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CHARACTERIZATION PROPERTIES FOR STARLIKENESS AND CONVEXITY OF SOME SUBCLASSES OF P-VALENT FUNCTIONS INVOLVING A CLASS OF INTEGRAL OPERATORS

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Abstract

This paper studies the sufficient conditions for the starlikeness and convexity of a class of fractional integral operators of certain analytic and p-valent functions in the open unit disk. Further characterization theorems associated with the Hadamard product (or convolution) are also considered.

Keywords: p-valent function, starlike function, convex function, fractional integral operators, Hadamard product.

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1. Introduction and Definitions

Let $\mathcal{A}(p)$ denote the class of functions defined by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in \mathbf{N}) \tag{1.1}$$

which are analytic and p-valent in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Then a function $f(z) \in \mathcal{A}(p)$ is called p-valent starlike of order α, if $f(z)$ satisfies the conditions

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \tag{1.2}$$

and

$$\int_0^{2\pi} \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} d\theta = 2p\pi \tag{1.3}$$

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for $0 \leq \alpha < p$, $p \in \mathbf{N}$ and $z \in \mathcal{U}$. We denote by $S^*(p,\alpha)$, the class of all p-valent starlike functions of order α. Also, a function $f(z) \in \mathcal{A}(p)$ is called p-valent convex of order α, if $f(z)$ satisfies the conditions

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \tag{1.4}$$

and

$$\int_0^{2\pi} \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} d\theta = 2p\pi \tag{1.5}$$

for $0 \leq \alpha < p$, $p \in \mathbf{N}$ and $z \in \mathcal{U}$. We denote by $K(p,\alpha)$, the class of all p-valent convex functions of order α. We note that

$$f(z) \in K(p,\alpha) \Leftrightarrow \frac{zf'(z)}{p} \in S^*(p,\alpha) \tag{1.6}$$

for $0 \leq \alpha < p$.

The classes $S^*(p,\alpha)$ and $K(p,\alpha)$ were introduced by Kapoor and Mishra [2] and studied by Patil and Thakare [5] and Owa [3]. For $\alpha = 0$, we get $S^*(p,0) = S^*(p)$ and $K(p,0) = K(p)$ are the classes of p-valent starlike functions and p-valent convex functions respectively which were introduced by Goodman [1]. If $p = 1$, we have $S^*(1,\alpha) = S^*(\alpha)$ and $K(1,\alpha) = K(\alpha)$ are the classes of starlike functions of order α and convex functions of order α respectively which were first introduced by Robertson [7] and studied by Silverman [9].

Let ${}_2F_1(a,b;c;z)$ be the Gauss hypergeometric function defined for $z \in \mathcal{U}$ by, see [10]

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \tag{1.7}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & \text{when } n = 0, \\ \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1) & \text{when } n \in \mathbf{N}. \end{cases} \tag{1.8}$$

for $\lambda \neq 0, -1, -2, \ldots$

We recall the following definitions of fractional integral operators as follows (see, [4, 11])

1.1. Definition. The fractional integral of order λ is defined by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\lambda}} d\xi \tag{1.9}$$

where $\lambda > 0$, $f(z)$ is analytic function in a simply- connected region of the z-plane containing the origin, and the multiplicity of $(z-\xi)^{\lambda-1}$ is removed by requiring $\log(z-\xi)$ to be real when $z - \xi > 0$.

1.2. Definition. For real $\lambda > 0, \mu$, and η, the fractional integral operator $I_{0,z}^{\lambda,\mu,\eta}$ is defined by

$$I_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{z^{-\lambda-\mu}}{\Gamma(\lambda)} \int_0^z (z-\xi)^{\lambda-1} f(\xi)\, {}_2F_1\left(\lambda+\mu, -\eta; \lambda; 1 - \frac{\xi}{z}\right) d\xi \tag{1.10}$$

where $f(z)$ is analytic function in a simply- connected region of the z-plane containing the origin, with the order $f(z) = O(|z|^{\varepsilon}),\ z \to 0$, where $\varepsilon > \max\{0, \mu - \eta\} - 1$ and the multiplicity of $(z-\xi)^{\lambda-1}$ is removed by requiring $\log(z-\xi)$ to be real when $z - \xi > 0$.

