

# COMMON FIXED POINTS FOR $\psi$ -CONTRACTIONS ON PARTIAL METRIC SPACES

Cristina Di Bari <sup>a †</sup> and Pasquale Vetro <sup>a ‡</sup>

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## Abstract

We prove some generalized versions of an interesting result of Matthews using conditions of different type in 0-complete partial metric spaces. We give, also, a homotopy result for operators on partial metric spaces.

**Keywords:** Points of coincidence, common fixed points, 0-complete partial metric space,  $\psi$ -contractions.

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## 1. Introduction

In the setting of domain theory, attempts were made in order to equip semantics domain with a notion of distance. In particular, Matthews [8] introduced the notion of a partial metric space as a part of the study of denotational semantics of data for networks, showing that the contraction mapping principle of Banach [2] can be generalized to the partial metric context for applications in program verification. Moreover, the existence of several connections between partial metrics and topological aspects of domain theory have been lately pointed by other authors as O'Neill [9], Bukatin and Scott [3], Bukatin and Shorina [4], Romaguera and Schellekens [13] and others.

After the definition of the concept of partial metric space, Matthews [8] obtained a Banach type fixed point theorem on complete partial metric spaces. In this paper to prove some generalized versions of the result of Matthews, we use conditions of different type in 0-complete partial metric spaces. In section 4, using our results, we give a homotopy result for operators on partial metric spaces.

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<sup>a</sup>Università degli Studi di Palermo, Dipartimento di Matematica e Informatica, Via Archirafi, 34 - 90123 Palermo (Italy).

<sup>†</sup>Email: [dibari@math.unipa.it](mailto:dibari@math.unipa.it)

<sup>‡</sup>Email: [vetro@math.unipa.it](mailto:vetro@math.unipa.it)

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## 2. Preliminaries

In this section, we recall some definitions and some properties of partial metric space [5, 8, 9, 10, 12, 14]. A partial metric on a nonempty set  $X$  is a function  $p : X \times X \rightarrow [0, +\infty[$  such that for all  $x, y, z \in X$ :

- (p<sub>1</sub>)  $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$ ,
- (p<sub>2</sub>)  $p(x, x) \leq p(x, y)$ ,
- (p<sub>3</sub>)  $p(x, y) = p(y, x)$ ,
- (p<sub>4</sub>)  $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$ .

A partial metric space is a pair  $(X, p)$  such that  $X$  is a nonempty set and  $p$  is a partial metric on  $X$ . It is clear that, if  $p(x, y) = 0$ , then from (p<sub>1</sub>) and (p<sub>2</sub>) it follows that  $x = y$ . But if  $x = y$ ,  $p(x, y)$  may not be 0. A basic example of a partial metric space is the pair  $([0, +\infty[, p)$ , where  $p(x, y) = \max\{x, y\}$  for all  $x, y \in [0, +\infty[$ . Other examples of partial metric spaces which are interesting from a computational point of view can be found in [8].

Each partial metric  $p$  on  $X$  generates a  $T_0$  topology  $\tau_p$  on  $X$  which has as a base the family of open  $p$ -balls  $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$ , where

$$B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\},$$

for all  $x \in X$  and  $\varepsilon > 0$ .

If  $p$  is a partial metric on  $X$ , then the function  $p^s : X \times X \rightarrow [0, +\infty[$  given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a metric on  $X$ . Let  $(X, p)$  be a partial metric space. Then:

A sequence  $\{x_n\}$  in a partial metric space  $(X, p)$  converges to a point  $x \in X$  if and only if  $p(x, x) = \lim_{n \rightarrow +\infty} p(x, x_n)$ .

A sequence  $\{x_n\}$  in a partial metric space  $(X, p)$  is called a Cauchy sequence if there exists (and is finite)  $\lim_{n, m \rightarrow +\infty} p(x_n, x_m)$ .

A partial metric space  $(X, p)$  is said to be complete if every Cauchy sequence  $\{x_n\}$  in  $X$  converges, with respect to  $\tau_p$ , to a point  $x \in X$  such that  $p(x, x) = \lim_{n, m \rightarrow +\infty} p(x_n, x_m)$ .

