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JORDAN TRIPLE $(\alpha,\beta)^*$-DERIVATIONS ON SEMIPRIME RINGS WITH INVOLUTION

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Abstract

Let R be a 2-torsion free semiprime $*$-ring. The aim of this paper is to show that every Jordan triple $(\alpha,\beta)^*$-derivation on R is a Jordan $(\alpha,\beta)^*$-derivation. Furthermore, every Jordan triple left α^*-centralizer on R is a Jordan left α^*-centralizer. Consequently, every generalized Jordan triple $(\alpha,\beta)^*$-derivation on R is a Jordan $(\alpha,\beta)^*$-derivation.

Keywords: ring, $*$-ring, $(\alpha,\beta)^*$- derivation, Jordan $(\alpha,\beta)^*$-derivation, Jordan triple $(\alpha,\beta)^*$-derivation, generalized Jordan triple $(\alpha,\beta)^*$-derivation.

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1. Introduction

Throughout this paper all rings will be associative. A ring R is n-torsion free, where $n > 1$ is an integer, in case $nx = 0$, $x \in R$, implies $x = 0$. As usual, commutator $ab - ba$ will be denoted by $[a,b]$ and anti-commutator $ab + ba$ will be denoted by $a \circ b$. Recall that a ring R is prime if for $a,b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$, and is semiprime in case $aRa = (0)$ implies $a = 0$. The center of a ring R will be denoted by $Z(R)$. A $*$-ring is a ring R equipped with an involution, that is an additive mapping $* : R \to R$ such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for all $a,b \in R$. An involution $*$ on a $*$-ring R is said to be positive definite if $a^*a = 0$ (with $a \in R$) implies $a = 0$.

An additive mapping $\delta : R \longrightarrow R$ is called a derivation (resp. a Jordan derivation) if $\delta(xy) = \delta(x)y + x\delta(y)$ (resp. $\delta(x^2) = \delta(x)x + x\delta(x)$) holds for all $x,y \in R$. If $\delta : R \to R$ is additive and if α and β are endomorphisms of R, then δ is said to be an (α,β)-derivation of R when for all $x,y \in R$, $\delta(xy) = \delta(x)\alpha(y) + \beta(x)\delta(y)$. Note that for I, the identity mapping on R, an (I,I)-derivation is just a derivation. An example of an (α,β)-derivation when R has a nontrivial central idempotent e is obtained if we let $\delta(x) = ex$, $\alpha(x) = x - ex$, and $\beta = I$ (or δ). Here, δ is not a derivation because $\delta(ee) = eee \neq 2eee = (ee)e + e(ee) = \delta(e)e + e\delta(e)$. In any ring

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with an endomorphism α, if we let $\delta = I - \alpha$, then δ is an (α, I)-derivation, but not a derivation when R is semiprime, unless $\alpha = I$. A famous result due to Herstein [12, Theorem 3.3], states that a Jordan derivation of a prime ring of characteristic not equal to 2 must be a derivation. A brief proof of Herstein's result can also be found in [6]. This result was extended to 2-torsion free semiprime rings by Cusack [8] and subsequently, by Brešar [3] .

Following [4], an additive mapping $\delta : R \to R$ is called a Jordan triple derivation if $\delta(xyx) = \delta(x)yx + x\delta(y)x + xy\delta(x)$ holds for all $x, y \in R$. One can easily prove that any Jordan derivation on an arbitrary 2-torsion free ring is a Jordan triple derivation (see [12, Lemma 3.5]). The famous result of Brešar in [4, Theorem 4.3] states that, if R is a 2-torsion free semiprime ring, any Jordan triple derivation $\delta : R \to R$ is a derivation.

Let R be a $*$-ring. An additive mapping $\delta : R \to R$ is called a Jordan $*$-derivation if

$$\delta(a \circ b) = \delta(a)b^* + a\delta(b) + \delta(b)a^* + b\delta(a) \text{ holds for all } a, b \in R.$$

