept of weak module amenability of Banach algebras by using $\mathfrak{A}$-module homomorphisms. On the other hand, for each pair $\mathfrak{A}$-module homomorphism σ and τ on $\mathcal{A}$, we define (σ,τ)-weak module amenability of Banach algebras and among other results, we study the relation between (σ,τ)-weak module amenability of $\mathcal{A}$ and $(\widehat{\sigma},\widehat{\tau})$-weak amenability of $\mathcal{A}/J$, where J is the closed ideal of $\mathcal{A}$ generated by elements of the form $(a\cdot\alpha)b - a(\alpha\cdot b)$, for $a\in\mathcal{A}$ and $\alpha\in\mathfrak{A}$ (see also [8]).

In the last part of this paper, we show that under some conditions, $\ell^1(S)$ is (σ,τ)-weakly module amenable for all $\ell^1(E)$-module homomorphisms σ and τ on $\ell^1(S)$. Finally by applying our results, we give an example that $\ell^1(S)$ $[C^*(S)]$ is (σ,σ)-weakly module amenable as an $\ell^1(E)$-bimodule [as an $C^*(E)$-bimodule]. These examples show that this new concept and module amenability on Banach algebras do not coincide.

2. Preliminaries and Notations

Throughout this paper, $\mathcal{A}$ and $\mathfrak{A}$ are Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with compatible actions as follows:

$$\alpha\cdot(ab)=(\alpha\cdot a)b,\quad (ab)\cdot\alpha=a(b\cdot\alpha)\quad (a,b\in\mathcal{A},\alpha\in\mathfrak{A}).$$

Let X be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule with the following compatible actions:

$$\alpha\cdot(a\cdot x)=(\alpha\cdot a)\cdot x,\ a\cdot(\alpha\cdot x)=(a\cdot\alpha)\cdot x,\ (\alpha\cdot x)\cdot a=\alpha\cdot(x\cdot a)\quad (a\in\mathcal{A},\alpha\in\mathfrak{A},x\in X)$$

and similar for the right or two-sided actions. Then we say that X is a Banach $\mathcal{A}$-$\mathfrak{A}$-module. Moreover, if $\alpha\cdot x=x\cdot\alpha$ for all $\alpha\in\mathfrak{A},x\in X$, then X is called a *commutative* $\mathcal{A}$-$\mathfrak{A}$-module. If X is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module, then so is X^*, where the actions of $\mathcal{A}$ and $\mathfrak{A}$ on X^* are defined as follows:

$$\langle f\cdot\alpha,x\rangle=\langle f,\alpha\cdot x\rangle,\ \langle f\cdot a,x\rangle=\langle f,a\cdot x\rangle,$$

$$\langle\alpha\cdot f,x\rangle=\langle f,x\cdot\alpha\rangle,\ \langle a\cdot f,x\rangle=\langle f,x\cdot a\rangle\quad (a\in\mathcal{A},\alpha\in\mathfrak{A},x\in X,f\in X^*).$$

One should remember that $\mathcal{A}$ is not an $\mathcal{A}$-$\mathfrak{A}$-module in general because $\mathcal{A}$ does not satisfy the compatibility condition $a\cdot(\alpha\cdot b)=(a\cdot\alpha)\cdot b$ for $\alpha\in\mathfrak{A},a,b\in\mathcal{A}$. But when $\mathcal{A}$ is a commutative $\mathfrak{A}$-module and acts on itself by multiplication from both sides, then it is also a Banach $\mathcal{A}$-$\mathfrak{A}$-module.

Let E and F be Banach algebras. We denote by $\mathrm{Hom}(E, F)$ the metric space of all bounded homomorphisms from E into F, with the metric derived from the bounded linear operators from E into F, and denote $\mathrm{Hom}(E, E)$ by $\mathrm{Hom}(E)$.

Now let $\mathcal{A}$ and $\mathcal{B}$ be $\mathfrak{A}$-bimodules. Then a $\mathfrak{A}$-module homomorphism from $\mathcal{A}$ to $\mathcal{B}$ is a bounded map $T : \mathcal{A} \longrightarrow \mathcal{B}$ with $T(a \pm b) = T(a) \pm T(b)$, and is multiplicative, that is $T(ab) = T(a)T(b)$ for all $a, b \in \mathcal{A}$, and

$$T(\alpha \cdot a) = \alpha \cdot T(a),\ T(a \cdot \alpha) = T(a) \cdot \alpha, \quad (a, \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

We denote by $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A}, \mathcal{B})$, the space of all such homomorphisms and denote $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A}, \mathcal{A})$ by $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$. Note that when $\mathfrak{A} = \mathbb{C}$, the set of complex numbers, then $\mathrm{Hom}_{\mathbb{C}}(\mathcal{A}, \mathcal{B}) = \mathrm{Hom}(\mathcal{A}, \mathcal{B})$. Although the elements of $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A}, \mathcal{B})$ are not necessarily linear, their boundedness still implies their norm continuity.

Let $\mathcal{A}$ and $\mathfrak{A}$ be as above and X be a Banach $\mathcal{A}$-$\mathfrak{A}$-module. Recall that the mapping $D : \mathcal{A} \longrightarrow X$ is bounded if there exists $M > 0$ such that $\|D(a)\| \leq M\|a\|$ for all $a \in \mathcal{A}$. Suppose that φ and ψ are in $\mathrm{Hom}_{\mathfrak{A}}(A)$. A bounded map $D : \mathcal{A} \longrightarrow X$ is called a *module* (φ, ψ)*-derivation* if

$$D(\alpha \cdot a) = \alpha \cdot D(a), \quad D(a \cdot \alpha) = D(a) \cdot \alpha \quad (a \in \mathcal{A}, \alpha \in \mathfrak{A})$$

and

$$D(a \pm b) = D(a) \pm D(b), \quad D(ab) = D(a) \cdot \varphi(b) + \psi(a) \cdot D(b) \quad (a, b \in \mathcal{A}).$$