A sequence  $\{x_n\}$  in a partial metric space  $(X, p)$  is called 0-Cauchy if  $\lim_{n, m \rightarrow +\infty} p(x_n, x_m) = 0$ . We say that  $(X, p)$  is 0-complete if every 0-Cauchy sequence in  $X$  converges, with respect to  $\tau_p$ , to a point  $x \in X$  such that  $p(x, x) = 0$ .

On the other hand, the partial metric space  $(\mathbb{Q} \cap [0, +\infty[, p)$ , where  $\mathbb{Q}$  denotes the set of rational numbers and the partial metric  $p$  is given by  $p(x, y) = \max\{x, y\}$ , provides an example of a 0-complete partial metric space which is not complete.

It is easy to see that, every closed subset of a complete (0-complete) partial metric space is complete (0-complete).

**2.1. Lemma** ([8, 10]). *Let  $(X, p)$  be a partial metric space.*

- (a)  $\{x_n\}$  is a Cauchy sequence in  $(X, p)$  if and only if it is a Cauchy sequence in the metric space  $(X, p^s)$ .
- (b) A partial metric space  $(X, p)$  is complete if and only if the metric space  $(X, p^s)$  is complete. Furthermore,  $\lim_{n \rightarrow +\infty} p^s(x_n, x) = 0$  if and only if

$$p(x, x) = \lim_{n \rightarrow +\infty} p(x_n, x) = \lim_{n, m \rightarrow +\infty} p(x_n, x_m).$$

**2.2. Lemma.** *Let  $(X, p)$  be a partial metric space and  $\{x_n\} \subset X$ . If  $x_n \rightarrow x \in X$  and  $p(x, x) = 0$ , then  $\lim_{n \rightarrow +\infty} p(x_n, z) = p(x, z)$  for all  $z \in X$ .*

*Proof.* By the triangle inequality

$$p(x, z) - p(x_n, x) \leq p(x_n, z) \leq p(x, z) + p(x_n, x).$$

Letting  $n \rightarrow +\infty$ , we obtain that  $p(x_n, z) \rightarrow p(x, z)$ . □

Define  $p(x, A) = \inf\{p(x, a) : a \in A\}$ . Then  $a \in \bar{A} \Leftrightarrow p(a, A) = p(a, a)$ , where  $\bar{A}$  denotes the closure of  $A$ . From

$$p^s(x, a) = 2p(x, a) - p(x, x) - p(a, a) \leq 2p(x, a)$$

for every  $a \in A$ , we deduce that  $p^s(x, A) \leq 2p(x, A)$ .

Let  $X$  be a non-empty set and  $T, f : X \rightarrow X$ . The mappings  $T, f$  are said to be weakly compatible if they commute at their coincidence point (i.e.  $Tfx = fTx$  whenever  $Tx = fx$ ). A point  $y \in X$  is called point of coincidence of  $T$  and  $f$  if there exists a point  $x \in X$  such that  $y = Tx = fx$ .

**2.3. Lemma.** *Let  $X$  be a non-empty set and the mappings  $T, f : X \rightarrow X$  have a unique point of coincidence  $v$  in  $X$ . If  $T$  and  $f$  are weakly compatible, then  $T$  and  $f$  have a unique common fixed point.*

*Proof.* Since  $v$  is a point of coincidence of  $T$  and  $f$ . Therefore,  $v = fu = Tu$  for some  $u \in X$ . By weakly compatibility of  $T$  and  $f$  we have

$$Tv = Tfu = fTu = fv.$$

It implies that  $Tv = fv = w$  (say). Then  $w$  is a point of coincidence of  $T$  and  $f$ . Therefore,  $v = w$  by uniqueness. Thus  $v$  is a unique common fixed point of  $T$  and  $f$ . □

### 3. Main results

Let  $(X, p)$  be a partial metric space and  $T, f : X \rightarrow X$  be such that  $TX \subset fX$ . For every  $x_0 \in X$  we consider the sequence  $\{x_n\} \subset X$  defined by  $fx_n = Tx_{n-1}$  for all  $n \in \mathbb{N}$  and we say that  $\{Tx_n\}$  is a  $T$ - $f$ -sequence of initial point  $x_0$ .

