Hacettepe Journal of Mathematics and Statistics
Volume 42 (6) (2013), 605 – 615

ON THE PARALLEL SURFACES IN GALILEAN SPACE

Mustafa Dede [a] [*], Cumali Ekici [b] [†] and A. Ceylan Çöken [c] [‡]

Received 22 : 02 : 2012 : Accepted 11 : 08 : 2012

Abstract

In this paper, first of all, the definition of parallel surfaces in Galilean space is given. Then, the relationship between the curvatures of the parallel surfaces in Galilean space is determined. Moreover, the first and second fundamental forms of parallel surfaces are found in Galilean space. Consequently, we obtained Gauss curvature and mean curvature of parallel surface in terms of those curvatures of the base surface.

Keywords: Parallel surfaces, Surface curvature, Galilean space

2000 AMS Classification: 53A35, 53Z05

1. Introduction

It is known that two surfaces with a common normal are called parallel surfaces. A large number of papers and books have been published in the literature which deal with parallel surfaces in both Minkowski space and Euclidean space such as [1, 4, 6, 7, 12, 13, 15]. However, this paper presents the differential properties of the parallel surfaces in three-dimensional Galilean space.

There are nine related plane geometries including Euclidean geometry, hyperbolic geometry and elliptic geometry. Galilean geometry is one of these geometries whose motions are the Galilean transformations of classical kinematics [16]. Differential geometry of the Galilean space $\mathbb{G}_3$ and especially the geometry of ruled surfaces in this space have been largely developed in O. Röschel's paper [14]. Some more results about ruled surfaces in $\mathbb{G}_3$ have been given in [8, 9, 10]. A. Öğrenmiş et al. obtained the characterizations of helix for a curve with respect to the Frenet frame in Galilean space [11]. In [3], curves

[a]Kilis 7 Aralık University, Department of Mathematics, 79000, Kilis/TURKEY.
[*]Corresponding author E-mail: `mustafadede@kilis.edu.tr`
[b]Eskişehir Osmangazi University, Department of Mathematics and Computer Science, 26480, Eski şehir/TURKEY.
[†]E-mail: `cekici@ogu.edu.tr`
[c]Süleyman Demirel University, Department of Mathematics, 32260, Isparta/TURKEY
[‡]E-mail: `ceylancoken@sdu.edu.tr`

explained in pseudo-Galilean space. C. Ekici and M. Dede [5] investigated Darboux vectors of ruled surfaces in pseudo-Galilean space. Recently, tubular surfaces in Galilean space introduced in [2].

The Galilean space $\mathbb{G}_3$ is a Cayley–Klein space equipped with the projective metric of signature $(0, 0, +, +)$. The absolute figure of the Galilean geometry consists of an ordered triple $\{\omega, f, I\}$, where ω is the real (absolute) plane, f is the real line (absolute line) in ω. I is the fixed elliptic involution of points of f.

1.1. Definition. A plane is called Euclidean if it contains f, otherwise it is called isotropic. Planes $x = constant$ are Euclidean and so is the plane ω. Other planes are isotropic. A vector $\mathbf{u} = (u_1, u_2, u_3)$ is said to be non-isotropic if $u_1 \neq 0$. All unit non-isotropic vectors are of the form $\mathbf{u} = (1, u_2, u_3)$. For isotropic vectors, $u_1 = 0$ holds [9]. Since $x = 0$ plane is Euclidean in Galilean space, it is easy to see that isotropic vectors are on the Euclidean planes.

1.2. Definition. Let $\mathbf{a} = (x, y, z)$ and $\mathbf{b} = (x_1, y_1, z_1)$ be vectors in Galilean space. The scalar product is defined by

$$< \mathbf{a}, \mathbf{b} >= x_1 x$$

The norm of $\mathbf{a}$ is defined by $\|\mathbf{a}\| = |x|$, and $\mathbf{a}$ is called a unit vector if $\|\mathbf{a}\| = 1$.

