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SOME APPLICATIONS OF AUGMENTATION QUOTIENTS

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Abstract

We give some applications of augmentation quotients of free group rings in group theory.

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1. Introduction

Let $\mathbb{Z}G$ denote the integral group ring of a group G and $\Delta(G)$ its augmentation ideal. Let $\{\gamma_n(G)\}_{n\geq 1}$ be the lower central series of G. We also write G' for $\gamma_2(G) = [G, G]$, the derived group of G. When G is free, then integral group ring is known as free group ring. Let $\Delta^n(G)$ denote the n-th associative power of $\Delta(G)$ with $\Delta^0(G) = \mathbb{Z}G$. The additive abelian group $\Delta^n(G)/\Delta^{n+1}(G)$ is known as the n-th augmentation quotient and has been intensively studied during the last forty years. Vermani[7] has given a notable survey article about work done on augmentation quotients. In this short note we are interested in the applications of augmentation quotients in group theory. Henceforth, unless or otherwise stated, F is a free group and R is a normal subgroup of F. Hurley and Sehgal[4] identified the subgroup $F \cap (1 + \Delta^2(F)\Delta^n(R))$ for all $n \geq 1$ and then using the fact that $\Delta(F)\Delta^n(R)/\Delta^2(F)\Delta^n(R)$ is free abelian for all $n \geq 1$ [1], they showed that the group $\gamma_{n+1}(R)/\gamma_{n+2}(R)\gamma_{n+1}(R \cap F')$ is a free abelian group for all $n \geq 1$. Gruenberg [1, Lemma III.5] proved that $\Delta^n(F)\Delta^m(R)/\Delta^{n+1}(F)\Delta^m(R)$ is a free abelian group for all $m, n \geq 1$. When R is an arbitrary subgroup of F, Karan and Kumar [5] proved that the groups $\Delta^n(F)\Delta^m(R)/\Delta^{n+1}(F)\Delta^m(R)$, $\Delta^n(F)\Delta^m(R)/\Delta^{n-1}(F)\Delta^{m+1}(R)$ and $\Delta^n(F)\Delta^m(R)/\Delta^n(F)\Delta^{m+1}(R)$ are free abelian for all $m, n \geq 1$. They gave the complete description of all these groups and explicit bases of first two groups. As a consequence of their results they proved that $R'/[R', R \cap F']$ is a free abelian group. Gumber et. al. [2] proved that $\Delta^p(R)\Delta^n(F)\Delta^q(R)/\Delta^p(R)\Delta^{n+1}(F)\Delta^q(R)$ is free abelian for all $p, q, n \geq 1$

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and as a consequence showed that $\gamma_3(R)/\gamma_4(R)[R\cap F', R\cap F', R]$ is a free abelian group. In section 3, we identify the subgroup

$$R \cap (1 + \Delta^{m+3}(R) + \Delta^{m+1}(R)\Delta(R\cap F') + \Delta^{m}(R)\Delta([R, R\cap F']))$$

for $m = 0, 1$, and 2, and then prove

Theorem A. The groups
(1) $R'/[R, R\cap F']$,
(2) $\gamma_3(R)/\gamma_4(R)[R, R\cap F', R\cap F']$, and
(3) $\gamma_4(R)/\gamma_5(R)[R, R\cap F', R\cap F', R\cap F'][\,[R, R\cap F'], [R, R\cap F']\,]$
are free abelian.

2. Preliminaries

Let G be a group and H be a normal subgroup of G such that G/H is free-abelian. Let $\{x_\delta H \mid \delta \in \Delta\}$ be a basis for G/H. We may suppose that the index set Δ is well ordered. As $G' \subset H$, S, the set consisting of elements of the form $x_{\delta_1}^{t_1} x_{\delta_2}^{t_2} \dots x_{\delta_n}^{t_n}$, $t_i \in Z, n \geq 1, \delta_1 < \delta_2 < \ldots < \delta_n$, is a transversal of H in G. Let L_n be the Z-submodule of $\Delta(G)$ generated by elements of the form

$$(x_{\delta_1}^{\epsilon_1} - 1)\dots(x_{\delta_n}^{\epsilon_n} - 1),\ \epsilon_i = 1 \text{ or } -1 \text{ for every } i \text{ and } \delta_1 \leq \delta_2 \leq \ldots \leq \delta_n.$$

For $m \geq 2$, let $L^{(m)} = \sum_{n \geq m} L_n$.

2.1. Theorem. [8] *For $n \geq 2$, $\Delta^n(G)$ is equal to*

$$\Delta^{n-1}(G)\Delta(H) + \Delta^{n-2}(G)\Delta(G') + \cdots + \Delta(G)\Delta(\gamma_{n-1}(G)) + \Delta(\gamma_n(G)) \ \oplus\ L^{(n)}.$$

Let U be a group and W be a left transversal of a subgroup V of U in U with $1 \in W$. Then every element of U can be uniquely written as wv, $w \in W$, $v \in V$. Let $\phi : ZU \to ZV$ be the onto homomorphism of right ZV-modules which on the elements of U is given by $\phi(wv) = v,\ w \in W,\ v \in V$. The homomorphism ϕ maps $\Delta(U)J$ onto $\Delta(V)J$ for every ideal J of ZV. In particular, by the choice of the transversal S of H in G, we have $\phi\,|_{L^{(n)}} = 0$. The homomorphism ϕ is usually called the filtration map.