When R is 2-torsion free, we define a Jordan $*$-derivation by merely insisting that

$$\delta(a^2) = \delta(a)a^* + a\delta(a) \text{ holds for all } a \in R.$$

These mappings are closely connected with the problem of the representability of quadratic functionals by sesquilinear forms. In 1989 Brešar and Vukman [5] studied some algebraic properties of Jordan $*$-derivations. An additive mapping $\delta : R \to R$ is said to be a Jordan triple $*$-derivation if $\delta(aba) = \delta(a)b^*a^* + a\delta(b)a^* + ab\delta(a)$ for all $a, b \in R$. Let α, β be endomorphisms of a $*$-ring R. An additive mapping $\delta : R \to R$ is called an $(\alpha, \beta)^*$-derivation if $\delta(ab) = \delta(a)\alpha(b^*) + \beta(a)\delta(b)$ for all $a, b \in R$. An additive mapping $\delta : R \to R$ is called a Jordan $(\alpha, \beta)^*$-derivation if

$$\delta(a \circ b) = \delta(a)\alpha(b^*) + \beta(a)\delta(b) + \delta(b)\alpha(a^*) + \beta(b)\delta(a) \text{ for all } a, b \in R.$$

If R is 2-torsion free, the definition of a Jordan $(\alpha, \beta)^*$-derivation is equivalent to the following condition

$$\delta(a^2) = \delta(a)\alpha(a^*) + \beta(a)\delta(a) \text{ for all } a \in R.$$

An additive mapping $\delta : R \to R$ is called a Jordan triple $(\alpha, \beta)^*$-derivation if $\delta(aba) = \delta(a)\alpha(b^*a^*) + \beta(a)\delta(b)\alpha(a^*) + \beta(ab)\delta(a)$ for all $a, b \in R$.

Obviously, every $(\alpha, \beta)^*$-derivation on a 2-torsion free $*$-ring is a Jordan triple $(\alpha, \beta)^*$-derivation, but converse need not be true in general. In [1] Shakir and Fošner, shows that the converse is true for 6-torsion free semiprime $*$-ring R. In Section 2 we remove the assumption that R is a 3-torsion free if the involution is positive definite and prove that every Jordan triple $(\alpha, \beta)^*$-derivation on a 2-torsion free semiprime $*$-ring with a positive definite involution is a Jordan $(\alpha, \beta)^*$-derivation.

An additive mapping $\Theta : R \longrightarrow R$ is said to be a generalized derivation (resp. a generalized Jordan derivation) on R if there exists a derivation $\delta : R \longrightarrow R$

such that $\Theta(xy) = \Theta(x)y + x\delta(y)$ (resp. $\Theta(x^2) = \Theta(x)x + x\delta(x)$) holds for all $x, y \in R$. An additive mapping $\Theta : R \longrightarrow R$ is said to be a generalized Jordan triple derivation on R if there exists a Jordan triple derivation $\delta : R \longrightarrow R$ such that $\Theta(xyx) = \Theta(x)yx + x\delta(y)x + xy\delta(x)$ holds for all $x, y \in R$. In 2003, Jing and Lu [20, Theorem 3.5] proved that every generalized Jordan triple derivation on a 2-torsion free prime ring R is a generalized derivation. Very recently, Vukman [24] extended Jing and Lu [20] result for 2-torsion free semiprime rings. The result was independently obtained by Fošner and Ilišević [10, Corollary 4.2]

An additive mapping $\Theta : R \to R$ is called a generalized (α, β)-derivation, for α and β endomorphisms of R, if there exists an (α, β)-derivation $\delta : R \to R$ such that $\Theta(xy) = \Theta(x)\alpha(y) + \beta(x)\delta(y)$ holds for all $x, y \in R$. Clearly, this notion include those of (α, β)-derivation when $\Theta = \delta$, of derivation when $\Theta = \delta$ and $\alpha = \beta = I$, and of generalized derivation, which is the case when $\alpha = \beta = I$. mappings of the form $\Theta(x) = ax + xb$ for $a, b \in R$ with $\delta(x) = xb - bx$ and $\alpha = \beta = I$ are generalized derivations, and more generally, mappings $\Theta(x) = a\alpha(x) + \beta(x)b$ are generalized (α, β)-derivations. To see this observe that $\Theta(xy) = a\alpha(x)\alpha(y) + \beta(x)\beta(y)b = (a\alpha(x) + \beta(x)b)\alpha(y) + \beta(x)(\beta(y)b - b\alpha(y))$, and as we have just seen above, $\delta(x) = b\alpha(x) - \beta(x)b$ is an (α, β)-derivation of R. As for derivation, a generalized Jordan (α, β)-derivation Θ assumes $x = y$ in the definition above; that is, we assume only that $\Theta(x^2) = \Theta(x)\alpha(x) + \beta(x)\delta(x)$, holds for all $x \in R$. An additive mapping $\Theta : R \to R$ is called a generalized Jordan triple (α, β)-derivation, for α and β endomorphisms of R, if there exists a Jordan triple (α, β)-derivation $\delta : R \to R$ such that $\Theta(xyx) = \Theta(x)\alpha(yx) + \beta(x)\delta(y)\alpha(x) + \beta(xy)\delta(x)$ holds for all $x, y \in R$. Clearly, this notion includes those of triple (α, β)-derivation when $\Theta = \delta$, of triple derivation when $\Theta = \delta$ and $\alpha = \beta = I$, and of generalized triple derivation which is the case $\alpha = \beta = I$. In 2007, Liu and Shiue [15] shows that on a 2-torsion free semiprime ring every Jordan triple (α, β)-derivation is an (α, β)-derivation and every generalized Jordan triple (α, β)-derivation is a generalized (α, β)-derivation.