On the other hand, as a consequence of Definition 1.1, we define the scalar product of two isotropic vectors, $\mathbf{p} = (0, y, z)$ and $\mathbf{q} = (0, y_1, z_1)$, as

$$< \mathbf{p}, \mathbf{q} >_1= yy_1 + zz_1$$

The orthogonality of isotropic vectors, $\mathbf{p} \perp_1 \mathbf{q}$, means that $< \mathbf{p}, \mathbf{q} >_1= 0$. The norm of $\mathbf{p}$ is defined by $\|\mathbf{p}\|_1 = \sqrt{y^2 + z^2}$, and $\mathbf{p}$ is called a unit isotropic vector if $\|\mathbf{p}\|_1 = 1$ [16].

1.3. Definition. Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be vectors in Galilean space [9]. The cross product of the vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as follows:

$$\mathbf{u} \wedge \mathbf{v} = \begin{vmatrix} 0 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (0, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1) \tag{1.1}$$

1.4. Definition. Let ε be a plane and $f(\varepsilon)$ the intersection of the absolute line f and ε. In Figure 1, the point $f(\varepsilon)$ is called the absolute point of ε. Then $f(\varepsilon)^{\perp} = I(f(\varepsilon))$ denotes the point on f orthogonal to $f(\varepsilon)$ according to the elliptic involution I. This is an elliptic involution because there is no line perpendicular to itself [14]. The elliptic involution in homogeneous coordinates is given by

$$(0 : 0 : x_2 : x_3) \to (0 : 0 : x_3 : -x_2) \tag{1.2}$$

1.5. Definition. If an admissible curve C of the class $C^r (r \geq 3)$ is given by the parametrization

$$r(x) = (x, y(x), z(x))$$

then x is a Galilean invariant of the arc length on C [8].

In Figure 2, the associated invariant moving trihedron is given by

$$\begin{aligned} \mathbf{t} &= (1, y'(x), z'(x)), \\ \mathbf{n} &= \frac{1}{\kappa}(0, y''(x), z''(x)), \\ \mathbf{b} &= \frac{1}{\kappa}(0, -z''(x), y''(x)) \end{aligned}$$

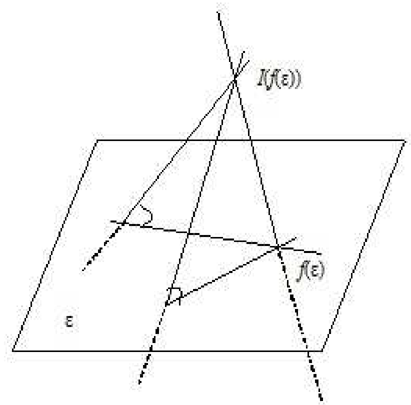

FIGURE 1

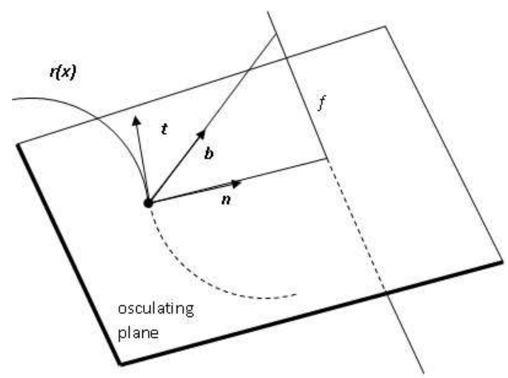

FIGURE 2

where $\kappa = \sqrt{y''(x)^2 + z''(x)^2}$ is the curvature and $\tau = \dfrac{1}{\kappa^2}\det[r'(x), r''(x), r'''(x)]$ is the torsion.

Frenet formulas may be written as

$$\frac{d}{dx}\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}$$

2. Surface Theory in Galilean Space

Assume that M is a surface in $\mathbb{G}_3$. The equation of M is given by the parametrization

$$\varphi(v^1, v^2) = (x(v^1, v^2), y(v^1, v^2), z(v^1, v^2)), \qquad v^1, v^2 \in \mathbb{R}$$

where $x(v^1, v^2), y(v^1, v^2), z(v^1, v^2) \in C^3$.