Notice that

$$I_{0,z}^{\lambda,-\lambda,\eta}f(z) = D_z^{-\lambda}f(z), \quad \lambda > 0 \tag{1.11}$$

With the aid of the above definitions, let us consider $N_{0,z}^{\lambda,\mu,\eta}f(z)$ the modification of the fractional integral operator of analytic and p-valent function which is defined in terms of $I_{0,z}^{\lambda,\mu,\eta}f(z)$ as follows:

$$N_{0,z}^{\lambda,\mu,\eta}f(z) = \frac{\Gamma(1-\mu+p)\Gamma(1+\lambda+\eta+p)}{\Gamma(1+p)\Gamma(1-\mu+\eta+p)}z^{\mu}I_{0,z}^{\lambda,\mu,\eta}f(z) \tag{1.12}$$

for $\lambda > 0, \mu < p+1, \eta > \max(-\lambda,\mu) - p - 1$ and $p \in \mathbf{N}$.

A general class of fractional integral operators involving the Gauss hypergeometric function was studied by Srivastava et al. [11]. Subsequently, this class was used to obtain some characterization theorems for starlikeness and convexity of certain analytic functions by Owa et al. [4].

This paper is devoted to the investigation of the sufficient conditions that are satisfied by a class of fractional integral operators of certain analytic and p-valent functions in the open unit disk to be starlike or convex. Further characterization properties associated with the Hadamard product (or convolution) are also considered.

2. Characterization Theorems

In order to prove our results we mention to the following known result which shall be used in the following (see [4, 11]).

2.1. Lemma. *Let $\lambda > 0, \mu$, and η be real, and let $k > \mu - \eta - 1$. Then*

$$I_{0,z}^{\lambda,\mu,\eta}z^k = \frac{\Gamma(k+1)\Gamma(k-\mu+\eta+1)}{\Gamma(k-\mu+1)\Gamma(k+\lambda+\eta+1)}z^{k-\mu} \tag{2.1}$$

For the classes $S^*(p,\alpha)$ and $K(p,\alpha)$, we shall need the following lemmas due to Owa [3]:

2.2. Lemma. *Let the function $f(z)$ defined by* (1.1). *If $f(z)$ satisfies*

$$\sum_{n=1}^{\infty}(p+n-\alpha)|a_{p+n}| \le p-\alpha \tag{2.2}$$

then $f(z)$ is in the class $S^*(p,\alpha)$.

2.3. Lemma. *Let the function $f(z)$ defined by* (1.1). *If $f(z)$ satisfies*

$$\sum_{n=1}^{\infty}(p+n)(p+n-\alpha)|a_{p+n}| \le p(p-\alpha) \tag{2.3}$$

then $f(z)$ is in the class $K(p,\alpha)$.

Now we prove

2.4. Lemma. *Let $\lambda, \mu, \eta \in \mathbf{R}$ such that*

$$\lambda > 0,\ \mu < p+1,\ \max(-\lambda,\mu) - p - 1 < \eta \le \lambda\left(\frac{p+2}{\mu} - 1\right),\ p \in \mathbf{N} \tag{2.4}$$

Also, let the function $f(z)$ defined by (1.1) *satisfies*

$$\sum_{n=1}^{\infty}\frac{(p+n-\alpha)}{(p-\alpha)}|a_{p+n}| \le \frac{(1-\mu+p)(1+\lambda+\eta+p)}{(1+p)(1-\mu+\eta+p)} \tag{2.5}$$

for $0 \le \alpha < p$. Then $N_{0,z}^{\lambda,\mu,\eta}f(z) \in S^*(p,\alpha)$

Proof. Applying Lemma 2.1, we have from (1.1) and (1.12) that

$$N_{0,z}^{\lambda,\mu,\eta} f(z) = z^p + \sum_{n=1}^{\infty} \psi(n) a_{p+n} z^{p+n} \tag{2.6}$$

where

$$\psi(n) = \frac{(1+p)_n (1-\mu+\eta+p)_n}{(1-\mu+p)_n (1+\lambda+\eta+p)_n} \tag{2.7}$$