If X is a commutative $\mathcal{A}$-$\mathfrak{A}$-module, then each $x \in X$ defines a module (φ, ψ)-derivation $D_x(a) = x \cdot \varphi(a) - \psi(a) \cdot x$ on $\mathcal{A}$. These are called *module* (φ, ψ)*-inner derivations*. Derivations of these forms are studied in [7]. A Banach algebra $\mathcal{A}$ is called *module* (φ, ψ)*-amenable* (as an $\mathfrak{A}$-module) if for any commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module X, each module (φ, ψ)-derivation $D : \mathcal{A} \longrightarrow X^*$ is (φ, ψ)-inner [7]. We use the notations $Z_{\mathfrak{A}}(\mathcal{A}, (X_{(\varphi,\psi)})^*)$ for the space of all module (φ, ψ)-derivations $D : \mathcal{A} \longrightarrow X^*$, $B_{\mathfrak{A}}(\mathcal{A}, (X_{(\varphi,\psi)})^*)$ for those which are inner (φ, ψ)-derivations, and $H_{\mathfrak{A}}(\mathcal{A}, (X_{(\varphi,\psi)})^*)$ for the quotient space which we call the first relative (to $\mathfrak{A}$) (φ, ψ)-cohomology group of $\mathcal{A}$ with coefficients in X^*. Hence $\mathcal{A}$ is module (φ, ψ)-amenable if and only if $H_{\mathfrak{A}}(\mathcal{A}, (X_{(\varphi,\psi)})^*) = \{0\}$ for all commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module X. Indeed, for any $\phi, \psi \in \mathrm{Hom}(\mathcal{A})$, a Banach algebra $\mathcal{A}$ is (ϕ, ψ)-weakly amenable if $H^1(\mathcal{A}, (\mathcal{A}_{(\phi,\psi)})^*) = \{0\}$ (for details see [8]).

3. (σ, τ)-weak module amenability of Banach algebras

Let Y be a subspace $\mathcal{A}^*$ as a vector space which is $\mathcal{A}$-submodule and commutative Banach $\mathfrak{A}$-submodule. From now on, such subspaces are called commutative Banach $\mathcal{A}$-$\mathfrak{A}$-submodule of $\mathcal{A}^*$.

3.1. Definition. Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module and $\sigma, \tau \in \mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$. Then $\mathcal{A}$ is called (σ, τ)*-weakly module amenable* (as an $\mathfrak{A}$-module) if for any commutative Banach $\mathcal{A}$-$\mathfrak{A}$-submodule Y of $\mathcal{A}^*$, each module derivation from $\mathcal{A}$ to $Y_{(\sigma,\tau)}$ is inner.

In other words, in the above definition the module actions on $\mathcal{A}$ are considered as follows:

$$a \cdot x := \sigma(a)x, \quad x \cdot a = x\tau(a) \quad (a, x \in \mathcal{A}).$$

Thus, the module actions $\mathcal{A}$ on $Y \subseteq \mathcal{A}^*$ are as follows:

$$\langle a \cdot y, b\rangle = \langle y, b\tau(a)\rangle, \quad \langle y \cdot a, b\rangle = \langle y, \sigma(a)b\rangle \quad (a, b \in \mathcal{A}, y \in Y).$$

Note that if σ and τ are the identity maps, then (σ, τ)-weak module amenability becomes weak module amenability (see [2]).

Consider the closed ideal J of $\mathcal{A}$ generated by elements of the form $(a\cdot\alpha)b-a(\alpha\cdot b)$ for $\alpha\in\mathfrak{A}, a,b\in\mathcal{A}$. The ideal J is both $\mathcal{A}$-submodule and $\mathfrak{A}$-submodules of $\mathcal{A}$. Hence the quotient Banach algebra $\mathcal{A}/J$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-module with compatible actions when $\mathcal{A}$ acts on $\mathcal{A}/J$ canonically. Now, if $\mathcal{A}/J$ is a commutative Banach $\mathfrak{A}$-module and σ,τ are epimorphisms in $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$, then $\mathcal{A}$ is (σ,τ)-weakly module amenable if and only if every module derivation from $\mathcal{A}$ to $(\mathcal{A}/J)^*$ is inner. In fact for each $\alpha\in\mathfrak{A}, a,b\in\mathcal{A}, y\in Y$, we have

$$\begin{aligned}\langle y,(\sigma(a)\cdot\alpha)\tau(b)-\sigma(a)(\alpha\cdot\tau(b))\rangle &= \langle y,(\sigma(a)\cdot\alpha)\tau(b)\rangle-\langle y,\sigma(a)(\alpha\cdot\tau(b))\rangle\\ &= \langle b\cdot y,\sigma(a)\cdot\alpha\rangle-\langle y\cdot a,\alpha\cdot\tau(b)\rangle\\ &= \langle \alpha\cdot(b\cdot y),\sigma(a)\rangle-\langle (y\cdot a).\alpha,\tau(b)\rangle\\ &= \langle (b\cdot y)\cdot\alpha,\sigma(a)\rangle-\langle \alpha\cdot(y\cdot a),\tau(b)\rangle\\ &= \langle b\cdot(y\cdot\alpha),\sigma(a)\rangle-\langle (\alpha\cdot y)\cdot a,\tau(b)\rangle\\ &= \langle y\cdot\alpha,\sigma(a)\tau(b)\rangle-\langle \alpha\cdot y,\sigma(a)\tau(b)\rangle\\ &= \langle y\cdot\alpha-\alpha\cdot y,\sigma(a)\tau(b)\rangle=0.\end{aligned}$$

Thus for $\alpha\in\mathfrak{A}, a,b\in\mathcal{A}, y\in Y$ with $\sigma(a_0)=a$ and $\tau(b_0)=b$, we get

$$\langle y,(a\cdot\alpha)b-a(\alpha\cdot b)\rangle=\langle y,(\sigma(a_0)\cdot\alpha)\tau(b_0)-\sigma(a_0)(\alpha\cdot\tau(b_0))\rangle=0.$$

By continuity of D, we see $D(a)\subseteq J^{\perp}=(\mathcal{A}/J)^*$. It immediately follows from the above definition that a module amenable Banach algebra $\mathcal{A}$ is (σ,τ)-weakly module amenable for all $\sigma,\tau\in\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$. As we will see later in section four with some examples, the converse is false. Here and subsequently, we denote $\overbrace{\sigma\circ\sigma...\circ\sigma}^{n-times}$ by σ^n for all $n\in\mathbb{N}$.