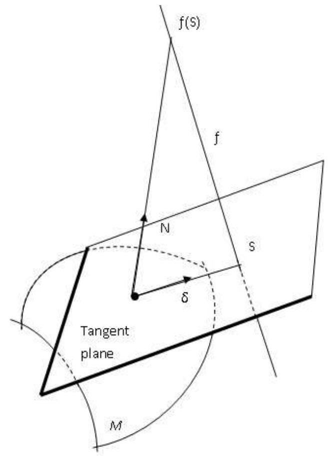

FIGURE 3

The isotropic unit normal vector field $\mathbf{N}$, shown in Figure 3, is given by

$$\mathbf{N} = \frac{\varphi_{,1} \wedge \varphi_{,2}}{\|\varphi_{,1} \wedge \varphi_{,2}\|_1} = \frac{(0, z_{,2}x_{,1} - z_{,1}x_{,2}, y_{,1}x_{,2} - y_{,2}x_{,1})}{\sqrt{(z_{,1}x_{,2} - z_{,2}x_{,1})^2 + (y_{,2}x_{,1} - y_{,1}x_{,2})^2}} \tag{2.1}$$

where partial differentiation with respect to v^1 and v^2 will be denoted by suffixes 1 and 2 respectively, that $\varphi_{,1} = \dfrac{\partial \varphi(v^1, v^2)}{\partial v^1}$ and $\varphi_{,2} = \dfrac{\partial \varphi(v^1, v^2)}{\partial v^2}$ [14].

Using (1.2), we obtain the isotropic unit vector δ in the tangent plane of surface as

$$\delta = \frac{(0, y_{,1}x_{,2} - y_{,2}x_{,1}, z_{,1}x_{,2} - z_{,2}x_{,1})}{w}$$

where

$$\langle \mathbf{N}, \delta \rangle_1 = 0, \quad \delta^2 = 1, \quad w = \|\varphi_{,1} \wedge \varphi_{,2}\|_1 \tag{2.2}$$

by means of Galilean geometry. Observe that a straightforward computation shows that δ can be expressed by

$$\delta = \frac{x_{,2}\varphi_{,1} - x_{,1}\varphi_{,2}}{w} \tag{2.3}$$

where $x_{,1}$ and $x_{,2}$ are the partial differentiation of the first component of the surface M with respect to v^1 and v^2, respectively

$$x_{,1} = \frac{\partial x(v^1, v^2)}{\partial v^1}, \quad x_{,2} = \frac{\partial x(v^1, v^2)}{\partial v^2} \tag{2.4}$$

Consequently to simplify the presentation (2.3), we may use Einstein summation convention, then δ may be rewritten as follows

$$\delta = g^i \varphi_{,i} = g^1 \varphi_{,1} + g^2 \varphi_{,2}$$

where

$$g_1 = x_{,1} \quad g_2 = x_{,2} \quad g_{ij} = g_i g_j \tag{2.5}$$

$$g^1 = \frac{x_{,2}}{w} \quad g^2 = -\frac{x_{,1}}{w} \quad g^{ij} = g^i g^j \tag{2.6}$$

From Definition 1.2, the first fundamental form I of the surface is given by

(2.7) $$I = (g_{ij} + \epsilon h_{ij}) dv^i dv^j$$

where h_{ij} and g_{ij} $(i, j = 1, 2)$ are called induced metric on the surface given by

(2.8) $$h_{ij} = \langle \varphi_{,i}, \varphi_{,j} \rangle_1 , \qquad g_{ij} = \langle \varphi_{,i}, \varphi_{,j} \rangle$$

and

$$\epsilon = \begin{cases} 0, & dv^1 : dv^2 \quad \text{non-isotropic} \\ 1, & dv^1 : dv^2 \quad \text{isotropic} \end{cases}$$

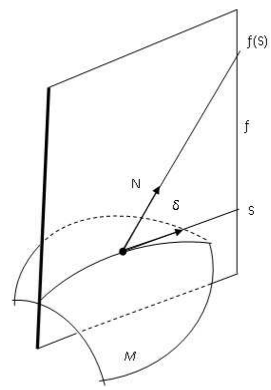

FIGURE 4

In Figure 4, isotropic curves are the intersections of the surface M with Euclidean planes [8]. All other curves on the surface are called non-isotropic curves.

Let $\alpha(s) = \varphi(v(s)^1, v(s)^2)$ be a non-isotropic curve in a surface patch φ, parametrized by the arc length s. From (2.5) it follows that

$$g_i v^{i\prime} = 1$$

where " $'$ " refers to $\dfrac{d}{ds}$.

