Hacettepe Journal of Mathematics and Statistics
Volume 42 (6) (2013), 653 – 658

INTEGRABILITY AND L_1-CONVERGENCE OF CERTAIN COSINE SUMS WITH THIRD QUASI HYPER CONVEX COEFFICIENTS

N. L. Braha *

Received 18 : 08 : 2010 : Accepted 25 : 09 : 2012

Abstract

In this paper criterion for L_1- convergence of a certain cosine sums with third quasi hyper-convex coefficients is obtained.

Keywords: cosine sums, L_1-convergence, third quasi hyper-convex null sequences.

2000 AMS Classification: 42A16, 40G05.

1. Introduction

It is well known that if a trigonometric series converges in L_1-metric to a function $f \in L_1$, then it is the Fourier series of the function f. Riesz [1] gave a counter example showing that in a metric space L_1 we cannot expect the converse of the above said result to hold true. This motivated the various authors to study L_1-convergence of the trigonometric series. During their investigations some authors introduced modified trigonometric sums as these sums approximate their limits better than the classical trigonometric series in the sense that they converge in L_1-metric to the sum of the trigonometric series whereas the classical series itself may not. In this contest we was introduced in [3], new modified cosine series given by relation

$$N_n^{(3)}(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^6}\sum_{k=1}^{n}\sum_{j=k}^{n}\left(\Delta^5 a_{j-3} - \Delta^5 a_{j-2}\right)\cos kx - \frac{a_1(15 - 6\cos x + \cos 2x)}{\left(2\sin\frac{x}{2}\right)^6} + \frac{a_2(6-\cos x)}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_3}{\left(2\sin\frac{x}{2}\right)^6}$$

and we will prove that this sums L_1-converges to $g(x)$, under conditions that coefficients (a_n) are third quasi hyper-convex. In the sequel we will briefly describe the notations and definitions which are used throughout the paper. In what follows we will denote by

$$g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx, \tag{1.1}$$

*Department of Mathematics and Computer Sciences, Avenue "Mother Theresa " 5, Prishtinë, 10000, Kosova
Email: nbraha@yahoo.com

with partial sums defined by

$$g_n(x)=\frac{a_0}{2}+\sum_{k=1}^{n}a_k\cos kx, \tag{1.2}$$

and

$$g(x)=\lim_{n\to\infty}g_n(x). \tag{1.3}$$

Dirichlet's kernels are denoted by

$$D_n(t)=\frac{1}{2}+\sum_{k=1}^{n}\cos kt=\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$

$$\widetilde{D}_n(t)=\sum_{k=1}^{n}\cos kt$$

$$\overline{\overline{D}}_n(t)=\sum_{k=1}^{n}\sin kt=\frac{\cos\frac{t}{2}-\cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$

$$\overline{D}_n(t)=-\frac{1}{2}\cot\frac{t}{2}+\overline{\overline{D}}_n(t)=-\frac{\cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$

In the following we will mention some known facts which will be very useful for us (see [6]):

$$S_n^0=S_n=a_0+a_1+\cdots+a_n$$

$$S_n^k=S_0^{k-1}+S_1^{k-1}+\cdots+S_n^{k-1}, k=1,2,\cdots;n=1,2,\cdots;$$

$$A_n^0=1, A_n^k=A_0^{k-1}+A_1^{k-1}+\cdots+A_n^{k-1}, k=1,2,\cdots;n=1,2,\cdots \tag{1.4}$$

The A_n's are called the binomial coefficients and are given by the following relation:

$$\sum_{k=0}^{\infty}A_k^{\alpha}x^k=(1-x)^{(-\alpha-1)}, \tag{1.5}$$

whereas S_n's are given by

$$\sum_{k=0}^{\infty}S_k^{\alpha}x^k=(1-x)^{-\alpha}\sum_{k=0}^{\infty}S_kx^k, \tag{1.6}$$

and

$$A_n^{\alpha}=\sum_{k=0}^{n}A_k^{\alpha-1}, A_n^{\alpha}-A_{n-1}^{\alpha}=A_n^{\alpha-1},$$

$$A_n^{\alpha}=\binom{n+\alpha}{n}\cong\frac{n^{\alpha}}{\Gamma(\alpha+1)}(\alpha\neq-1,-2,\cdots). \tag{1.7}$$

In what follows we will consider that $\alpha>0$.