Let R be a $*$-ring. An additive mapping $\Theta : R \to R$ is called a generalized $*$-derivation, for α, β endomorphisms of R, if there exists a $*$-derivation $\delta : R \to R$ such that $\Theta(ab) = \Theta(a)b^* + a\delta(b)$ for all $a, b \in R$. An additive mapping $\Theta : R \to R$ is called a generalized Jordan $*$-derivation if there exists a Jordan $*$-derivation δ such that

$$\Theta(a \circ b) = \Theta(a)b^* + a\delta(b) + \Theta(b)a^* + b\delta(a) \text{ holds for all } a, b \in R.$$

When R is 2-torsion free, we define a generalized Jordan $*$-derivation by

$$\Theta(a^2) = \Theta(a)a^* + a\delta(a) \text{ holds for all } a \in R.$$

An additive mapping $\Theta : R \to R$ is called a generalized Jordan triple $*$-derivation if there exists a Jordan triple $*$-derivation $\delta : R \to R$ such that

$$\Theta(aba) = \Theta(a)(b^*a^*) + a\delta(b)a^* + ab\delta(a) \text{ holds for all } a, b \in R.$$

An additive mapping $\Theta : R \to R$ is called a generalized $(\alpha, \beta)^*$-derivation if there exists an $(\alpha, \beta)^*$-derivation $\delta : R \to R$ such that $\Theta(ab) = \Theta(a)\alpha(b^*) +$

$\beta(a)\delta(b)$ for all $a, b \in R$. An additive mapping $\Theta : R \to R$ is called a generalized Jordan $(\alpha,\beta)^*$-derivation if there exists a Jordan $(\alpha,\beta)^*$-derivation δ such that

$$\Theta(a\circ b) = \Theta(a)\alpha(b^*)+\beta(a)\delta(b)+\Theta(b)\alpha(a^*)+\beta(b)\delta(a) \text{ holds for all } a, b \in R.$$

When R is 2-torsion free, the definition of a generalized Jordan $(\alpha,\beta)^*$-derivation is equivalent to the following condition

$$\Theta(a^2) = \Theta(a)\alpha(a^*) + \beta(a)\delta(a) \text{ holds for all } a \in R.$$

An additive mapping $\Theta : R \to R$ is called a generalized Jordan triple $(\alpha,\beta)^*$-derivation if there exists a Jordan triple $(\alpha,\beta)^*$-derivation $\delta : R \to R$ such that

$$\Theta(aba) = \Theta(a)\alpha(b^*a^*) + \beta(a)\delta(b)\alpha(a^*) + \beta(ab)\delta(a) \text{ holds for all } a, b \in R.$$

A generalized Jordan triple $(I, I)^*$-derivation is just a generalized triple $*$-derivation. It can be easily seen that on a 2-torsion free $*$-ring every generalized Jordan $*$-derivation is a generalized Jordan triple $*$-derivation. But converse need not be true in general. Thus, the concept of a generalized Jordan triple $(\alpha,\beta)^*$-derivation covers the concepts of a Jordan triple $(\alpha,\beta)^*$-derivation and a Jordan triple left α^*-centralizer, that is an additive mapping $\Psi : R \to R$ satisfying $\Psi(aba) = \Psi(a)\alpha(b^*a^*)$ for all $a, b \in R$. In Section 3, we show that every Jordan triple left α^*-centralizer on a 2-torsion free semiprime $*$-ring is a Jordan α^*-centralizer. Daif and Tamman [9] established that on a 6-torsion free semiprime $*$-ring every generalized Jordan triple $*$-derivation is a generalized Jordan $*$-derivation. In 2008, Fošner and Ilišević [10] obtained the above mention result without the assumption of 3-torsion free. In Section 4, we extend Fošner and Ilišević result for a generalized Jordan triple $(\alpha,\beta)^*$-derivation.