3.2. Proposition. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule and $\sigma,\tau,\mu\in\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$. If μ is an epimorphism and $\mathcal{A}$ is $(\sigma\circ\mu,\tau\circ\mu)$-weakly module amenable, then $\mathcal{A}$ is (σ,τ)-weakly module amenable. The converse is true if μ^2 is the identity map.*

Proof. Let Y be a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-submodule of $\mathcal{A}^*$ and let $D:\mathcal{A}\to Y_{(\sigma,\tau)}$ be a module (σ,τ)-derivation. Then $D\circ\mu$ is a module $(\sigma\circ\mu,\tau\circ\mu)$-derivation. So there exists $y\in Y_{(\sigma\circ\mu,\tau\circ\mu)}$ such that for each $a\in\mathcal{A}$, $D(a)=y\cdot(\sigma\circ\mu)(a)-(\tau\circ\mu)(a)\cdot y$. Given $b\in\mathcal{A}$. Then there exists $a\in\mathcal{A}$ such that $\mu(a)=b$ and hence

$$D(b)=D(\mu(a))=y\cdot\sigma(\mu(a))-\tau(\mu(a))\cdot y=y\cdot\sigma(b)-\tau(b)\cdot y.$$

Thus D is (σ,τ)-inner.

Conversely, suppose that $D:\mathcal{A}\to Y_{(\sigma\circ\mu,\tau\circ\mu)}$ is a module $(\sigma\circ\mu,\tau\circ\mu)$-derivation. It is easy to show that $\widetilde{D}=D\circ\mu^{-1}$ is in $Z_{\mathfrak{A}}(\mathcal{A},(Y_{(\sigma,\tau)}))$. Thus there exists $y\in Y_{(\sigma,\tau)}$ so that for each $a\in\mathcal{A}$, $D(a)=y\cdot\sigma(a)-\tau(a)\cdot y$. We have

$$D(a)=D(\mu^{-1}(\mu(a)))=\widetilde{D}(\mu(a))=y\cdot(\sigma\circ\mu)(a)-(\tau\circ\mu)(a)\cdot y,$$

for all $a\in\mathcal{A}$. Therefore D is $(\sigma\circ\mu,\tau\circ\mu)$-inner. □

3.3. Corollary. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module and $\sigma\in\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$. Then the following statements hold:*

(i) *If σ is an epimorphism and $\mathcal{A}$ is (σ^n,σ^n)-weakly module amenable for some $n\in\mathbb{N}$, then $\mathcal{A}$ is weakly module amenable;*
(ii) *If $\mathcal{A}$ is weakly module amenable and σ^2 is the identity map, then $\mathcal{A}$ is (σ,σ)-weakly module amenable.*

3.4. Proposition. *Let $\sigma,\tau\in\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$ such that σ be an epimorphism and let the restriction of σ on the set $\{ab-ba\mid a,b\in\mathcal{A}\}$ be the identity map. If $\mathcal{A}$ is (τ,τ)-weakly module amenable, then $\mathcal{A}$ is $(\sigma\circ\tau,\sigma\circ\tau)$-weakly module amenable.*

Proof. Let Y be a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-submodule of $\mathcal{A}^*$ and let $D : \mathcal{A} \to Y_{(\sigma\circ\tau,\sigma\circ\tau)}$ be a module $(\sigma\circ\tau,\sigma\circ\tau)$-derivation. Define $\widetilde{D} : \mathcal{A} \to Y_{(\tau,\tau)}$ via $\langle \widetilde{D}(a), b\rangle := \langle D(a), \sigma(b)\rangle$. It is easy to check that $\widetilde{D}$ is a module (τ,τ)-derivation and thus there exists $y \in Y_{(\tau,\tau)}$ such that $\widetilde{D}(a) = y\cdot\tau(a) - \tau(a)\cdot y$ for every $a \in \mathcal{A}$. Take $x \in \mathcal{A}$. Since σ is an epimorphism, there exists $b \in \mathcal{A}$ such that $x = \sigma(b)$. Then for each $a \in \mathcal{A}$, we get

$$\begin{aligned}\langle D(a), x\rangle &= \langle \widetilde{D}(a), b\rangle = \langle y\cdot\tau(a) - \tau(a)\cdot y, b\rangle\\ &= \langle y, \sigma(\tau(a)b - b\tau(a))\rangle\\ &= \langle y\cdot\sigma\circ\tau(a) - \sigma\circ\tau(a)\cdot y, x\rangle.\end{aligned}$$

It follows that D is an $(\sigma\circ\tau,\sigma\circ\tau)$-inner derivation. □

3.5. Corollary. *Let $\sigma \in \mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$ such that σ is an epimorphism and let the restriction of σ on $\widetilde{\mathcal{A}} = \{ab - ba \mid a, b \in \mathcal{A}\}$ be the identity map. If $\mathcal{A}$ is weakly module amenable, then $\mathcal{A}$ is (σ^n,σ^n)-weakly module amenable for all $n \in \mathbb{N}$.*

Recall that $\mathfrak{A}$ has a bounded approximate identity for $\mathcal{A}$ if there is a bounded net $\{\alpha_j\}$ in $\mathfrak{A}$ such that $\|\alpha_j\cdot a - a\| \to 0$ and $\|a\cdot\alpha_j - a\| \to 0$, for each $a \in \mathcal{A}$.