**3.1. Lemma.** *For every function  $\psi : [0, +\infty[ \rightarrow [0, +\infty[$ , let  $\psi^n$  be the  $n$ th iterate of  $\psi$ . Then the following holds: if  $\psi$  is nondecreasing, then for each  $t > 0$ ,  $\lim_{n \rightarrow +\infty} \psi^n(t) = 0$  implies  $\psi(t) < t$ .*

The following theorem of common fixed point in partial metric space is first main result.

**3.2. Theorem.** *Let  $(X, p)$  be a partial metric space and  $T, f$  be mappings on  $X$  with  $TX \subset fX$ . Assume that*

$$(3.1) \quad p(Tx, Ty) \leq \psi(\max\{p(fx, fy), p(fx, Tx), p(fy, Ty), \frac{1}{2}[p(fx, Ty) + p(fy, Tx)]\})$$

for all  $x, y \in X$ , where  $\psi : [0, +\infty[ \rightarrow [0, +\infty[$  is right continuous, nondecreasing function such that  $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$  for all  $t > 0$ . If  $TX$  or  $fX$  is a 0-complete subspace of  $X$ , then  $T$  and  $f$  have a unique point of coincidence. Moreover, if  $T$  and  $f$  are weakly compatible, then  $T$  and  $f$  have a unique fixed point.

*Proof.* Fix  $x_0 \in X$ . We prove that the  $T$ - $f$ -sequence  $\{Tx_n\}$  of initial point  $x_0$  is a Cauchy sequence in  $TX$ . If  $Tx_n = Tx_{n-1}$  for some  $n \in \mathbb{N}$ , then  $Tx_n = Tx_m$  for all  $m \in \mathbb{N}$  with  $m > n$  and so  $\{Tx_n\}$  is a Cauchy sequence.

Suppose that  $Tx_n \neq Tx_{n-1}$  for all  $n \in \mathbb{N}$ . We have

$$p(Tx_{n+1}, Tx_n) \leq \psi(\max\{p(fx_{n+1}, fx_n), p(fx_{n+1}, Tx_{n+1}), p(fx_n, Tx_n)\},$$

$$\begin{aligned} & \frac{1}{2}[p(fx_{n+1}, Tx_n) + p(fx_n, Tx_{n+1})] \\ = & \psi(\max\{p(Tx_{n-1}, Tx_n), p(Tx_n, Tx_{n+1}), \\ & \frac{1}{2}[p(Tx_n, Tx_n) + p(Tx_{n-1}, Tx_{n+1})]\}). \end{aligned}$$

Since, by  $(p_4)$ ,  $p(Tx_n, Tx_n) + p(Tx_{n-1}, Tx_{n+1}) \leq p(Tx_{n-1}, Tx_n) + p(Tx_n, Tx_{n+1})$ , we obtain

$$p(Tx_{n+1}, Tx_n) \leq \psi(\max\{p(Tx_{n-1}, Tx_n), p(Tx_n, Tx_{n+1})\}).$$

Now,  $\max\{p(Tx_{n-1}, Tx_n), p(Tx_n, Tx_{n+1})\} = p(Tx_n, Tx_{n+1})$  implies a contradiction and so

$$p(Tx_{n+1}, Tx_n) \leq \psi(p(Tx_{n-1}, Tx_n)), \quad \text{for all } n \in \mathbb{N},$$

and hence

$$(3.2) \quad p(Tx_{n+1}, Tx_n) \leq \psi^n(p(Tx_1, Tx_0)), \quad \text{for all } n \in \mathbb{N}.$$

Fix  $\varepsilon > 0$  and choose  $n(\varepsilon) \in \mathbb{N}$  such that

$$\sum_{n \geq n(\varepsilon)} \psi^n(p(Tx_1, Tx_0)) < \varepsilon.$$

Now, for all  $n \geq n(\varepsilon)$  and all  $k \in \mathbb{N}$ , we have

$$\begin{aligned} p(Tx_{n+k}, Tx_n) & \leq \sum_{j=1}^k p(Tx_{n+j}, Tx_{n+j-1}) - \sum_{j=1}^{k-1} p(Tx_{n+j}, Tx_{n+j}) \\ & \leq \sum_{j=1}^k p(Tx_{n+j}, Tx_{n+j-1}) \\ & \leq \sum_{n \geq n(\varepsilon)} \psi^n(p(Tx_1, Tx_0)) < \varepsilon. \end{aligned}$$

This implies that  $\lim_{n, m \rightarrow +\infty} p(Tx_n, Tx_m) = 0$  and hence  $\{Tx_n\}$  is a 0-Cauchy sequence in the partial metric space  $(X, p)$ .