The coefficients L_{ij} of second fundamental form are given by

(2.9) $$L_{ij} = \left\langle \frac{\varphi_{,ij} x_{,1} - x_{,ij} \varphi_{,1}}{x_{,1}}, \mathbf{N} \right\rangle_1$$

The Christoffel symbols of the surface are given by

$$\Gamma^1_{ij} = \left\langle \frac{\varphi_{,ij} x_{,2} - x_{,ij} \varphi_{,2}}{w}, \delta \right\rangle_1 , \qquad \Gamma^2_{ij} = \left\langle \frac{\varphi_{,ij} x_{,1} - x_{,ij} \varphi_{,1}}{w}, \delta \right\rangle_1$$

2.1. Theorem. *Let M be a surface in Galilean space.*

(2.10) $$\varphi_{,ij} = \Gamma^k_{ij} \varphi_{,k} + L_{ij} \mathbf{N}$$

is called the Gauss equation of the surface [14].

2.2. Theorem. *Let M be a surface in Galilean space. The Weingarten equation is given by*

(2.11) $$\mathbf{N}_{,i} = B_i \delta + C_i \mathbf{N}$$

where $C_i = 0$, $B_i = -g^k L_{ki}$ [14]. *Moreover, from (2.2) and (2.11), we have*

$$\delta_{,i} = g^k L_{ki}\mathbf{N} \tag{2.12}$$

2.3. Theorem. *Let* $\alpha(s) = \varphi(v(s)^1, v(s)^2)$ *be non-isotropic curve on the surface, parametrized by the arc length* s. *The equation of normal curvature* k_n *and geodesic curvature* k_g *of the surface are given by, respectively*

$$k_n = L_{ij} v^{i\prime} v^{j\prime}, \quad k_g = \frac{\Gamma^k_{ij} v^{i\prime} v^{j\prime} + v^{k\prime\prime}}{g^k} \tag{2.13}$$

where " ′ " refers to $\dfrac{d}{ds}$.

In addition, let ϕ *be the Euclidean angle between the isotropic vectors, the surface normal* $\mathbf{N}$ *and the curve normal* $\mathbf{n}$, *we have*

$$\cos\phi = \frac{\langle \mathbf{N}, \varphi'' \rangle_1}{\|\varphi''\|_1}\ , \quad \sin\phi = \frac{\langle \delta, \varphi'' \rangle_1}{\|\varphi''\|_1}$$

Consequently, k_n *and* k_g *are obtained by, respectively*

$$k_n = \kappa\cos\phi\ , \quad k_g = \kappa\sin\phi$$

where $\kappa = \|\varphi''\|_1$ *is the curvature of* $\alpha(s)$ [14].

2.4. Corollary. *The equation of the asymptotic lines are given by*

$$L_{ij} v^{i\prime} v^{j\prime} = 0$$

2.5. Corollary. *Since* K_1 *corresponding value of the normal curvature may be found by making use of Lagrange's multipliers, we have*

$$K_1 = \frac{L_{11}L_{22} - (L_{12})^2}{w^2 g^{ij} L_{ij}} \tag{2.14}$$

This implies the following theorem.

2.6. Theorem. *Let* M *be a surface in* $\mathbb{G}_3$ [14]. *The Gauss curvature* K *and the mean curvature* H *of the surface are given by, respectively*

$$K = \frac{\det L_{ij}}{w^2}, \quad 2H = g^{ij} L_{ij} \tag{2.15}$$

The following corollary is clear from (2.14) and (2.15).

2.7. Corollary. K_1 *can be expressed by*

$$K_1 = \frac{K}{2H}$$

3. Parallel Surfaces in Galilean Space

3.1. Definition. Let M and M^λ be two surfaces in Galilean space $\mathbb{G}_3$ and $\lambda \in \mathbb{R}$, $\forall p \in M$. The function

$$\begin{array}{llll} f: & \varphi(v^1, v^2) & \longrightarrow & \varphi^\lambda(v^1, v^2) \\ & p & \longrightarrow & f(p) = [p_1, p_2 + \lambda a_2(p), p_3 + \lambda a_3(p)] \end{array}$$

is called the parallelization function between M and M^λ where $p = (p_1, p_2, p_3)$ and

$$\mathbf{N} = \sum_{i=2}^{3} a_i \frac{\partial}{\partial x_i} = (0, a_2, a_3)$$

is the isotropic unit normal vector field on M and furthermore M^λ is called parallel surface to M in $\mathbb{G}_3$ where λ is a given positive real number.