3.6. Proposition. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module and $\sigma,\tau \in \mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$. If $\mathfrak{A}$ has a bounded approximate identity, then (σ,τ)-weak amenability of $\mathcal{A}$ implies its (σ,τ)-weak module amenability.*

Proof. Let Y be a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-submodule of $\mathcal{A}^*$ and let $D : \mathcal{A} \to Y_{(\sigma,\tau)}$ be a module (σ,τ)-derivation. If $\{\alpha_j\}$ is a bounded approximate identity for $\mathfrak{A}$, then by the Cohen factorization theorem [11], it is a bounded approximate identity for $\mathcal{A}$. Thus for each $a \in \mathcal{A}$ there are $\beta \in \mathfrak{A}$ and $b \in \mathcal{A}$ such that $a = \beta\cdot b$. Hence for each $a \in \mathcal{A}$ and $\rho \in \mathbb{C}$, we deduce that

$$\sigma(\rho a) = \sigma(\rho(\beta\cdot b)) = \lim_j \sigma(\rho(\alpha_j\beta)\cdot b) = \lim_j \sigma(\rho\alpha_j\cdot a) = \lim_j \rho\alpha_j\cdot\sigma(a) = \rho\sigma(a).$$

Therefore σ is $\mathbb{C}$-linear. Similarly, $\tau \in \mathrm{Hom}(\mathcal{A})$. To complete of the proof, it is enough to show that D is $\mathbb{C}$-linear. Again, by the Cohen factorization theorem for each $a \in \mathcal{A}$ there are $\gamma \in \mathfrak{A}$ and $y \in Y$ such that $D(a) = \gamma\cdot y$. Then

$$\begin{aligned}D(\rho a) &= D(\rho(\beta\cdot b)) = \lim_j D(\rho(\alpha_j\beta)\cdot b)\\ &= \lim_j D(\rho\alpha_j\cdot a) = \lim_j \rho\alpha_j\cdot D(a)\\ &= \lim_j \rho\alpha_j\cdot(\gamma\cdot y) = \rho(\gamma\cdot y) = \rho D(a).\end{aligned}$$

for all $a \in \mathcal{A}$ and $\rho \in \mathbb{C}$. □

3.7. Proposition. *Let $\mathcal{A}$ be a commutative Banach algebra and a commutative Banach $\mathfrak{A}$-bimodule. Suppose that $\sigma \in \mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$ such that $\sigma^2 = \sigma$, and the range of σ is a closed ideal of $\mathcal{A}$. If $\mathcal{A}$ is weakly module amenable and $\mathfrak{A}$ has a bounded approximate identity for $\mathcal{A}$, then $\mathcal{A}$ is (σ,σ)-weakly module amenable.*

Proof. Let Y be a Banach $\mathcal{A}$-$\mathfrak{A}$-submodule of $\mathcal{A}^*$ and let $D : \mathcal{A} \to Y_{(\sigma,\sigma)}$ be a module (σ,σ)-derivation. It is easily verified that the mapping $\overline{D} : \mathcal{A} \to Y$ is defined by $\langle \overline{D}(a), b\rangle := \langle D(a), \sigma(b)\rangle$, is a module derivation. Thus there exists $y \in Y$ such that $\overline{D}(a) = y\cdot a - a\cdot y$. Since $\mathcal{A} = \ker(\sigma) \oplus \mathrm{Im}(\sigma)$, it follows from [4, Proposition 2.1] that $\mathcal{A}/\mathrm{Im}(\sigma) \cong \ker(\sigma)$ is a weakly module amenable Banach algebra. For every $a \in \mathcal{A}$, we put $a = a_1 + a_2$ in which $a_1 \in \ker(\sigma)$ and $a_2 \in \mathrm{Im}(\sigma)$. By [4, Proposition 2.4] and the

Cohen factorization theorem, $(\ker(\sigma))^2$ is dense in $\ker(\sigma)$. Hence, there is a bounded net $(a_l b_l)_l \subset (\ker(\sigma))^2$ such that $a_l b_l \to a_1$, and

$$D(a_1) = \lim_l D(a_l b_l) = \lim_l (D(a_l) \cdot \sigma(b_l) - \sigma(a_l) \cdot D(b_l)) = 0.$$

This shows that $D(a) = D(\sigma(a))$ for all $a \in \mathcal{A}$. Now, suppose that $b \in \mathcal{A}$ such that $b = b_1 + b_2$ where $b_1 \in \ker(\sigma)$ and $b_2 \in \mathrm{Im}(\sigma)$. Take $a \in \mathcal{A}$ and the bounded nets $(a_{l_1} b_{l_2})_l \subset (\ker(\sigma))^2$ and $(a_{k_1} b_{k_2})_k \subset \mathcal{A}^2$ such that $a_{l_1} b_{l_2} \to b_1$ and $a_{k_1} b_{k_2} \to a$. Then, we have

$$\begin{aligned}
\langle D(a), b_1 \rangle &= \lim_l \lim_k \langle D(a_{k_1} b_{k_2}), a_{l_1} b_{l_2} \rangle \\
&= \lim_l \lim_k \langle D(a_{k_1}) \cdot \sigma(b_{k_2}) + \sigma(b_{k_1}) \cdot D(b_{k_2}), a_{l_1} b_{l_2} \rangle \\
&= \lim_l \lim_k \langle D(a_{k_1}), \sigma(b_{k_2}) a_{l_1} b_{l_2} \rangle + \lim_l \lim_k \langle D(b_{k_2}), a_{l_1} b_{l_2} \sigma(b_{k_1}) \rangle = 0.
\end{aligned}$$