Suppose that  $TX$  is a 0-complete subspace of  $(X, p)$ , then there exists  $y \in TX \subset fX$  such that

$$p(y, y) = \lim_{n \rightarrow +\infty} p(Tx_n, y) = \lim_{n \rightarrow +\infty} p(fx_n, y) = \lim_{n, m \rightarrow +\infty} p(Tx_n, Tx_m) = 0.$$

This holds also if  $fX$  is a 0-complete subspace of  $(X, p)$  with  $y \in fX$ .

Let  $u \in X$  be such that  $y = fu$ . We show that  $y$  is a point of coincidence of  $T$  and  $f$ . If not  $p(fu, Tu) > 0$ , since

$$\max\{p(fx_n, fu), p(fx_n, Tx_n), p(fu, Tu), \frac{1}{2}[p(fx_n, Tu) + p(fu, Tx_n)]\} \geq p(fu, Tu)$$

for all  $n \in \mathbb{N}$ , from

$$p(Tx_n, Tu) \leq \psi(\max\{p(fx_n, fu), p(fx_n, Tx_n), p(fu, Tu), \frac{1}{2}[p(fx_n, Tu) + p(fu, Tx_n)]\})$$

as  $n \rightarrow +\infty$ , by Lemma 2.2 and by the right continuity of  $\psi$ , we have

$$p(fu, Tu) \leq \psi(p(fu, Tu)) < p(fu, Tu).$$

This implies  $p(fu, Tu) = 0$ , that is  $Tu = fu$ . Hence, we have shown that  $y = fu = Tu$  is a point of coincidence of  $T$  and  $f$ . If  $z \in X$ , with  $z = fs = Ts$ , is another point of coincidence of  $T$  and  $f$ , then  $z = y$ . Assume  $z \neq y$ , from

$$\begin{aligned} p(Tu, Ts) &\leq \psi(\max\{p(fs, fu), p(fu, Tu), p(fs, Ts), \frac{1}{2}[p(fu, Ts) + p(fs, Tu)]\}) \\ &= \psi(p(Tu, Ts)) < p(Tu, Ts) \end{aligned}$$

we deduce that  $y = z$ , and so  $y$  is a unique point of coincidence of  $T$  and  $f$ . By Lemma 2.3, we deduce that  $y$  is a unique common fixed point of  $T$  and  $f$ .  $\square$

If in Theorem 3.2 we take  $\psi(t) = kt$  with  $k \in [0, 1[$ , we obtain the following theorem.

**3.3. Theorem.** *Let  $(X, p)$  be a partial metric space and  $T, f$  be mappings on  $X$  with  $TX \subset fX$ . Assume that*

$$(3.3) \quad p(Tx, Ty) \leq k \max\{p(fx, fy), p(fx, Tx), p(fy, Ty), \frac{1}{2}[p(fx, Ty) + p(fy, Tx)]\}$$

for all  $x, y \in X$ , where  $k \in [0, 1[$ . If  $TX$  or  $fX$  is a 0-complete subspace of  $X$ , then  $T$  and  $f$  have a unique point of coincidence. Moreover, if  $T$  and  $f$  are weakly compatible, then  $T$  and  $f$  have a unique fixed point.

From Theorem 3.3, we deduce the following theorem.

**3.4. Theorem.** *Let  $(X, p)$  be a partial metric space and  $T, f$  be mappings on  $X$  with  $TX \subset fX$ . Assume that*

$$(3.4) \quad p(Tx, Ty) \leq ap(fx, fy) + b[p(fx, Tx) + p(fy, Ty)] + c[p(fx, Ty) + p(fy, Tx)]$$

for all  $x, y \in X$ , where  $a, b, c \geq 0$  and  $a + 2b + 2c < 1$ . If  $TX$  or  $fX$  is a 0-complete subspace of  $X$ , then  $T$  and  $f$  have a unique point of coincidence. Moreover, if  $T$  and  $f$  are weakly compatible, then  $T$  and  $f$  have a unique fixed point.