The Cesaro means T_k^{α} of order α are denoted by $T_k^{\alpha}=\frac{S_k^{\alpha}}{A_k^{\alpha}}$. Also for $0<x\leq\pi$, let

$$S_n^0(x)=\widetilde{D}_n(x)=\cos x+\cos 2x+\cdots+\cos nx$$

$$S_n^1(x)=S_0(x)+S_1(x)+\cdots+S_n(x),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$S_n^k(x)=S_0^{k-1}(x)+S_1^{k-1}(x)+\cdots+S_n^{k-1}(x). \tag{1.8}$$

The Cesaro means $T_k^{\alpha}(x)$ of order α are denoted by $T_k^{\alpha}(x)=\frac{S_k^{\alpha}(x)}{A_k^{\alpha}}$.

1.1. Lemma. *(see [2]) If* $\alpha \geq 0, p \geq 0,\ \epsilon_n = o(n^{-p}),$ *and* $\sum_{n=0}^{\infty} A_n^{\alpha+p}|\Delta^{\alpha+1}\epsilon_n| < \infty,$ *then*

$$\sum_{n=0}^{\infty} A_n^{\lambda+p}|\Delta^{\lambda+1}\epsilon_n| < \infty,$$

for $-1 \leq \lambda \leq \alpha,\ A_n^{\lambda+p}\Delta^{\lambda}\epsilon_n$ *is of bounded variation for* $0 \leq \lambda \leq \alpha$ *and tends to zero as* $n \to \infty$.

The same holds with 0 in place of o in $\epsilon_n = o(n^{-p})$(see Lemma 2 in [2]).

1.2. Definition. A sequence of scalars (a_n) is said to be quasi-convex if $a_n \to 0$ as $n \to \infty$, and

$$\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1}| < \infty, (a_0 = 0), \tag{1.9}$$

where $\Delta a_n = a_n - a_{n+1},\ \Delta^n = \Delta(\Delta^{n-1})$.

1.3. Definition ([5])**.** A sequence of scalars (a_n) is said to be quasi hyper-convex if $a_n \to 0$ as $n \to \infty$, and

$$\sum_{n=1}^{\infty} n^{\alpha}|\Delta^{\alpha+1} a_{n-1}| < \infty, (a_0 = 0), \tag{1.10}$$

for $\alpha > 0$. For $\alpha = 1$, this class reduces to the class defined in Definition 1.2.

1.4. Definition. A sequence of scalars (a_n) is said to be third quasi hyper-convex if $a_n \to 0$ as $n \to \infty$, and

$$\sum_{n=1}^{\infty} n^{3\alpha}|\Delta^{3\alpha-1} a_{n-1}| < \infty, (a_0 = a_{-1} = a_{-2} = 0). \tag{1.11}$$

1.5. Definition. [4] A sequence of scalars (a_n) is said to be third generalized semi-convex if $a_n \to 0$ as $n \to \infty$, and

$$\sum_{n=1}^{\infty} n^{3\alpha}|\Delta^{3\alpha-1} a_{n-1} + \Delta^{3\alpha-1} a_n| < \infty, (a_0 = a_{-1} = a_{-2} = 0). \tag{1.12}$$

1.6. Remark. If (a_n) is a third quasi hyper-convex null scalar sequence, then it is third generalized semi-convex scalars sequence too.

2. Results

In this paper we consider the modified cosine sums defined in [3] as follows:

$$N_n^{(3)}(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^6}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta^5 a_{j-3} - \Delta^5 a_{j-2})\cos kx - \frac{a_1(15 - 6\cos x + \cos 2x)}{\left(2\sin\frac{x}{2}\right)^6} + \frac{a_2(6 - \cos x)}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_3}{\left(2\sin\frac{x}{2}\right)^6} \tag{2.1}$$

and we will prove that this sums L_1-converges to $g(x)$, under conditions that coefficients (a_n) are third quasi hyper-convex and $\alpha \in \mathbb{N}$. In paper [3], was proved that the above modified cosine sums L_1-converges to $g(x)$ under condition that coefficients (a_n) are third semi-convex. We will use this trivial fact:

2.1. Lemma. *If (a_n) is a third quasi hyper-convex null sequence of scalars, then the following relation holds*

$$\sum_{k=1}^{\infty} k^{3\alpha} |(\Delta^{3\alpha-1} a_{k-1} - \Delta^{3\alpha-1} a_k)| < \infty.$$

2.2. Theorem. *Let (a_n) be a third quasi hyper-convex null sequence, then $N_n^{(3)}(x)$ converges to $g(x)$ in L_1 norm.*

Proof. Let us start from the modified cosine sums:

$$N_n^{(3)}(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^6} \sum_{k=1}^{n}\sum_{j=k}^{n} (\Delta^5 a_{j-3} - \Delta^5 a_{j-2}) \cos kx - \\ - \frac{a_1(15 - 6\cos x + \cos 2x)}{\left(2\sin\frac{x}{2}\right)^6} + \frac{a_2(6 - \cos x)}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_3}{\left(2\sin\frac{x}{2}\right)^6}$$

In what follows we will prove that

$$||g(x) - N_n^{(3)}(x)||_{L_1} \to 0, n \to \infty,$$

where (a_n) are third quasi hyper-convex null coefficients, taking in consideration Cesaro's mean of integral order.

Applying Abel's transformation, we have

$$N_n^{(3)}(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^6} \sum_{k=1}^{n-1} (\Delta^5 a_{k-3} - \Delta^5 a_{k-2}) \widetilde{D}_k(x) + \frac{\Delta^5 a_{n-3}\cdot \widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^6} + \\ + \frac{\Delta^5 a_{n-2}\cdot \widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_1(15 - 6\cos x + \cos 2x)}{\left(2\sin\frac{x}{2}\right)^6} + \frac{a_2(6 - \cos x)}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_3}{\left(2\sin\frac{x}{2}\right)^6}$$

If we use Abel's transformation $3\alpha - 5$ times, we obtain:

$$N_n^{(3)}(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^6} \sum_{k=1}^{n-3\alpha+5} (\Delta^{3\alpha-1} a_{k-3} - \Delta^{3\alpha-1} a_{k-2}) S_k^{3\alpha-6}(x) - \\ \sum_{k=1}^{3\alpha-6} \frac{(\Delta^{k+4} a_{n-k-3} - \Delta^{k+4} a_{n-k-2}) S_{n-k}^k(x)}{\left(2\sin\frac{x}{2}\right)^6} + \frac{\Delta^5 a_{n-3} + \Delta^5 a_{n-2}}{\left(2\sin\frac{x}{2}\right)^6} \widetilde{D}_n(x) \\ - \frac{a_1(15 - 6\cos x + \cos 2x)}{\left(2\sin\frac{x}{2}\right)^6} + \frac{a_2(6 - \cos x)}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_3}{\left(2\sin\frac{x}{2}\right)^6}. \tag{2.2}$$

Since $S_n^k(x)$, $T_n(x)$, $\widetilde{D}_n(x)$ are uniformly bounded in any segment $[\epsilon, \pi - \epsilon]$, for any $\epsilon > 0$, and $T_n^k(x) = \frac{S_n^k(x)}{A_n^k}$ we have (see [3])

$$g(x) = \lim_{n\to\infty} N_n^{(3)}(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^6} \sum_{k=1}^{\infty} (\Delta^{3\alpha-1} a_{k-3} - \Delta^{3\alpha-1} a_{k-2}) S_k^{3\alpha-6}(x) - \\ \frac{a_1(15 - 6\cos x + \cos 2x)}{\left(2\sin\frac{x}{2}\right)^6} + \frac{a_2(6 - \cos x)}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_3}{\left(2\sin\frac{x}{2}\right)^6}. \tag{2.3}$$

From relations (2.2) and (2.3) we have:

$$g(x) - N_n^{(3)}(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^6} \sum_{k=n-(3\alpha-5)}^{\infty} (\Delta^{3\alpha-1} a_{k-3} - \Delta^{3\alpha-1} a_{k-2}) S_k^{3\alpha-6}(x) +$$

$$\sum_{k=1}^{3\alpha-6}\frac{(\Delta^{k+4}a_{n-k-3}-\Delta^{k+4}a_{n-k-2})S_{n-k}^{k}(x)}{\left(2\sin\frac{x}{2}\right)^{6}}-\frac{\Delta^{5}a_{n-3}+\Delta^{5}a_{n-2}}{\left(2\sin\frac{x}{2}\right)^{6}}\widetilde{D}_n(x).$$