We observe that the functions $\psi(n)$ satisfy the inequality $\psi(n+1) \leq \psi(n)$, $\forall n \in \mathbf{N}$, provided that $\eta \leq \lambda \left(\frac{p+2}{\mu} - 1\right)$. Thereby, we deduced that $\psi(n)$ is non-increasing. Thus under conditions stated in (2.4), we have

$$0 < \psi(n) \leq \psi(1) = \frac{(1+p)(1-\mu+\eta+p)}{(1-\mu+p)(1+\lambda+\eta+p)} \tag{2.8}$$

Therefore, (2.5) and (2.8) yield

$$\sum_{n=1}^{\infty} \frac{(p+n-\alpha)}{(p-\alpha)} \psi(n) |a_{p+n}| \leq \psi(1) \sum_{n=1}^{\infty} \frac{(p+n-\alpha)}{(p-\alpha)} |a_{p+n}| \leq 1 \tag{2.9}$$

Hence, by Lemma 2.2, we have

$$N_{0,z}^{\lambda,\mu,\eta} f(z) \in S^*(p,\alpha)$$

and the proof is complete. □

2.5. Remark. The equality in (2.5) is attained for the function $f(z)$ defined by

$$f(z) = z^p + \frac{(p-\alpha)(1-\mu+p)(1+\lambda+\eta+p)}{(p+1-\alpha)(1+p)(1-\mu+\eta+p)} z^{p+1} \tag{2.10}$$

Similarly, we can prove with the help of Lemma 2.3, the following result which characterizes the class $K(p,\alpha)$.

2.6. Lemma. *Under the conditions stated in* (2.4)*, let the function* $f(z)$ *defined by* (1.1) *satisfies*

$$\sum_{n=1}^{\infty} \frac{(p+n)(p+n-\alpha)}{p(p-\alpha)} |a_{p+n}| \leq \frac{(1-\mu+p)(1+\lambda+\eta+p)}{(1+p)(1-\mu+\eta+p)} \tag{2.11}$$

for $0 \leq \alpha < p$*. Then* $N_{0,z}^{\lambda,\mu,\eta} f(z) \in K(p,\alpha)$

2.7. Remark. The equality in (2.11) is attained for the function $f(z)$ defined by

$$f(z) = z^p + \frac{p(p-\alpha)(1-\mu+p)(1+\lambda+\eta+p)}{(1+p)^2(p+1-\alpha)(1-\mu+\eta+p)} z^{p+1} \tag{2.12}$$

3. Characterization Theorems Involving The Hadamard Product

Let $f_i(z) \in \mathcal{A}(p)$ $(i = 1, 2)$ be given by

$$f_i(z) = z^p + \sum_{n=1}^{\infty} a_{p+n,i} z^{p+n} \quad (p \in \mathbf{N}) \tag{3.1}$$

Then, the Hadamard product (or convolution) $(f_1 * f_2)(z)$ of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p + \sum_{n=1}^{\infty} a_{p+n,1} a_{p+n,2} z^{p+n} \quad (p \in \mathbf{N}) \tag{3.2}$$

Now to prove our next characterization theorem, we state here the following result due to Ruscheweyh and Sheil-Small [8], see also [4, 6]

3.1. Theorem. *Let $\varphi(z)$ and $g(z)$ be analytic in $|z| < 1$ and satisfy $\varphi(0) = g(0) = 0, \varphi'(0) \neq 0$, and $g'(0) \neq 0$. Also, suppose that*

$$\varphi(z) * \left\{ \frac{1+abz}{1-bz} g(z) \right\} \neq 0, \quad 0 < |z| < 1 \tag{3.3}$$

for a and b on the unit circle. Then for a function $F(z)$ analytic in $|z| < 1$ such that $\mathrm{Re}\{F(z)\} > 0$ satisfies the inequality

$$\mathrm{Re}\left\{ \frac{(\varphi * Fg)(z)}{(\varphi * g)(z)} \right\} > 0, \quad |z| < 1. \tag{3.4}$$