2. Jordan Triple $(\alpha,\beta)^*$-Derivations

One can easily prove that every Jordan $*$-derivation on a 2-torsion free $*$-ring is a Jordan triple $*$-derivation. However, the converse is not true in general. The following example due to Shakir and Fošner [1] justifies this fact:

2.1. Example. Let S be any commutative ring, and let $R = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z \in S \right\}$.

Define mappings $\delta : R \to R$ and $\alpha, \beta, * : R \to R$ as follows: $\delta \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\alpha \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -x & y \\ 0 & 0 & -z \\ 0 & 0 & 0 \end{pmatrix}$, $\beta \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -x & -y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}$ and $* \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & z & y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}$. Then it is straightforward to check that δ is a Jordan triple $(\alpha,\beta)^*$-derivation, but not a Jordan $(\alpha,\beta)^*$-derivation.

In [23] Vukman established the converse for a 6-torsion free semiprime ring. Recently, Fošner and Ilišević [10] prove that any Jordan triple $*$-derivation on a 2-torsion free semiprime ring is a Jordan $*$-derivation. In 2010, Shakir and

Fošner [1] obtained the Vukman [23] result for Jordan triple $(\alpha,\beta)^*$-derivation. In the present section, we obtained an analogue of the result due to Fošner and Ilišević [10, Theorem 5.2] for Jordan triple $(\alpha,\beta)^*$-derivation. In fact we obtain the following result:

2.2. Theorem. *Let R be a 2-torsion free semiprime $*$-ring with a positive definite involution. Let α, β be $*$-automorphisms of R, and let $\delta : R \to R$ be an additive mapping . Then the following conditions are mutually equivalent:*

(i) *δ is a Jordan $(\alpha,\beta)^*$-derivation;*

(ii) *$\delta(aba) = \delta(a)\alpha(b^*a^*) + \beta(a)\delta(b)\alpha(a^*) + \beta(ab)\delta(a)$ for all $a, b \in R$.*

We begin our discussion with the following lemma which is a generalization of the result due to Ilišević [14].

2.3. Lemma. *Let R be a semiprime $*$-ring with a positive definite involution, and let α, β be $*$-automorphisms of R. If there exists an element $r \in R$ such that $\beta(a)r\alpha(a^*) = 0$ for all $a \in R$, then $r = 0$.*

Proof. By linearization we get

$$\beta(a)r\alpha(b^*) = -\beta(b)r\alpha(a^*) \qquad \text{for all } a, b \in R.$$

Notice that $\beta(a)r\alpha(a^*) = 0$, since α and β are $*$-automorphisms, implies $\alpha(a)r^*\beta(a^*) = 0$ for all $a \in R$, thus $\alpha(a^*)r^*\beta(a) = 0$ for all $a \in R$. Now we have, for all $a, b \in R$,

$$(r^*\beta(a)r)\alpha(b)(r^*\beta(a)r) = r^*(\beta(a)r\alpha(b))(r^*\beta(a)r) =$$
$$= r^*(-\beta(b^*)r\alpha(a^*))(r^*\beta(a)r) = -r^*\beta(b^*)r(\alpha(a^*)r^*\beta(a))r = 0.$$

Since α is surjective, we have

$$(r^*\beta(a)r)R(r^*\beta(a)r) = 0 \qquad \text{for all } a \in R.$$

Semiprimeness of R yields $r^*\beta(a)r = 0$ for all $a \in R$. Since β is surjective, we have $r^*Rr = 0$. This implies $r^*r = 0$. Since R is a $*$-ring with a positive definite involution, this implies $r = 0$. $\square$

2.4. Lemma. *Let R be a 2-torsion free $*$-ring, and let α, β be endomorphisms of R. If $\delta : R \to R$ is a Jordan $(\alpha,\beta)^*$-derivation, then for arbitrary $a, b, c \in R$, we have*

$$\begin{array}{lll} (I) \quad \delta(aba) & = & \delta(a)\alpha(b^*a^*) + \beta(a)\delta(b)\alpha(a^*) + \beta(ab)\delta(a). \\ (II) \quad \delta(abc+cba) & = & \delta(a)\alpha(b^*c^*) + \beta(a)\delta(b)\alpha(c^*) + \beta(ab)\delta(c) + \delta(c)\alpha(b^*a^*) \\ & & +\beta(c)\delta(b)\alpha(a^*) + \beta(cb)\delta(a). \end{array}$$