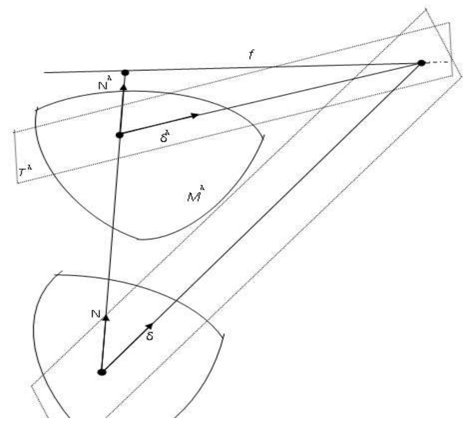

FIGURE 5

In Figure 5, T and T^λ are the isotropic tangent planes of parallel surfaces M and M^λ, respectively.

Note that from the definition of parallel surfaces, we have $\mathbf{N}(p) = \pm\mathbf{N}^\lambda(f(p))$. Moreover this leads to the fact that $\delta(p) = \pm\delta^\lambda(f(p))$ in Galilean space.

3.2. Definition. Let M and M^λ be parallel surfaces in Galilean space. We define the parallel surface M^λ to base surface M at distance λ as

$$\varphi^\lambda(v^1, v^2) = \varphi(v^1, v^2) + \lambda\mathbf{N} \tag{3.1}$$

where $\mathbf{N}$ is normal vector of the base surface.

3.3. Theorem. *Let M and M^λ be parallel surfaces in Galilean space. The relationship between the $w = \|\varphi_{,1} \wedge \varphi_{,2}\|_1$ and $w^\lambda = \|\varphi^\lambda_{,1} \wedge \varphi^\lambda_{,2}\|_1$ can be given as follows:*

$$w^\lambda = w(1 - 2\lambda H) \tag{3.2}$$

Proof. Taking the partial derivatives of M^λ gives

$$\varphi^\lambda_{,1} = \varphi_{,1} + \lambda\mathbf{N}_{,1}, \qquad \varphi^\lambda_{,2} = \varphi_{,2} + \lambda\mathbf{N}_{,2} \tag{3.3}$$

Thus, by (1.1), we see $\mathbf{N}_{,1} \wedge \mathbf{N}_{,2} = 0$. In addition, by (2.11), (2.15) and (3.3), we get

$$\varphi^\lambda_{,1} \wedge \varphi^\lambda_{,2} = (\varphi_{,1} \wedge \varphi_{,2})(1 - 2\lambda H)$$

Taking norm of the both sides, we have

$$w^\lambda = w(1 - 2\lambda H)$$

□

3.4. Theorem. *Let M and M^λ be parallel surfaces in Galilean space. The first fundamental form I^λ of the parallel surface is given by*

$$I^\lambda = \begin{cases} I & dv^1 : dv^2 \quad \textit{non-isotropic} \\ \\ I - \lambda(2L_{ij} - \lambda g^k L_{ki} g^k L_{kj}) dv^i dv^j & dv^1 : dv^2 \quad \textit{isotropic} \end{cases}$$

Proof. Let us now consider $\epsilon = 0$ in (2.7), it follows that

$$I^{\lambda} = g_{ij}^{\lambda} dv^i dv^j \tag{3.4}$$

From (2.1), (2.4) and (3.1), we obtained the partial differentiation of the first component of the surface M^{λ} as

$$x_{,1}^{\lambda} = x_{,1}, \quad x_{,2}^{\lambda} = x_{,2} \tag{3.5}$$

Substituting (3.5) into (2.5), we have

$$g_i^{\lambda} = g_i \tag{3.6}$$

By using (3.6) and (3.4), I^{λ} is obtained by

$$I^{\lambda} = g_{ij} dv^i dv^j = I$$

We now consider $\epsilon = 1$. In this case, the first fundamental form I^{λ} is

$$I^{\lambda} = h_{ij}^{\lambda} dv^i dv^j$$

Differentiating (3.1) then, using (2.8) gives

$$\left\langle \varphi_{,i}^{\lambda}, \varphi_{,j}^{\lambda} \right\rangle_1 = h_{ij} + 2\lambda \left\langle \mathbf{N}_{,i}, \varphi_{,j} \right\rangle_1 + \lambda^2 \left\langle \mathbf{N}_{,i}, \mathbf{N}_{,j} \right\rangle_1 \tag{3.7}$$