The last equality follows from the fact that $\sigma(b_{k_2}) a_{l_1} b_{l_2}$ and $a_{l_1} b_{l_2} \sigma(b_{k_1})$ are in $\ker(\sigma) \cap \mathrm{Im}(\sigma) = \{0\}$. Also,

$$\begin{aligned}
\langle D(a), b_2 \rangle &= \langle D(a), \sigma(b_2) \rangle = \langle D(\sigma(a)), \sigma(b_2) \rangle \\
&= \langle \overline{D}(\sigma(a)), b_2 \rangle = \langle y \cdot \sigma(a) - \sigma(a) \cdot y, b_2 \rangle \\
&= \langle y, \sigma(a) b_2 - b_2 \sigma(a) \rangle = \langle \overline{D}(-b_2), \sigma(a) \rangle \\
&= \langle D(-\sigma(b_2)), \sigma^2(a) \rangle = \langle \overline{D}(-\sigma(b_2)), \sigma(a) \rangle \\
&= \langle y \cdot \sigma(a) - \sigma(a) \cdot y, b_2 \rangle.
\end{aligned}$$

The above computations show that $D \in B_{\mathfrak{A}}(\mathcal{A}, Y_{(\sigma,\sigma)})$. Therefore $\mathcal{A}$ is (σ, σ)-weakly module amenable. □

Let $\mathcal{A}$ and $\mathfrak{A}$ be as in the previous section and X be a Banach $\mathcal{A}$-$\mathfrak{A}$-module with the compatible actions, and J be the corresponding closed ideals of $\mathcal{A}$. Let $\sigma \in \mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$. Then for each $a, b \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$, we have

$$\sigma((a \cdot \alpha) b - a(\alpha \cdot b)) = (\sigma(a) \cdot \alpha) \sigma(b) - \sigma(a)(\alpha \cdot \sigma(b)) \in J.$$

Since J is a closed ideal of $\mathcal{A}$ and σ is continuous, $\sigma(J) \subseteq J$. Therefore, the mapping $\widehat{\sigma} : \mathcal{A}/J \longrightarrow \mathcal{A}/J$ is defined by $\widehat{\sigma}(a + J) = \sigma(a) + J$ is well defined.

Recall that a left Banach $\mathcal{A}$-module X is called a *left essential* $\mathcal{A}$-module if the linear span of $\mathcal{A} \cdot X = \{a \cdot x : a \in \mathcal{A}, x \in X\}$ is dense in X. Right essential $\mathcal{A}$-modules and (two-sided) essential $\mathcal{A}$-bimodules are defined similarly. We remark that if $\mathcal{A}$ is an essential left (right) $\mathfrak{A}$-module, then every $\mathfrak{A}$-module homomorphism σ is also a linear homomorphism. If $a \in \mathcal{A}$, then there is a sequence $(b_n) \subseteq \mathcal{A} \cdot \mathfrak{A}$ such that $\lim_n b_n = a$. Assume that $b_n = \sum_{m=1}^{K_n} \alpha_{n,m} a_{n,m}$ for some finite sequences $(a_{n,m})_{m=1}^{m=K_n} \subseteq \mathcal{A}$ and $(\alpha_{n,m})_{m=1}^{m=K_n} \subseteq \mathfrak{A}$. Let $t \in \mathbb{C}$. Then

$$\begin{aligned}
\sigma(t b_n) &= \sigma(t \textstyle\sum_{m=1}^{K_n} \alpha_{n,m} \cdot a_{n,m}) = \sum_{m=1}^{K_n} \sigma((t\alpha_{n,m}) \cdot a_{n,m}) \\
&= \textstyle\sum_{m=1}^{K_n} (t\alpha_{n,m}) \cdot \sigma(a_{n,m}) = \sum_{m=1}^{K_n} t\sigma(\alpha_{n,m} \cdot a_{n,m}) = t\sigma(b_n),
\end{aligned}$$

and so by the continuity of σ, $\sigma(ta) = t\sigma(a)$. By definition of $\widehat{\sigma}$, it is also $\mathbb{C}$-linear.

We say the Banach algebra $\mathfrak{A}$ acts trivially on $\mathcal{A}$ from left (right) if for each $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$, $\alpha \cdot a = \phi(\alpha) a$ ($a \cdot \alpha = \phi(\alpha) a$), where ϕ is a continuous linear functional on $\mathfrak{A}$. The following lemma is proved in [3, Lemma 3.1].

3.8. Lemma. *Let $\mathcal{A}$ be a Banach algebra and Banach $\mathfrak{A}$-module with compatible actions, and J_0 be a closed ideal of $\mathcal{A}$ such that $J \subseteq J_0$. If $\mathcal{A}/J_0$ has a left or right identity $e + J_0$,* *then for each $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$ we have $a \cdot \alpha - \alpha \cdot a \in J_0$, i.e., $\mathcal{A}/J_0$ is a commutative Banach $\mathfrak{A}$-module.*

The concept of $(\widehat{\sigma}, \widehat{\tau})$-weak amenability of $\mathcal{A}/J$ has been investigated in [8]. Relating to this, we now prove the main result in this section which gives the sufficient conditions for being (σ, τ)-weakly module amenable of a Banach algebra.

3.9. Theorem. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module with trivial left action, and let σ, τ be in $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$ and $\mathcal{A}/J$ has an identity. If $\mathcal{A}$ is a right essential $\mathfrak{A}$-module, then $(\widehat{\sigma}, \widehat{\tau})$-weak amenability of $\mathcal{A}/J$ implies (σ, τ)-weak module amenability of $\mathcal{A}$. The converse is true if σ and τ are epimorphisms.*

Proof. Let Y be a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-submodule of $\mathcal{A}^*$, and let $D : \mathcal{A} \to Y_{(\sigma,\tau)}$ be a module (σ, τ)-derivation. For $y \in Y, a, b \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$, we get

$$\begin{aligned} ((a \cdot \alpha)b - a(\alpha \cdot b)) \cdot y &= (a \cdot \alpha) \cdot (b \cdot y) - a \cdot ((\alpha \cdot b) \cdot y) \\ &= a \cdot (\alpha \cdot (b \cdot y)) - a \cdot (\alpha \cdot (b \cdot y)) = 0. \end{aligned}$$

Hence, $J \cdot Y = \{0\}$. Similarly, we have $Y \cdot J = \{0\}$. Therefore, the following module actions are well-defined

$$(a + J) \cdot y := a \cdot y, \quad y \cdot (a + J) := y \cdot a \quad (y \in Y, a \in \mathcal{A}).$$