We shall also need the following results:

2.2. Lemma. [9] *Let G be a group, K a subgroup of G, and J an ideal of $\mathbb{Z}G$ containing $\Delta^2(K)$. Then $G \cap (1 + J + \Delta(K)) = (G \cap (1 + J))K$.*

2.3. Theorem. [8] *Let G be a group with a normal subgroup H such that G/H is free abelian. Then $G \cap (1 + \Delta^n(G) + \Delta(G)\Delta(H)) = \gamma_n(G)H'$ for all $n \geq 1$.*

3. Proof of Theorem A

To avoid repeated and prolonged expressions, we write

$$\begin{aligned} A &= \Delta^4(R) + \Delta^2(R)\Delta(R\cap F') + \Delta(R)\Delta([R, R\cap F']) \\ B &= \Delta^5(R) + \Delta^3(R)\Delta(R\cap F') + \Delta^2(R)\Delta([R, R\cap F']). \end{aligned}$$

3.1. Proposition. *The group*

$$\frac{\gamma_{m+2}(R)}{\gamma_{m+2}(R) \cap (1 + \Delta^{m+3}(R) + \Delta^{m+1}(R)\Delta(R\cap F') + \Delta^m(R)\Delta([R, R\cap F']))}$$

is free-abelian for all $m \geq 0$.

Proof. It follows from the proof of Theorem 1.1 and Corollary 2.4 of [6] that

$$\Delta^3(F) \cap \Delta^2(R) = \Delta^3(R) + \Delta(R)\Delta(R \cap F') + \Delta([R, R \cap F']),$$

and since $\Delta(R)\mathbb{Z}F$ is a free right $\mathbb{Z}F$-module [3, Proposition I.1.12], we have

$$\Delta^m(R)\Delta^3(F) \cap \Delta^{m+2}(R) = \Delta^{m+3}(R) + \Delta^{m+1}(R)\Delta(R \cap F') + \Delta^m(R)\Delta([R, R \cap F'])$$

for all $m \geq 0$. The natural homomorphism

$$\eta : \Delta^{m+2}(R) \to \Delta^m(R)\Delta^2(F)/\Delta^m(R)\Delta^3(F)$$

has $ker\, \phi = \Delta^{m+3}(R) + \Delta^{m+1}(R)\Delta(R \cap F') + \Delta^m(R)\Delta([R, R \cap F'])$ in view of the above intersection. Thus $\Delta^{m+2}(R)/(\Delta^{m+3}(R) + \Delta^{m+1}(R)\Delta(R \cap F') + \Delta^m(R)\Delta([R, R \cap F']))$ is free-abelian. Again, the homomorphism

$$\theta : \gamma_{m+2}(R) \to \frac{\Delta^{m+2}(R)}{\Delta^{m+3}(R) + \Delta^{m+1}(R)\Delta(R \cap F') + \Delta^m(R)\Delta([R, R \cap F'])}$$

defined as $x \to \overline{(x-1)}\,, x \in \gamma_{m+2}(R)$ has $ker\, \theta$ equal to

$$\gamma_{m+2}(R) \cap (1 + \Delta^{m+3}(R) + \Delta^{m+1}(R)\Delta(R \cap F') + \Delta^m(R)\Delta([R, R \cap F'])).$$

Therefore

$$\frac{\gamma_{m+2}(R)}{\gamma_{m+2}(R) \cap (1 + \Delta^{m+3}(R) + \Delta^{m+1}(R)\Delta(R \cap F') + \Delta^m(R)\Delta([R, R \cap F']))}$$

is free-abelian for all $m \geq 0$.

□

3.2. Proposition. $R \cap (1 + \Delta^3(R) + \Delta(R)\Delta(R \cap F') + \Delta([R, R \cap F'])) = [R, R \cap F']$.

Proof. Proof is easy and follows by Lemma 2.2 and Theorem 2.3. □

3.3. Proposition. $R'/[R, R \cap F']$ *is free-abelian.*

Proof. The proof follows by putting $m = 0$ in Proposition 3.1 and then using Proposition 3.2. □

3.4. Proposition. $R \cap (1 + A) = \gamma_4(R)[R, R \cap F', R \cap F']$.

Proof. Since $\gamma_4(R) - 1 \subset \Delta^4(R)$ and $[R, R \cap F', R \cap F'] - 1 \subset \Delta^2(R)\Delta(R \cap F') + \Delta(R)\Delta([R, R \cap F'])$, it follows that $\gamma_4(R)[R, R \cap F', R \cap F'] \subset R \cap (1 + A)$. For the reverse inequality, we let $w \in R$ such that $w - 1 \in A$ and proceed to show that $w \equiv 1 \pmod{\gamma_4(R)[R, R \cap F', R \cap F']}$. Since $R/R \cap F'$ is free-abelian, using Theorem 2.1 repeatedly we have

$$\begin{aligned} A &= \Delta(\gamma_4(R)) + L^{(4)} + \Delta(R)\Delta^2(R \cap F') + \Delta(R')\Delta(R \cap F') \\ &\quad + L^{(2)}\Delta(R \cap F') + \Delta(R)\Delta([R, R \cap F']). \end{aligned}$$