**3.5. Remark.** Assume that for all  $x, y \in X$  the following condition holds

$$(3.5) \quad p(Tx, Ty) \leq ap(fx, fy) + bp(fx, Tx) + cp(fy, Ty) + dp(fx, Ty) + ep(fy, Tx),$$

where  $a, b, c, d, e \geq 0$  and  $a + b + c + d + e < 1$ .

Then, we get

$$p(Ty, Tx) \leq ap(fy, fx) + bp(fy, Ty) + cp(fx, Tx) + dp(fy, Tx) + ep(fx, Ty).$$

It follows that

$$p(Tx, Ty) \leq ap(fx, fy) + \frac{b+c}{2}[p(fx, Tx) + p(fy, Ty)] + \frac{d+e}{2}[p(fx, Ty) + p(fy, Tx)],$$

$x, y \in X$ .

We deduce that Theorem 3.4 holds also if we use condition (3.5) instead of (3.4) and so we obtain a fixed point theorem of Hardy-Rogers type [6].

From Theorem 3.4, we deduce the following corollaries.

**3.6. Corollary** (Banach type). *Let  $(X, p)$  be a partial metric space and  $T, f$  be mappings on  $X$  with  $TX \subset fX$ . Assume that*

$$(3.6) \quad p(Tx, Ty) \leq kp(fx, fy)$$

for all  $x, y \in X$ , where  $0 \leq k < 1$ . If  $TX$  or  $fX$  is a 0-complete subspace of  $X$ , then  $T$  and  $f$  have a unique point of coincidence. Moreover, if  $T$  and  $f$  are weakly compatible, then  $T$  and  $f$  have a unique fixed point.

**3.7. Corollary** (Kannan type [7]). *Let  $(X, p)$  be a partial metric space and  $T, f$  be mappings on  $X$  with  $TX \subset fX$ . Assume that*

$$(3.7) \quad p(Tx, Ty) \leq k \max\{p(fx, Tx), p(fy, Ty)\}$$

*for all  $x, y \in X$ , where  $0 \leq k < 1$ . If  $TX$  or  $fX$  is a 0-complete subspace of  $X$ , then  $T$  and  $f$  have a unique point of coincidence. Moreover, if  $T$  and  $f$  are weakly compatible, then  $T$  and  $f$  have a unique fixed point.*

Using Remark 1, we state the following corollary.

**3.8. Corollary** (Reich type [11]). *Let  $(X, p)$  be a partial metric space and  $T, f$  be mappings on  $X$  with  $TX \subset fX$ . Assume that*

$$(3.8) \quad p(Tx, Ty) \leq ap(fx, fy) + bp(fx, Tx) + cp(fy, Ty)$$

*for all  $x, y \in X$ , where  $a, b, c \geq 0$  and  $a + b + c < 1$ . If  $TX$  or  $fX$  is a 0-complete subspace of  $X$ , then  $T$  and  $f$  have a unique point of coincidence. Moreover, if  $T$  and  $f$  are weakly compatible, then  $T$  and  $f$  have a unique fixed point.*

**3.9. Example.** Let  $X = [0, +\infty[ \cap \mathbb{Q}$  and  $p(x, y) = \max\{x, y\}$ , then  $(X, p)$  is a 0-complete partial metric space. Let  $T, f : X \rightarrow X$ , with  $Tx = x^2/(1+x)$  and  $fx = x$  for all  $x \in X$ . Let  $\psi : [0, +\infty[ \rightarrow [0, +\infty[$  with  $\psi(t) = t^2/(1+t)$ . Then,  $\psi$  is continuous and nondecreasing and  $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$  for all  $t > 0$ . From

$$p(Tx, Ty) = \max\left\{\frac{x^2}{1+x}, \frac{y^2}{1+y}\right\} = \psi(p(x, y)),$$

we deduce that  $T$  has a unique fixed point. We note that  $(X, p^s)$  is not complete and hence it is not possible to deduce the existence of a fixed point of  $T$  using the result of Boyd and Wong [1].

## 4. A homotopy result

In this section, we give a homotopy result using a condition of Kannan type.