Finally, substituting (2.7), (2.10) and (2.11) into (3.7), we have

$$I^{\lambda} = I - \lambda(2L_{ij} - \lambda g^k L_{ki} g^k L_{kj}) dv^i dv^j$$

□

3.5. Theorem. *Let M and M^{λ} be two parallel surfaces. The coefficients L_{ij}^{λ} of second fundamental form of the parallel surface are given by*

$$L_{ij}^{\lambda} = L_{ij} - \lambda g^k L_{ki} g^k L_{kj} \tag{3.8}$$

Proof. Differentiating (3.1), we get

$$\varphi_{,ij}^{\lambda} = \varphi_{,ij} + \lambda \mathbf{N}_{,ij} \tag{3.9}$$

Substituting (3.5) and (3.9) into (2.9) then, using $\langle \mathbf{N}_{,ij}, \mathbf{N} \rangle_1 = - \langle \mathbf{N}_{,i}, \mathbf{N}_{,j} \rangle_1$ implies that

$$L_{ij}^{\lambda} = L_{ij} - \lambda \langle \mathbf{N}_{,i}, \mathbf{N}_{,j} \rangle_1 \tag{3.10}$$

From (2.2), (2.11) and (3.10) , we have

$$L_{ij}^{\lambda} = L_{ij} - \lambda g^k L_{ki} g^k L_{kj}$$

□

3.6. Corollary. *Asymptotic lines of the parallel surface M^{λ} are given by*

$$L_{ij}^{\lambda} = L_{ij} - \lambda g^k L_{ki} g^k L_{kj} = 0$$

3.7. Theorem. *Let M and M^{λ} be two parallel surfaces in $\mathbb{G}_3$, and $\alpha^{\lambda}(s) = \varphi^{\lambda}(v(s)^1, v(s)^2)$ be a non-isotropic curve on the parallel surface, parametrized by the arc length s, given by*

$$g_i^{\lambda} v^{i\prime} = 1$$

where " $\prime$ " refers to $\dfrac{d}{ds}$. The normal curvature k_n^{λ} of parallel surface is given by

$$k_n^{\lambda} = k_n - \lambda(g^k L_{ki} g^k L_{kj}) v^{i\prime} v^{j\prime}$$

where k_n is the normal curvature of M.

Proof. Differentiating (3.1) with respect to s gives

$$\varphi^{\lambda\prime} = \varphi_{,i}v^{i\prime} + \lambda \mathbf{N}_{,i}v^{i\prime}$$

and

$$\varphi^{\lambda\prime\prime} = \varphi_{,ij}v^{i\prime}v^{j\prime} + \varphi_{,k}v^{k\prime\prime} + \lambda(\mathbf{N}_{,ij}v^{i\prime}v^{j\prime} + \mathbf{N}_{,k}v^{k\prime\prime}) \tag{3.11}$$

Substituting (2.10) into (3.11), we get

$$\varphi^{\lambda\prime\prime} = (\Gamma^k_{ij}v^{i\prime}v^{j\prime} + v^{k\prime\prime})\varphi_{,k} + L_{ij}v^{i\prime}v^{j\prime}\mathbf{N} + \lambda(\mathbf{N}_{,ij}v^{i\prime}v^{j\prime} + \mathbf{N}_{,k}v^{k\prime\prime}) \tag{3.12}$$

Taking scalar product of both sides of (3.12) with $\mathbf{N}$ gives

$$\left\langle \varphi^{\lambda\prime\prime}, \mathbf{N}\right\rangle_1 = (L_{ij} + \lambda\left\langle \mathbf{N}, \mathbf{N}_{,ij}\right\rangle_1)v^{i\prime}v^{j\prime}$$