Applying Theorem 3.1, we shall prove

3.2. Theorem. *Let the conditions stated in* (2.4) *hold, and let the function $f(z)$ defned by* (1.1) *be in the class $S^*(p, \alpha)$, and satisfies:*

$$h(z) * \left\{ \frac{1+abz}{1-bz} f(z) \right\} \neq 0, \quad z \in \mathcal{U} - \{0\} \tag{3.5}$$

for a and b on the unit circle, where

$$h(z) = z^p + \sum_{n=1}^{\infty} \frac{(1+p)_n (1-\mu+\eta+p)_n}{(1-\mu+p)_n (1+\lambda+\eta+p)_n} z^{p+n}, \quad (p \in \mathbf{N}) \tag{3.6}$$

Then $N_{0,z}^{\lambda,\mu,\eta} f(z)$ is in the class $S^*(p, \alpha)$.

Proof. Using (2.6) and (3.6), we have

$$\begin{aligned} N_{0,z}^{\lambda,\mu,\eta} f(z) &= z^p + \sum_{n=1}^{\infty} \frac{(1+p)_n (1-\mu+\eta+p)_n}{(1-\mu+p)_n (1+\lambda+\eta+p)_n} a_{p+n} z^{p+n} \\ &= (h * f)(z) \end{aligned} \tag{3.7}$$

By setting $\varphi(z) = h(z), g(z) = f(z)$ and $F(z) = \frac{zf'(z)}{f(z)} - \alpha$, in Lemma 3.1, we find with the help of (3.7) that

$$\begin{aligned} &\mathrm{Re}\left\{ \frac{(\varphi * Fg)(z)}{(\varphi * g)(z)} \right\} > 0 \\ &\qquad \Rightarrow \quad \mathrm{Re}\left\{ \frac{(h * zf')(z)}{(h * f)(z)} \right\} - \alpha > 0 \\ &\qquad \Rightarrow \quad \mathrm{Re}\left\{ \frac{z(h * f)'(z)}{(h * f)(z)} \right\} - \alpha > 0 \\ &\qquad \Rightarrow \quad \mathrm{Re}\left\{ \frac{z(N_{0,z}^{\lambda,\mu,\eta} f(z))'}{N_{0,z}^{\lambda,\mu,\eta} f(z)} \right\} - \alpha > 0 \\ &\qquad \Rightarrow \quad N_{0,z}^{\lambda,\mu,\eta} f(z) \in S^*(p, \alpha) \end{aligned}$$

and the proof is complete. □

3.3. Theorem. *Let the conditions stated in* (2.4) *hold, and let the function $f(z)$ defined by* (1.1) *be in the class $K(p, \alpha)$, and satisfies:*

$$h(z) * \left\{ \frac{1+abz}{1-bz} zf'(z) \right\} \neq 0, \quad z \in \mathcal{U} - \{0\} \tag{3.8}$$

for a and b on the unit circle, where $h(z)$ is given by (3.6). *Then $N_{0,z}^{\lambda,\mu,\eta} f(z)$ is also in the class $K(p, \alpha)$.*

Proof. Using (1.6) and Theorem 3.2, we observe that

$$\begin{aligned} f(z) \in K(p,\alpha) \quad &\Leftrightarrow \quad \frac{zf'(z)}{p} \in S^*(p,\alpha) \\ &\Rightarrow \quad N_{0,z}^{\lambda,\mu,\eta}\left(\frac{zf'(z)}{p}\right) \in S^*(p,\alpha) \\ &\Leftrightarrow \quad \left(h * \frac{zf'}{p}\right)(z) \in S^*(p,\alpha) \\ &\Leftrightarrow \quad \frac{z(h*f)'(z)}{p} \in S^*(p,\alpha) \\ &\Leftrightarrow \quad (h*f)(z) \in K(p,\alpha) \\ &\Leftrightarrow \quad N_{0,z}^{\lambda,\mu,\eta} f(z) \in K(p,\alpha) \end{aligned}$$

which completes the proof of Theorem 3.3. □

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