Proof. (I) Since δ is a Jordan $(\alpha,\beta)^*$-derivation, we have

$$\delta(ab+ba) = \delta(a)\alpha(b^*) + \beta(a)\delta(b) + \delta(b)\alpha(a^*) + \beta(b)\delta(a) \text{ for all } a, b \in R. \tag{2.1}$$

Replacing b by $ab + ba$ in (2.1) we get

$$\begin{aligned} \delta(a(ab+ba) + (ab+ba)a) = {} & \delta(a)\alpha(a^*b^*) + \delta(a)\alpha(b^*a^*) + \beta(a)\delta(a)\alpha(b^*) \\ & + \beta(a^2)\delta(b) + \beta(a)\delta(b)\alpha(a^*) + \beta(ab)\delta(a) \\ & + \delta(a)\alpha(b^*a^*) + \beta(a)\delta(b)\alpha(a^*) + \delta(b)\alpha({a^*}^2) \\ & \beta(b)\delta(a)\alpha(a^*) + \beta(ab)\delta(a) + \beta(ba)\delta(a). \end{aligned} \tag{2.2}$$

On the other hand, we have

$$\begin{aligned} \delta(a(ab+ba)+(ab+ba)a) &= \delta(a^2b+ba^2)+2\delta(aba) \\ &= \delta(a)\alpha(a^*b^*)+\beta(a)\delta(a)\alpha(b^*)+\beta(a^2)\delta(b) \\ &\quad \delta(b)\alpha({a^*}^2)+\beta(b)\delta(a)\alpha(a^*)+\beta(ba)\delta(a)+2\delta(aba). \end{aligned} \tag{2.3}$$

Comparing (2.2) & (2.3), and using the fact that R is 2-torsion free, we get the required result.

(II) We compute $W=\delta((a+c)b(a+c))$ in two different ways. On one hand, we find that $W=\delta(a+c)\alpha(b^*a^*+b^*c^*)+\beta(a+c)\delta(b)\alpha(a^*+c^*)+\beta(ab+cb)\delta(a)$, and on the other hand $W=\delta(aba)+\delta(abc+cba)+\delta(cbc)$. Comparing two expressions, we obtain the required result.

□

Proof of the Theorem 2.2. The implication $(i) \implies (ii)$ is clear by Lemma 2.4 (I). $(ii) \implies (i)$. We use the ideal of [10, Theorem 5.2]. By Lemma 2.4, for any $a,b\in R$, we have

$$\begin{aligned} \delta((ab)^2) &= \delta(abab)=\delta(ab(ab)+(ab)ba)-\delta(ab^2a) \\ &= \big(\delta(a)\alpha(b^*)\alpha((ab)^*)+\beta(a)\delta(b)\alpha(ab)^*)+\beta(ab)\delta(ab) \\ &\quad +\delta(ab)\alpha(b^*)\alpha(a^*)+\beta(ab)\delta(b)\alpha(a^*)+\beta(ab^2)\delta(a)\big) \\ &\quad -\big(\delta(a)\alpha(b^*)^2\alpha(a^*)+\beta(a)\delta(b^2)\alpha(a^*)+\beta(ab^2)\delta(a)\big) \\ &= \beta(a)\delta(b)\alpha(b^*)\alpha(a^*)+\beta(ab)\delta(ab) \\ &\quad +\delta(ab)\alpha(b^*)\alpha(a^*)+\beta(ab)\delta(b)\alpha(a^*)-\beta(a)\delta(b^2)\alpha(a^*), \end{aligned}$$

that is,

$$\big(\delta((ab)^2)-\delta(ab)\alpha(b^*a^*)-\beta(ab)\delta(ab)\big)+\beta(a)\left(\delta(b^2)-\delta(b)\alpha(b^*)-\beta(b)\delta(b)\right)\alpha(a^*)=0.$$

Let us define a mapping $\Lambda: R\to R$ by

$$\Lambda(a)=\delta(a^2)-\delta(a)\alpha(a^*)-\beta(a)\delta(a).$$

Thus, we find that

$$\Lambda(ab)+\beta(a)\Lambda(b)\alpha(a^*)=0.$$

Now using the above identities three times for all $a,b,c\in R$, we find that

$$\begin{aligned} 2\beta(cb)\Lambda(a)\alpha(b^*c^*) &= \beta(c)(\beta(b)\Lambda(a)\alpha(b^*))\alpha(c^*)+\beta(cb)\Lambda(a)\alpha(cb)^* \\ &= \beta(c)(-\Lambda(ba))\alpha(c^*)-\Lambda((cb)a) \\ &= -\beta(c)\Lambda(ba)\alpha(c^*)-\Lambda(cba) \\ &= \Lambda(cba)-\Lambda(cba) \\ &= 0. \end{aligned}$$