Now since $R \cap (1 + A) \subset R \cap F'$, using the filtration map $\phi : ZR \to Z(R \cap F')$, it follows that

$$\begin{aligned} R \cap (1 + A) &\subset (R \cap F') \cap (1 + \Delta^3(R \cap F') + \Delta(R')\Delta(R \cap F') \\ &\qquad + \Delta(R \cap F')\Delta([R, R \cap F']) + \Delta(\gamma_4(R))) \\ &\subset (R \cap F') \cap (1 + \Delta^3(R \cap F') + \Delta(R')\Delta(R \cap F') \\ &\qquad + \Delta([R, R \cap F', R \cap F']) + \Delta(\gamma_4(R))) \\ &= (R \cap F') \cap (1 + \Delta^3(R \cap F') + \Delta(R')\Delta(R \cap F')) \\ &\qquad [R, R \cap F', R \cap F']\gamma_4(R) \\ &= [R, R \cap F', R \cap F']\gamma_4(R), \end{aligned}$$

where last equality follows by Theorem 2.4 and second last equality follws by Lemma 2.3. □

3.5. Proposition.

$$R\cap(1+B)=\gamma_5(R)[R,R\cap F',R\cap F',R\cap F'][\,[R,R\cap F'],[R,R\cap F']\,].$$

Proof. As in the above proposition, it is sufficient to prove that if $w\in R$ is such that $w-1\in B$, then

$$w\equiv 1 \pmod{\gamma_5(R)[R,R\cap F',R\cap F',R\cap F'][\,[R,R\cap F'],[R,R\cap F']\,]}.$$

Using Theorem 2.1 repeatedly, we have

$$\begin{aligned}
&\Delta^5(R)+\Delta^3(R)\Delta(R\cap F')+\Delta^2(R)\Delta([R,R\cap F'])\\
=\ &\Delta(R)\Delta(\gamma_4(R))+\Delta(\gamma_5(R)+L^{(5)}+\Delta(R)\Delta^3(R\cap F')\\
&\quad+\Delta(R')\Delta^2(R\cap F')+L^{(2)}\Delta^2(R\cap F')+\Delta(R)\Delta(R')\Delta(R\cap F')\\
&\quad+\Delta(\gamma_3(R))\Delta(R\cap F')+L^{(3)}\Delta(R\cap F')+\Delta(R)\Delta(R\cap F')\\
&\quad\Delta([R,R\cap F'])+\Delta(R')\Delta([R,R\cap F'])+L^{(2)}\Delta([R,R\cap F']).
\end{aligned}$$

Applying filtration map $\phi: ZR\to Z(R\cap F')$, we have

$$\begin{aligned}
&R\cap(1+\Delta^5(R)+\Delta^3(R)\Delta(R\cap F')+\Delta^2(R)\Delta([R,R\cap F']))\\
=\ &(R\cap F')\cap(1+\Delta^4(R\cap F')+\Delta(R')\Delta^2(R\cap F')+\Delta(\gamma_3(R))\Delta(R\cap F')\\
&\quad+\Delta^2(R\cap F')\Delta([R,R\cap F'])+\Delta(R')\Delta([R,R\cap F']))\gamma_5(R)\\
\subset\ &(R\cap F')\cap(1+\Delta^4(R\cap F')+\Delta(R')\Delta^2(R\cap F')+\Delta(\gamma_3(R))\Delta(R\cap F')\\
&\quad+\Delta(R')\Delta([R,R\cap F']))\gamma_5(R)[R,R\cap F',R\cap F',R\cap F'].
\end{aligned}$$

Now since $R\cap F'/R'$ is free-abelian, a use of similar arguments with left replaced by right and the left ZR'-homomorphism $\phi: Z(R\cap F')\to ZR'$ implies that

$$\begin{aligned}
&(R\cap F')\cap(1+\Delta^4(R\cap F')+\Delta(R')\Delta^2(R\cap F')+\Delta(\gamma_3(R))\Delta(R\cap F')\\
&\quad+\Delta(R')\Delta([R,R\cap F']))\gamma_5(R)[R,R\cap F',R\cap F',R\cap F']\\
=\ &R'\cap(1+\Delta^3(R')+\Delta(R')\Delta([R,R\cap F']))\gamma_5(R)\\
&\quad[R,R\cap F',R\cap F',R\cap F']\\
=\ &\gamma_5(R)[R,R\cap F',R\cap F',R\cap F'][\,[R,R\cap F'],[R,R\cap F']\,],
\end{aligned}$$

since $R'/[R,R\cap F']$ is free-abelian by Proposition 3.3. □

Proof. (**Proof of Theorem A:**) The proof of (1) follows by Proposition 3.3 and the proofs of (2) and (3) follow by putting $m=1,2$ in Proposition 3.1 and using Propositions 3.4 and 3.5 respectively. □

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