Using (2.11), (2.13) and $\left\langle \mathbf{N}_{,ij}, \mathbf{N}\right\rangle_1 = -\left\langle \mathbf{N}_{,i}, \mathbf{N}_{,j}\right\rangle_1$ implies that the relation between the normal curvatures of two parallel surfaces is

$$k^\lambda_n = k_n - \lambda(g^k L_{ki} g^k L_{kj})v^{i\prime}v^{j\prime}$$

□

3.8. Theorem. *Let M and M^λ be two parallel surfaces in $\mathbb{G}_3$. The relation between the geodesic curvatures of two parallel surfaces is given by*

$$k^\lambda_g = k_g - \lambda g^k L_{ki} v^{k\prime\prime}$$

Proof. Taking scalar product of both sides of (3.12) with δ gives

$$\left\langle \varphi^{\lambda\prime\prime}, \delta\right\rangle_1 = (\Gamma^k_{ij}v^{i\prime}v^{j\prime} + v^{k\prime\prime})\left\langle \varphi_{,k}, \delta\right\rangle_1 + \lambda(\left\langle \mathbf{N}_{,ij}, \delta\right\rangle_1 v^{i\prime}v^{j\prime} + \left\langle \mathbf{N}_{,k}, \delta\right\rangle_1 v^{k\prime\prime}) \tag{3.13}$$

Substituting (2.11) and (2.13) into (3.13), we have

$$k^\lambda_g = k_g + \lambda(\left\langle \mathbf{N}_{,ij}, \delta\right\rangle_1 v^{i\prime}v^{j\prime} - g^k L_{ki} v^{k\prime\prime})$$

From (2.11) and (2.12), we have $\left\langle \mathbf{N}_{,ij}, \delta\right\rangle_1 = 0$ which implies that

$$k^\lambda_g = k_g - \lambda g^k L_{ki} v^{k\prime\prime}$$

□

3.9. Theorem. *Let M and M^λ be two parallel surfaces in $\mathbb{G}_3$. The relations between the Gauss curvatures and the mean curvatures of two parallel surfaces are*

$$K^\lambda = \frac{K}{1 - 2\lambda H} \tag{3.14}$$

and

$$H^\lambda = \frac{H}{1 - 2\lambda H} \tag{3.15}$$

respectively.

Proof. Substituting (3.2) and (3.8) into (2.15) gives

$$K^\lambda = \frac{\det[L_{ij} - \lambda g^k L_{ki} g^k L_{kj}]}{w^2(1 - 2\lambda H)^2}$$

Simple calculation implies that

$$K^\lambda = \frac{(L_{11}L_{22} - L^2_{12})(1 - \lambda(g^{11}L_{11} + 2g^{12}L_{12} + g^{22}L_{22}))}{w^2(1 - 2\lambda H)^2} \tag{3.16}$$

Combining (2.15) and (3.16), we have

$$K^\lambda = \frac{K}{1 - 2\lambda H}$$

Using (2.6), (3.2) and (3.5) implies that

$$(3.17) \quad (g^{ij})^{\lambda} = \frac{g^{ij}}{(1-2\lambda H)^2}$$

Taking account of (2.15), (3.8) and (3.17), we find that

$$(3.18) \quad 2H^{\lambda} = \frac{g^{ij}}{(1-2\lambda H)^2}(L_{ij} - \lambda g^k L_{ki} g^k L_{kj})$$

Finally, substituting (2.15) into (3.18) then, H^{λ} can be written as

$$H^{\lambda} = \frac{H}{1-2\lambda H}$$

□

Now we shall consider some particular cases of the results (3.14) and (3.15).