**4.1. Theorem.** *Let  $(X, p)$  be a 0-complete partial metric space,  $U$  an open subset of  $X$ . Suppose that  $H : \bar{U} \times [0, 1] \rightarrow X$  be such that*

- (i)  $x \neq H(x, \lambda)$  for all  $x \in \partial U$  and  $\lambda \in [0, 1]$ , where  $\partial U$  is the boundary of  $U$  in  $X$ ;
- (ii) There exist  $k \in [0, 1/2[$  and a continuous function  $\xi : [0, 1] \rightarrow \mathbb{R}$  such that

$$p(H(x, \lambda), H(y, \mu)) \leq k[p(x, H(x, \lambda)) + p(y, H(y, \mu))] + |\xi(\lambda) - \xi(\mu)|$$

$$\text{for all } x, y \in \bar{U} \text{ and } \lambda, \mu \in [0, 1].$$

*If  $H(\cdot, 0)$  has a fixed point in  $U$ , then  $H(\cdot, \lambda)$  has a fixed point in  $U$  for every  $\lambda \in [0, 1]$ .*

*Proof.* Consider the set

$$\Lambda = \{\lambda \in [0, 1] : x = H(x, \lambda) \text{ for some } x \in U\}.$$

We note that  $0 \in \Lambda$ , since  $H(\cdot, 0)$  has a fixed point in  $U$ . We will show that  $\Lambda$  is both closed and open in  $[0, 1]$  and hence by connectedness we have that  $\Lambda = [0, 1]$ .

We show that  $\Lambda$  is closed in  $[0, 1]$ . Let  $\{\lambda_n\} \subset \Lambda$  with  $\lambda_n \rightarrow \lambda \in [0, 1]$  as  $n \rightarrow +\infty$ . Now, for each  $n \in \mathbb{N}$  there is  $x_n \in U$  such that  $x_n = H(x_n, \lambda_n)$ . Then

$$\begin{aligned} p(x_n, x_m) &= p(H(x_n, \lambda_n), H(x_m, \lambda_m)) \\ &\leq kp(x_n, H(x_n, \lambda_n)) + kp(x_m, H(x_m, \lambda_m)) + |\xi(\lambda_n) - \xi(\lambda_m)| \\ &= kp(x_n, x_n) + kp(x_m, x_m) + |\xi(\lambda_n) - \xi(\lambda_m)| \\ &\leq 2kp(x_n, x_m) + |\xi(\lambda_n) - \xi(\lambda_m)|. \end{aligned}$$

It implies that

$$p(x_n, x_m) \leq \frac{1}{1-2k} |\xi(\lambda_n) - \xi(\lambda_m)|,$$

and so  $\{x_n\}$  is a 0-Cauchy sequence, since  $\lim_{n,m \rightarrow +\infty} p(x_n, x_m) = 0$ . As  $X$  is 0-complete there exists  $x \in \bar{U}$  with

$$p(x, x) = \lim_{n \rightarrow +\infty} p(x, x_n) = \lim_{n,m \rightarrow +\infty} p(x_n, x_m) = 0.$$

We have

$$\begin{aligned} p(x_n, H(x, \lambda)) &= p(H(x_n, \lambda_n), H(x, \lambda)) \\ &\leq kp(x_n, H(x_n, \lambda_n)) + kp(x, H(x, \lambda)) + |\xi(\lambda_n) - \xi(\lambda)| \\ &= kp(x_n, x_n) + kp(x, H(x, \lambda)) + |\xi(\lambda_n) - \xi(\lambda)| \\ &\leq kp(x, x_n) + kp(x_n, H(x, \lambda)) + |\xi(\lambda_n) - \xi(\lambda)|. \end{aligned}$$

Consequently

$$p(x_n, H(x, \lambda)) \leq \frac{k}{1-k} p(x, x_n) + \frac{1}{1-k} |\xi(\lambda_n) - \xi(\lambda)|$$

and letting  $n \rightarrow +\infty$ , we obtain  $p(x, H(x, \lambda)) = 0$ . This implies  $x = H(x, \lambda)$  and so  $\lambda \in \Lambda$ , that is  $\Lambda$  is closed in  $[0, 1]$ .