Thus Y is a Banach $\mathcal{A}/J$-$\mathfrak{A}$-module. Define $\widetilde{D} : \mathcal{A}/J \longrightarrow Y \subseteq J^{\perp} = ((\mathcal{A}/J)_{(\widehat{\sigma},\widehat{\tau})})^*$ via $\widetilde{D}(a + J) = D(a)$. For each $\alpha \in \mathfrak{A}$ and $a, b \in \mathcal{A}$ we have

$$\begin{aligned} D((a \cdot \alpha)b - a(\alpha \cdot b)) &= D((a \cdot \alpha)b) - D(a(\alpha \cdot b)) \\ &= D(a \cdot \alpha) \cdot \sigma(b) + \tau(a \cdot \alpha) \cdot D(b) \\ &- (D(a) \cdot \sigma(\alpha \cdot b) - \tau(a) \cdot D(\alpha \cdot b)) \\ &= (D(a) \cdot \alpha) \cdot \sigma(b) - D(a) \cdot (\alpha \cdot \sigma(b)) \\ &+ (\tau(a) \cdot \alpha) \cdot D(b) - \tau(a) \cdot (\alpha \cdot D(b)) = 0. \end{aligned}$$

It means that D vanishes on J. Therefore $\widetilde{D}$ is well-defined. For each a, b in $\mathcal{A}$ we have

$$\begin{aligned} \tilde{D}(ab + J) = D(ab) &= D(a) \cdot \sigma(b) + \tau(a) \cdot D(b) \\ &= \tilde{D}(a + J) \cdot (\sigma(b) + J) + (\tau(a) + J) \cdot \tilde{D}(b + J) \\ &= \tilde{D}(a + J) \cdot \widehat{\sigma}(b + J) + \widehat{\tau}(a + J) \cdot \tilde{D}(b + J). \end{aligned}$$

Since $\mathcal{A}$ is a right essential $\mathfrak{A}$-module, $\widehat{\sigma}$ and $\widehat{\tau}$ are homomorphism. Thus $\widehat{\sigma}, \widehat{\tau} \in \mathrm{Hom}(\mathcal{A}/J)$. Now, it follows from the above discussion that $\tilde{D}$ is also $\mathbb{C}$-linear, and so it is $(\widehat{\sigma}, \widehat{\tau})$-inner. Hence there exists $y \in Y$ such that

$$D(a) = \tilde{D}(a + J) = y \cdot \widehat{\sigma}(a + J) - \widehat{\tau}(a + J) \cdot y = y \cdot \sigma(a) - \tau(a) \cdot y.$$

Therefore D is a module (σ, τ)-inner derivation.

Conversely, suppose that $\sigma, \tau \in \mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$ are epimorphisms, and $D : \mathcal{A}/J \longrightarrow ((\mathcal{A}/J)_{(\widehat{\sigma},\widehat{\tau})})^*$ is a $(\widehat{\sigma}, \widehat{\tau})$-derivation. We define $\tilde{D} : \mathcal{A} \longrightarrow ((\mathcal{A}/J)_{(\sigma,\tau)})^*$ by $\tilde{D}(a) = D(a + J)$, for all $a \in \mathcal{A}$. Lemma 3.8 shows that when $\mathfrak{A}$ acts on $\mathcal{A}$ trivially from left or right, then $\mathcal{A}/J$ is a commutative $\mathfrak{A}$-module and thus $Y = J^{\perp} \subseteq \mathcal{A}^*$. Hence $\tilde{D}$ could be considered as a map from $\mathcal{A}$ to Y. Now, for each $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$ we have

$$\tilde{D}(\alpha \cdot a) = D(\alpha \cdot a + J) = D(\phi(\alpha)a + J) = \phi(\alpha)D(a + J) = \alpha \cdot \tilde{D}(a)$$

and

$$\tilde{D}(a \cdot \alpha) = D(a \cdot \alpha + J) = D(\phi(\alpha)a + J) = \phi(\alpha)D(a + J) = \tilde{D}(a) \cdot \alpha.$$

Also, for $a, b \in \mathcal{A}$ we obtain $\tilde{D}(ab) = \tilde{D}(a) \cdot \sigma(b) + \tau(a) \cdot \tilde{D}(b)$. Thus $\tilde{D}$ is a (σ, τ)-module derivation. Due to (σ, τ)-weak module amenability of $\mathcal{A}$, there exists $y \in Y \cong$

$((\mathcal{A}/J)_{(\sigma,\tau)})^*$ such that $\tilde{D}(a) = \sigma(a)\cdot y - y\cdot\tau(a)$, and so $D(a+J) = \widehat{\sigma}(a+J)\cdot y - y\cdot\widehat{\tau}(a+J)$. □

The Banach algebras with compatible $\mathfrak{A}$-module structure could be considered as objects of a category $\mathfrak{C}_{\mathfrak{A}}$ whose morphisms are bounded $\mathfrak{A}$-module maps. We are interested in the case where $\mathfrak{A}$ is an injective object in $\mathfrak{C}_{\mathfrak{A}}$, that is for any objects $A, B \in \mathfrak{C}_{\mathfrak{A}}$ and monomorphism $\theta : B \longrightarrow A$ and morphism $\mu : B \longrightarrow \mathfrak{A}$, there exists a morphism $\widetilde{\mu} : A \longrightarrow \mathfrak{A}$ such that $\mu = \widetilde{\mu}\circ\theta$. This is the case when $\mathfrak{A} = \mathbb{C}$ (Hahn Banach Theorem).