Since R is 2-torsion free $*$-ring, we find that $\beta(c)\beta(b)\Lambda(a)\alpha(b^*)\beta(c^*)=0$. Therefore, using Lemma 2.3 two times, we get $\Lambda(a)=0$. Hence δ is a Jordan $(\alpha,\beta)^*$-derivation on R. This completes the proof of our theorem. □

3. Jordan triple left α^*- centralizer

According to Zalar [25], an additive mapping $\Psi : R \to R$ is called a left (resp. right) centralizer of R if $\Psi(ab) = \Psi(a)b$ (resp. $\Psi(ab) = a\Psi(b)$) holds for all $a, b \in R$. If $a \in R$, then $L_x(a) = xa$ is a left centralizer and $R_x(a) = ax$ is a right centralizer. If Ψ is both left and right centralizer, then Ψ is called a centralizer. An additive mapping $\Psi : R \to R$ is called a left (resp. right) Jordan centralizer in case $\Psi(a^2) = \Psi(a)a$ (resp. $\Psi(a^2) = a\Psi(a)$) holds for all $a \in R$. For an endomorphism α of R, an additive mapping $\Psi : R \to R$ is called a left α-centralizer (resp. Jordan right α-centralizer) if $\Psi(a^2) = \Psi(a)\alpha(a)$ (resp. $\Psi(a^2) = \alpha(a)\Psi(a)$) holds for all $a \in R$.

Let R be a $*$-ring and let α be an endomorphism of R. An additive mapping $\Psi : R \to R$ is called a Jordan left α^*-centralizer if $\Psi(a^2) = \Psi(a)\alpha(a^*)$ for all $a \in R$. For $\alpha = I$, identity mapping on R, then we have usual definition of a Jordan left $*$-centralizer. An additive mapping $\Psi : R \to R$ is called a Jordan triple left α^*-centralizer if $\Psi(aba) = \Psi(a)\alpha(b^*a^*)$ holds for all $a, b \in R$. It is easy to see that every Jordan left α^*-centralizer on a 2-torsion free $*$-ring is a Jordan triple left α^*-centralizer. But the converse need not to be true in general. In 2010, Shakir and Fošner [1] shows that the converse is true if the underlying ring is 6-torsion free. In the present section we prove the result of Shakir and Fošner [1] without the restriction of 3-torsion free. In fact we obtain the following result:

3.1. Theorem. *Let R be a 2-torsion free semiprime $*$-ring with a positive definite involution, and let α be a $*$-automorphism of R. Let $\Psi : R \to R$ be an additive mapping. Then the following conditions are equivalent:*

(i) Ψ is a Jordan left α^*-centralizer;

(ii) $\Psi(aba) = \Psi(a)\alpha(b^*)\alpha(a^*)$ for all $a, b \in R$.

Proof. It is easy to prove that $(i) \implies (ii)$. Now we have to prove $(ii) \implies (i)$. We have

$$\Psi(aba) = \Psi(a)\alpha(b^*)\alpha(a^*) \text{ for all } a, b \in R. \tag{3.1}$$

A straightforward linearization on a yields that

$$\Psi(abc + cba) = \Psi(a)\alpha(b^*)\alpha(c^*) + \Psi(c)\alpha(b^*)\alpha(a^*). \tag{3.2}$$

Replacing c by a^2 in (3.2), we get

$$\Psi(aba^2 + a^2ba) = \Psi(a)\alpha(a^*b^*a^{2^*}) + \Psi(a^2)\alpha(b^*a^*). \tag{3.3}$$

Now, replace b by $ab + ba$ in the relation (3.1), to get

$$\Psi(a^2ba + aba^2) = \Psi(a)\alpha(b^*a^*) + \Psi(a)\alpha(b^*a^{*2}). \tag{3.4}$$

From (3.3) and (3.4), we obtain

$$\{\Psi(a^2) - \Psi(a)\alpha(a^*)\}\alpha(b^*a^*) = 0 \text{ for all } a, b \in R.$$