Now, we show that  $\Lambda$  is open in  $[0, 1]$ . Let  $\lambda_0 \in \Lambda$  and  $x_0 \in U$  with  $x_0 = H(x_0, \lambda_0)$ . Let  $s = p(x_0, \partial U) = \inf\{p(x_0, x) : x \in \partial U\} > p(x_0, x_0)$ . Fix  $\varepsilon \in ]0, (1-2k)s[$  and let  $\rho > 0$  such that for each  $\lambda \in ]\lambda_0 - \rho, \lambda_0 + \rho[$  holds  $|\xi(\lambda) - \xi(\lambda_0)| < \varepsilon$ . Now, for fixed  $\lambda \in ]\lambda_0 - \rho, \lambda_0 + \rho[$  and  $x \in Y = \{x \in X : p(x, x_0) \leq s\}$ , we have

$$\begin{aligned} p(x_0, H(x, \lambda)) &= p(H(x_0, \lambda_0), H(x, \lambda)) \\ &\leq kp(x_0, H(x_0, \lambda_0)) + kp(x, H(x, \lambda)) + |\xi(\lambda_0) - \xi(\lambda)| \\ &= kp(x_0, x_0) + kp(x, H(x, \lambda)) + |\xi(\lambda_0) - \xi(\lambda)| \\ &\leq kp(x, x_0) + kp(x_0, H(x, \lambda)) + |\xi(\lambda_0) - \xi(\lambda)|. \end{aligned}$$

Consequently

$$p(x_0, H(x, \lambda)) \leq \frac{k}{1-k} p(x, x_0) + \frac{1}{1-k} |\xi(\lambda_0) - \xi(\lambda)| \leq s.$$

Then for each fixed  $\lambda \in ]\lambda_0 - \rho, \lambda_0 + \rho[$  the mapping  $H(\cdot, \lambda) : Y \rightarrow Y$  has a fixed point in  $\bar{U}$ . But this fixed point must be in  $U$  by condition (i). This implies that  $\Lambda$  is open in  $[0, 1]$ . Then  $\Lambda = [0, 1]$  and  $H(\cdot, \lambda)$  has a fixed point in  $U$  for every  $\lambda \in [0, 1]$ .  $\square$

## References

- [1] Boyd, D. W. and Wong, J. S. W. *On nonlinear contractions*, Proc. Amer. Math. Soc., **20**, 458–464, 1969.
- [2] Banach, S. *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., **3**, 133–181, 1922.
- [3] Bukatin, M. A. and Scott, J. S. *Towards computing distances between programs via Scott domains*, in: Logical Foundations of Computer Science, Lecture Notes in Computer Science, vol. **1234**, Springer, 33–43, 1997.
- [4] Bukatin, M. A. and Shorina, S. Y. *Partial metrics and co-continuous valuations*, in: Foundations of Software Science and Computation Structures, Lecture Notes in Computer Science, vol. **1378**, Springer, 33–43, 1998.

- [5] Ćirić, Lj., Samet, B., Aydi, H. and Vetro, C. *Common fixed points of generalized contractions on partial metric spaces and an application*, Appl. Math. Comput., **218**, 2398–2406, 2011.
- [6] Hardy, G. E. and Rogers, T. D. *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull., **16**, 201–206, 1973.
- [7] Kannan, R. *Some results on fixed points*, Bull. Calcutta Math. Soc., **60**, 71–76, 1968.
- [8] Matthews, S. G. *Partial metric topology*, in: Proc. 8th Summer Conference on General Topology and Applications. Ann. New York Acad. Sci., **728**, 183–197, 1994.
- [9] O’Neill, S. J. *Partial metrics, valuations and domain theory*, in: Proc. 11th Summer Conference on General Topology and Applications. Ann. New York Acad. Sci., **806**, 304–315, 1996.
- [10] Oltra, S. and Valero, O. *Banach’s fixed point theorem for partial metric spaces*, Rend. Istit. Mat. Univ. Trieste, **36**, 17–26, 2004.
- [11] Reich, S. *Some remarks concerning contraction mappings*, Canad. Math. Bull., **14**, 121–124, 1971.
- [12] Romaguera, S. *A Kirk type characterization of completeness for partial metric spaces*, Fixed Point Theory Appl., Volume **2010**, Article ID 493298, 6 pages.
- [13] Romaguera, S. and Schellekens, M. *Partial metric monoids and semivaluation spaces*, Topology Appl., **153**, 948–962, 2005.
- [14] Valero, O. *On Banach fixed point theorems for partial metric spaces*, Appl. Gen. Topol., **6**, 229–240, 2005.