3.10. Theorem. *Let M and M^{λ} be two parallel surfaces in $\mathbb{G}_3$. If the base surface is minimal, then the parallel surface is minimal.*

Proof. Since M is minimal surface, $H = 0$. Therefore, from (3.15) we have

$$H^{\lambda} = 0$$

□

3.11. Theorem. *Let M and M^{λ} be two parallel surfaces in $\mathbb{G}_3$. If base surface is Weingarten, then parallel surface is Weingarten.*

Proof. Since base surface is Weingarten, it satisfies the following condition

$$(3.19) \quad H_{,1}K_{,2} - H_{,2}K_{,1} = 0$$

On the other hand, differentiating (3.14) and (3.15) with respect to v^1 and v^2, we get

$$K^{\lambda}_{,1} = \frac{(1-2\lambda H)K_{,1} + 2\lambda K H_{,1}}{(1-2\lambda H)^2}, \quad H^{\lambda}_{,1} = \frac{(1-2\lambda H)H_{,1} + 2\lambda H H_{,1}}{(1-2\lambda H)^2}$$

$$K^{\lambda}_{,2} = \frac{(1-2\lambda H)K_{,2} + 2\lambda K H_{,2}}{(1-2\lambda H)^2}, \quad H^{\lambda}_{,2} = \frac{(1-2\lambda H)H_{,2} + 2\lambda H H_{,2}}{(1-2\lambda H)^2}$$

Thus,

$$(3.20) \quad H^{\lambda}_{,1}K^{\lambda}_{,2} - H^{\lambda}_{,2}K^{\lambda}_{,1} = \frac{H_{,1}K_{,2} - H_{,2}K_{,1}}{(1-2\lambda H)^3}$$

Combining (3.19) and (3.20) gives

$$H^{\lambda}_{,1}K^{\lambda}_{,2} - H^{\lambda}_{,2}K^{\lambda}_{,1} = 0$$

This completes the proof. □

References

[1] Çöken, A. C., Çiftçi, Ü. and Ekici, C. *On parallel timelike ruled surfaces with timelike rulings*, Kuwait Journal of Science & Engineering, **35** (1A), 21–31, 2008.

[2] Dede, M. *Tubular surfaces in Galilean space*, Math. Commun., **18**, 209–217, 2013.

[3] Divjak, B. *Curves in pseudo-Galilean geometry*, Annales Univ. Sci. Budapest, **41**, 117–128, 1998.

[4] Eisenhart, L. P. *A treatise on the differential geometry of curves and surfaces*, (Dover Publication, Inc., New York, 1960).

[5] Ekici, C. and Dede, M. *On the Darboux vector of ruled surfaces in pseudo-Galilean space*, Mathematical and Computational Applications, **16**(4), 830–838, 2011.

[6] Görgülü, A. and Çöken A. C. *The Euler theorem for parallel pseudo-Euclidean hpersurfaces in pseudo-Euclidean space* E_1^{n+1}, Journ. Inst. Math. and Comp. Sci. (Math. Series), **6.2**, 161–165, 1993.

[7] Guggenheimer, H. W. *Differential geometry*, (McGraw-Hill, New York, 1963).

[8] Kamenarovic, I. *Existence theorems for ruled surfaces in the Galilean space* $\mathbb{G}_3$, Rad Hrvatske Akad. Znan. Umj. Mat., **456**(10), 183–196, 1991.

[9] Milin-Sipus, Z. *Ruled Weingarten surfaces in Galilean space*, Periodica Mathematica Hungarica, **56**(2), 213–225, 2008.

[10] Milin-Sipus, Z. and Divjak, B. *Special curves on ruled surfaces in Galilean and pseudo-Galilean spaces*, Acta Math. Hungar., **98**(3), 203–215, 2003.

[11] Öğrenmiş, A., Ergüt, M. and Bektaş, M. *On the helices in the Galilean space* $\mathbb{G}_3$, Iranian Journal of Science & Technology, Transaction A, **31**(A2), 177–181, 2007.

[12] Roberts, S. *On Parallel surfaces*, Proc. London Math. Soc., **1-4**(1), 218–236, 1871.

[13] Roberts, S. *On the parallel surfaces of developables and curves of double curvature*, Proc. London Math. Soc., **1-5**(1), 90–94, 1873.

[14] Röschel, O. *Die geometrie des Galileischen raumes*, (Habilitationsschrift, Leoben, 1984).

[15] Sağel, M. K. and Hacısalihoğlu, H. H. *On the parallel hypersurfaces with constant curvature*, Commun. Fac. Sci. Univ. Ank. Series A, **40**, 1–5, 1991.

[16] Yaglom, I. M. *A simple non-Euclidean geometry and its physical basis*, (Springer-Verlag, New York Inc., 1979).