3.10. Proposition. *Let $\mathcal{A}$ be a commutative $\mathfrak{A}$-module and let σ, τ be in $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$ such that $\sigma(a)b = a\tau(b)$ for all $a, b \in \mathcal{A}$. Also let $\mathfrak{A}$ be injective and has a bounded approximate identity. If $\mathcal{A}$ is (σ,τ)-weakly module amenable, then span $(\mathcal{A}\mathfrak{A}\mathcal{A})$ is dense in $\mathcal{A}$.*

Proof. Let B be the linear span of $(\mathcal{A}\mathfrak{A}\mathcal{A})$. Suppose that $\overline{B} \neq \mathcal{A}$. Take $a_0 \in \mathcal{A}\backslash\overline{B}$ and $f_1 \in \mathcal{A}^*$ such that $f_1(a_0) = 1$ and $f_1|_{\overline{B}} = 0$. Since a_0 is not in $\overline{B}$, similar to the proof of [2, lemma 2.1] we can construct an epimorphism $f_2 : \mathcal{A} \longrightarrow \mathfrak{A}$ such that $f_2|_{\overline{B}} = 0$ and $f_2(a_0) = 1$. Define $D : \mathcal{A} \longrightarrow ((\mathcal{A})_{(\sigma,\tau)})^*$ via $D(a) = f_2(a)\cdot f_1$ for all $a \in \mathcal{A}$. Then D is (σ,τ)-module derivation and hence there exists $g \in (\mathcal{A}_{(\sigma,\tau)})^*$ such that $D(a) = g\cdot\sigma(a) - \tau(a)\cdot g$, for all $a \in \mathcal{A}$. Thus, we have

$$\begin{aligned}1 = f_2(a_0)f_1(a_0) &= \langle D(a_0), a_0\rangle \\ &= \langle g\cdot\sigma(a_0) - \tau(a_0)\cdot g, a_0\rangle \\ &= \langle g, \sigma(a_0)a_0 - \tau(a_0)a_0\rangle = 0,\end{aligned}$$

which is a contradiction. □

3.11. Corollary. *With the hypotheses of the above Proposition, $\mathcal{A}$ is $(0,0)$-weakly module amenable if and only if span $(\mathcal{A}\mathfrak{A}\mathcal{A})$ is dense in $\mathcal{A}$.*

Proof. Let $D : \mathcal{A} \to (\mathcal{A}_{(0,0)})^*$ be a $(0,0)$-module derivation. Then we have $D(\mathcal{A}\mathfrak{A}\mathcal{A}) = \{0\}$. Since D is continuous, we have $D = 0$. So D is $(0,0)$-inner. Conversely, let $\mathcal{A}$ be $(0,0)$-weakly amenable. Then by Proposition (3.10), $\overline{\mathcal{A}\mathfrak{A}\mathcal{A}} = \mathcal{A}$. □

3.12. Remark. Let $\mathcal{A}$ be a commutative $\mathfrak{A}$-module and let $\sigma, \tau \in \mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$ such that $\sigma(a)b = a\tau(b)$ for all $a, b \in \mathcal{A}$. Then the second adjoints σ'' and τ'' belong to $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A}^{**})$ and are also w^*-w^*-continuous. We thus can show that $\sigma''(F)\Box G = F\Box\tau''(G)$, where $\Box$ is the first Arens product on the second dual $\mathcal{A}^{**}$ (for more information about this product see [10]). Now, if $\mathcal{A}^{**}$ is (σ'',τ'')-weakly amenable then by Proposition 3.10, $\overline{\mathcal{A}^{**}\mathfrak{A}\mathcal{A}^{**}} = \mathcal{A}^{**}$. It follows from the proof of [2, Proposition 3.6] that $\overline{\mathcal{A}\mathfrak{A}\mathcal{A}} = \mathcal{A}$. Therefore $\mathcal{A}$ is $(0,0)$-weakly amenable by Corollary 3.11.

4. (σ,τ)-weak module amenability of semigroup algebras

Let S be an (discrete) inverse semigroup with the set of idempotents E_S (or E), where the order of E is defined by

$$e \le d \Longleftrightarrow ed = e \quad (e, d \in E).$$

It is easy to show that E is a (commutative) subsemigroup of S [17, Theorem V.1.2]. In particular $\ell^1(E)$ could be regarded as a subalgebra of $\ell^1(S)$, and thereby $\ell^1(S)$ is a Banach algebra and a Banach $\ell^1(E)$-module with compatible actions [1]. We consider the following module actions $\ell^1(E)$ on $\ell^1(S)$:

$$\delta_e\cdot\delta_s = \delta_s,\ \delta_s\cdot\delta_e = \delta_{se} = \delta_s * \delta_e \quad (s \in S, e \in E). \tag{4.1}$$

If ϕ is a continuous linear function on $\ell^1(E)$, then for each $e \in E$ we have $\phi(\delta_e) = 1$. So for each $f = \sum_{e\in E} f(e)\delta_e \in \ell^1(E)$ and $g = \sum_{s\in S} g(s)\delta_s \in \ell^1(S)$, we get

$$\begin{aligned} f \cdot g &= (\sum_{e\in E} f(e)\delta_e) \cdot (\sum_{s\in S} g(s)\delta_s) = \sum_{s\in S, e\in E} f(e)g(s)\delta_e \cdot \delta_s \\ &= \sum_{s\in S, e\in E} f(e)g(s) \cdot \delta_s = (\sum_{e\in E} f(e))(\sum_{s\in S} g(s)\delta_s) = \phi(f)g. \end{aligned}$$

Therefore multiplication from left is trivial. In this case, the ideal J (see section 3) is the closed linear span of $\{\delta_{set} - \delta_{st} : s, t \in S, e \in E\}$. We consider an equivalence relation on S as follows:

$$s \approx t \Longleftrightarrow \delta_s - \delta_t \in J \quad (s, t \in S).$$

For an inverse semigroup S, the quotient $S/\approx$ is a discrete group (see [3] and [23]). As in [24, Theorem 3.3], we may observe that $\ell^1(S)/J \cong \ell^1(S/\approx)$. We consider the following module actions $\ell^1(E)$ on $\ell^1(S)/J \cong \ell^1(S/\approx)$:

$$\delta_e \cdot (\delta_s + J) = \delta_s + J, \ (\delta_s + J) \cdot \delta_e = \delta_{se} + J \quad (s \in S, e \in E).$$