Let us define $\Delta : R \to R$ by

$$\Delta(a) = \Psi(a^2) - \Psi(a)\alpha(a^*). \tag{3.5}$$

We have

(3.6) $\quad \Delta(a)\alpha(b^*a^*) = 0.$

Again replace b by b^* in (3.6), to get

(3.7) $\quad \Delta(a)\alpha(b)\alpha(a^*) = 0.$

Now, replacing b by $a^*b\alpha^{-1}(\Delta(a))$ in (3.7) we get $\Delta(a)\alpha(a^*)\alpha(b)\Delta(a)\alpha(a^*) = 0$ for all $a, b \in R$, that is, $\Delta(a)\alpha(a^*)R\Delta(a)\alpha(a^*) = (0)$, and semiprimeness of R yields that

(3.8) $\quad \Delta(a)\alpha(a^*) = 0$ for all $a \in R$.

Now, multiplying the relation (3.7) from the left side by $\alpha(a^*)$ and from right by $\Delta(a)$, we obtain $\alpha(a^*)\Delta(a)R\alpha(a^*)\Delta(a) = (0)$. Thus, again by semiprimeness of R, it follows that

(3.9) $\quad \alpha(a^*)\Delta(a) = 0$ for all $a \in R$.

The linearization of (3.8) gives that

$$\{\Psi(a+b)^2 - \Psi(a+b)\alpha((a+b)^*)\}\alpha((a+b)^*) = 0.$$

Now, we define $\lambda : R \times R \to R$ by

$$\lambda(a, b) = \Psi(ab + ba) - \Psi(a)\alpha(b^*) - \Psi(b)\alpha(a^*).$$

Thus, above equation can be rewritten as

(3.10) $\quad \Delta(a)\alpha(b^*) + \lambda(a, b)\alpha(a^*) + \Delta(b)\alpha(a^*) + \lambda(a, b)\alpha(b^*) = 0.$

Now, replacing a by $-a$ in (3.10) we get

(3.11) $\quad \Delta(a)\alpha(b^*) + \lambda(a, b)\alpha(a^*) - \Delta(b)\alpha(a^*) - \lambda(a, b)\alpha(b^*) = 0.$

Adding (3.10) and (3.11) and using the fact that R is 2-torsion free, we find that

$$\Delta(a)\alpha(b^*) + \lambda(a, b)\alpha(a^*) = 0 \text{ for all } a, b \in R.$$

Now, multiply the above equation by $\Delta(a)$ from the right and use (3.9) to get $\Delta(a)\alpha(b^*)\Delta(a) = 0$. Again replacing b by b^* and using the fact that α is automorphism, we find that $\Delta(a)R\Delta(a) = (0)$ for all $a \in R$. Since R is semiprime, we find that $\Delta(a) = 0$ for all $a \in R$. This proves that $\Psi(a^2) = \Psi(a)\alpha(a^*)$ holds for all $a \in R$. In other words, Ψ is a Jordan left α^* centralizer. □

4. Generalized Jordan triple $*$-derivations

It is obvious to see that if R is 2-torsion free, then any generalized Jordan $(\alpha, \beta)^*$-derivation $\Theta : R \to R$ with related Jordan $(\alpha, \beta)^*$-derivation $\delta : R \to R$, is a generalized Jordan triple $(\alpha, \beta)^*$-derivation, but the converse need not to be true in general. The following example shows that:

4.1. Example. Consider the rings S, R and $\alpha, \beta, *$ as in Example 2.1. Define mapping $\Theta : R \to R$ such that $\Theta \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then we can find an associated Jordan triple $(\alpha, \beta)^*$-derivation $d : R \to R$ such that

$\delta\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & y & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. It can be easily seen that Θ is a generalized Jordan triple $(\alpha,\beta)^*$-derivation associated with a Jordan triple $(\alpha,\beta)^*$-derivation δ, but not a generalized Jordan $(\alpha,\beta)^*$-derivation.

Motivated by Theorem 2.2, in the present section we show that on a 2-torsion free semiprime $*$-ring R, every generalized Jordan triple $(\alpha,\beta)^*$-derivation associated with a Jordan triple $(\alpha,\beta)^*$-derivation is a generalized Jordan $(\alpha,\beta)^*$-derivation.

We begin our discussion with the following lemma.