Indeed $\delta_s - \delta_{se} \in J$ if and only if $\delta_{st} - \delta_{set} \in J$, for all $s, t \in S, e \in E$. Therefore $\ell^1(S/\approx)$ is a commutative $\ell^1(E)$-bimodule. For each $\sigma \in \mathrm{Hom}_{\ell^1(E)}(\ell^1(S))$, we define $\widehat{\sigma}$ in $\mathrm{Hom}(\ell^1(S/\approx))$ by $\widehat{\sigma}(\delta_{[s]}) = \delta_{[\sigma(s)]}$ and extend by linearity, where $[s]$ denote the equivalence class of s in $S/\approx$ (see the explanations after Proposition 3.7). We see that all conditions of Theorem 3.9 hold for $\sigma, \tau \in \mathrm{Hom}_{\ell^1(E)}(\ell^1(S))$ which are also epimorphism. Now, if $\ell^1(S)$ is (σ, τ)-weakly module amenable then $\ell^1(S/\approx)$ is $(\widehat{\sigma}, \widehat{\tau})$-weakly amenable. We are now going to prove the main result in this section.

4.1. Theorem. *Let S be an inverse semigroup with the set of idempotents E. Then for each σ and τ in $\mathrm{Hom}_{\ell^1(E)}(\ell^1(S))$, the semigroup algebra $\ell^1(S)$ is (σ, τ)-weakly module amenable as an $\ell^1(E)$-module, with trivial left action.*

Proof. Suppose firstly that σ or τ is zero map. Since $S/\approx$ is a discrete group, the group algebra $\ell^1(S/\approx)$ has an identity, and thus $\ell^1(S/\approx)$ is $(\widehat{\sigma}, 0)$ and $(0, \widehat{\sigma})$-weakly amenable by [8, Example 4.2]. With the actions considered in (4.1), for each $f \in \ell^1(S)$, we have

$$f = \sum_{s\in S} f(s)\delta_s = \sum_{s\in S} f(s)\delta_s * \delta_{s^*s} = \sum_{s\in S} f(s)\delta_s \cdot \delta_{s^*s}.$$

Consequently f belongs to the closed linear span of $\ell^1(S) \cdot \ell^1(E) = \{\delta_s \cdot \delta_e : e \in E, s \in S\}$. This shows that $\ell^1(S)$ is a right essential $\ell^1(E)$-module. For $\mathcal{A} = \ell^1(S)$ and $\mathfrak{A} = \ell^1(E)$, the result of this case follows from Theorem 3.9. For the case that both σ and τ are non-zero homomorphisms, it is proved in [14, Theorem 2.5] that for any locally compact group G, the group algebra $L^1(G)$ is (φ, ψ)-weakly amenable for all $\varphi, \psi \in \mathrm{Hom}(L^1(G))$. In particular, $\ell^1(S/\approx)$ is $(\widehat{\sigma}, \widehat{\tau})$-weakly amenable. Now, Theorem 3.9 again shows that $\ell^1(S)$ is (σ, τ)-weakly module amenable. □

Note that for an amenable inverse semigroup S, $\ell^1(S)$ is module $\ell^1(E)$-amenable [1, Theorem 3.1] and so, it is module (σ, τ)-amenable [7, Corollary 2.3]. We close this section by two examples.

4.2. Example. *Let S be a commutative inverse semigroup. Then $\ell^1(S)$ is a commutative Banach algebra and commutative Banach $\ell^1(E)$-module with the following actions:*

$$\delta_e \cdot \delta_s = \delta_s \cdot \delta_e = \delta_{es} \quad (s \in S, e \in E).$$

We consider the mapping σ as follows:

$$\sigma : \ell^1(S) \longrightarrow \ell^1(S); \ \sum_{s\in S} f(s)\delta_s \mapsto \sum_{s\in S} \overline{f(s)}\delta_{s^*} \quad (s \in S),$$

where $\overline{f(s)}$ *is the complex conjugate of* $f(s)$*. Obviously* $\sigma \in Hom_{\ell^1(E)}(\ell^1(S))$*. Also,* σ *is also* $\mathbb{C}$*-linear and* σ^2 *is the identity map. It is shown in* [4, Theorem 3.1] *that* $\ell^1(S)$ *is weakly module amenable. Now it follows from Corollary 3.3 that* $\ell^1(S)$ *is* (σ, σ)*-weakly module amenable. Note that if* S *is not amenable,* $\ell^1(S)$ *is not module amenable* [1, Theorem 3.1].

4.3. Example. *Let* S *be an inverse semigroup with the set of idempotents* E*. Let* $C^*(S)$ *be the enveloping* C^**-algebra of* $\ell^1(S)$ *(see* [13]*). Then by continuity, the action of* $\ell^1(E)$ *on* $\ell^1(S)$ *extends to an action of* $C^*(E)$ *on* $C^*(S)$*. The* C^**-algebra* $C^*(E)$ *has a bounded approximate identity, and so it is* $(\sigma, 0)$ *and* $(0, \sigma)$*-weakly module amenable by Proposition 3.6 and* [8, Example 4.2]*, for all* $\sigma \in Hom_{C^*(E)}(C^*(S))$ *. Now, suppose that* σ^2 *is the identity map (see Example 4.2). Since* $C^*(S)$ *is weakly amenable* [16, Theorem 1.10]*,* $C^*(S)$ *is* (σ, σ)*-weakly module amenable by Corollary 3.3. However, if* $C^*(S)$ *is nuclear then it is amenable* [15]*. By* [1, Proposition 2.1]*,* $C^*(S)$ *is module amenable as an* $C^*(E)$*-module. Therefore* $C^*(S)$ *is module* (σ, τ)*-amenable, for all* $\sigma, \tau \in Hom_{C^*(E)}(C^*(S))$ *by* [7, Corollary 2.3].

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