4.2. Lemma. *Let R be a 2-torsion free $*$-ring, and let α,β be endomorphisms of R. If $\Theta : R \to R$ is a generalized Jordan $(\alpha,\beta)^*$-derivation associated with a Jordan $(\alpha,\beta)^*$-derivation $\delta : R \to R$. Then for arbitrary $a,b,c \in R$, we have*

$$\begin{array}{llcl} (I) & \Theta(aba) & = & \Theta(a)\alpha(b^*a^*) + \beta(a)\delta(b)\alpha(a^*) + \beta(ab)\delta(a). \\ (II) & \Theta(abc+cba) & = & \Theta(a)\alpha(b^*c^*) + \beta(a)\delta(b)\alpha(c^*) + \beta(ab)\delta(c) + \Theta(c)\alpha(b^*a^*) \\ & & & +\beta(c)\delta(b)\alpha(a^*) + \beta(cb)\delta(a). \end{array}$$

Proof. Using similar arguments as used in the proof of Lemma 2.4, we obtain the assertion of the lemma. □

Now we are well equipped to prove the main theorem of this section.

4.3. Theorem. *Let R be a 2-torsion free semiprime $*$-ring with a positive definite involution, and let α, β be $*$-automorphisms of R. Let $\Theta,\delta : R \to R$ be additive mappings. Then the following conditions are equivalent:*

(i) Θ is a generalized Jordan $(\alpha,\beta)^*$-derivation;

(ii) $\Theta(aba) = \Theta(a)\alpha(b^*a^*) + \beta(a)\delta(a)\alpha(a^*) + \beta(ab)\delta(a)$ for all $a,b \in R$.

Proof. In view of Lemma 4.2, it is clear that $(i) \implies (ii)$. Let us prove the reverse. If $\delta = 0$, then Θ is a Jordan triple left α^*-centralizer on R. Thus, by Theorem 3.1, Θ is a Jordan left α^*-centralizer. Hence, for $\delta = 0$, Θ is a generalized Jordan $(\alpha,\beta)^*$-derivation.

Now assume that the associated Jordan triple $(\alpha,\beta)^*$-derivation δ is nonzero. Therefore by Theorem 2.2, δ is a Jordan $(\alpha,\beta)^*$-derivation on R. Now set $\Psi = \Theta - \delta$. Thus, we find that

$$\begin{aligned} \Psi(aba) &= \Theta(aba) - \delta(aba) \\ &= \Theta(a)\alpha(b^*a^*) + \beta(a)\delta(b)\alpha(a^*) + \beta(ab)\delta(a) \\ &- \delta(a)\alpha(b^*a^*) - \beta(a)\delta(b)\alpha(a^*) - \beta(ab)\delta(a) \\ &= (\Theta(a) - \delta(a))\alpha(b^*a^*) \\ &= \Psi(a)\alpha(b^*a^*). \end{aligned}$$

This implies that Ψ is a Jordan triple left α^*-centralizer on R. Hence, by Theorem 3.1, one can conclude that Ψ is a Jordan left α^*-centralizer on R. Therefore

$$\begin{aligned}\Theta(a^2) &= \Psi(a^2) + \delta(a^2)\\ &= \Psi(a)\alpha(a^*) + \delta(a)\alpha(a^*) + \beta(a)\delta(a)\\ &= (\Psi(a) + \delta(a))\alpha(a^*) + \beta(a)\delta(a)\\ &= \Theta(a)\alpha(a^*) + \beta(a)\delta(a).\end{aligned}$$

This shows that Θ is a generalized Jordan $(\alpha,\beta)^*$-derivation associated with a Jordan $(\alpha,\beta)^*$-derivation δ on R. This completes the proof of the theorem. □

Combining Theorem 2.2 and Theorem 4.3, we get the following result:

4.4. Theorem. *Let R be a 2-torsion free semiprime $*$-ring with a positive definite involution and let α,β be automorphisms of R. Let $\Theta,\delta : R \to R$ be additive mappings. Then the following conditions are mutually equivalent:*

(i) *for all $a,b \in R$,*

$$\Theta(aba) = \Theta(a)\alpha(b^*a^*) + \beta(a)\delta(b)\alpha(a^*) + \beta(ab)\delta(a)$$
$$\delta(aba) = \delta(a)\alpha(b^*a^*) + \beta(a)\delta(b)\alpha(a^*) + \beta(ab)\delta(a)$$

(ii) *for all $a,b \in R$,*

$$\Theta(a^2) = \Theta(a)\alpha(a^*) + \beta(a)\delta(b)$$
$$\delta(a^2) = \delta(a)\alpha(a^*) + \beta(a)\delta(b).$$

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