Hence

$$||g(x)-N_n^{(3)}(x)||\le\left\|\frac{1}{\left(2\sin\frac{x}{2}\right)^{6}}\sum_{k=n-(3\alpha-5)}^{\infty}(\Delta^{3\alpha-1}a_{k-3}-\Delta^{3\alpha-1}a_{k-2})S_k^{3\alpha-6}(x)\right\|+$$

$$\left\|\frac{1}{\left(2\sin\frac{x}{2}\right)^{6}}\sum_{k=1}^{3\alpha-6}\Delta^{k+4}a_{n-k-3}S_{n-k}^{k}(x)\right\|+\left\|\frac{1}{\left(2\sin\frac{x}{2}\right)^{6}}\sum_{k=1}^{3\alpha-6}\Delta^{k+4}a_{n-k-2}S_{n-k}^{k}(x)\right\|$$

$$+\left\|\frac{\Delta^{5}a_{n-3}\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^{6}}\right\|+\left\|\frac{\Delta^{5}a_{n-2}\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^{6}}\right\|$$

Finally we will have

$$||g(x)-N_n^{(3)}(x)||\le C_1\int_0^{\pi}\left|\sum_{k=n-(3\alpha-5)}^{\infty}(\Delta^{3\alpha-1}a_{k-3}-\Delta^{3\alpha-1}a_{k-2})S_k^{3\alpha-6}(x)\right|dx+$$

$$C_1\int_0^{\pi}\left|\sum_{k=1}^{3\alpha-6}\Delta^{k+4}a_{n-k-3}S_{n-k}^{k}(x)\right|dx+C_1\int_0^{\pi}\left|\sum_{k=1}^{3\alpha-6}\Delta^{k+4}a_{n-k-2}S_{n-k}^{k}(x)\right|dx+$$

$$C_1\int_0^{\pi}\left|\Delta^{5}a_{n-3}\widetilde{D}_n(x)\right|dx+C_1\int_0^{\pi}\left|\Delta^{5}a_{n-2}\widetilde{D}_n(x)\right|dx$$

$$\le C_1\cdot\sum_{k=n-(3\alpha-5)}^{\infty}A_k^{3\alpha-6}\left|(\Delta^{3\alpha-5}a_{k-3}-\Delta^{3\alpha-5}a_{k-2})\right|\int_0^{\pi}\left|T_k^{\alpha-1}(x)\right|dx+$$

$$C_1\cdot\sum_{k=1}^{3\alpha-6}A_{n-k}^{k}\left|\Delta^{k+4}a_{n-k-3}\right|\int_0^{\pi}\left|T_{n-k}^{k}(x)\right|dx+C_1\cdot\sum_{k=1}^{3\alpha-6}A_{n-k}^{k}\left|\Delta^{k+4}a_{n-k-2}\right|\int_0^{\pi}\left|T_{n-k}^{k}(x)\right|dx+$$

$$C_1\cdot A_n^0\cdot|\Delta^{5}a_{n-3}|\int_0^{\pi}|T_n^0(x)|dx+C_1\cdot A_n^0\cdot|\Delta^{5}a_{n-2}|\int_0^{\pi}|T_n^0(x)|dx. \tag{2.4}$$

The second summand in the relation (2.4) can be expressed as follows:

$$\sum_{k=1}^{3\alpha-6}A_{n-k}^{k}\left|\Delta^{k+4}a_{n-k-2}\right|\int_0^{\pi}\left|T_{n-k}^{k}(x)\right|dx=$$

$$\sum_{k=1}^{3\alpha-6}A_{n-k}^{k}\left|\Delta^{k}a_{n-k-3}-4\Delta^{k}a_{n-k-2}+6\Delta^{k}a_{n-k-1}-4\Delta^{k}a_{n-k}+\Delta^{k}a_{n-k+1}\right|\int_0^{\pi}\left|T_{n-k}^{k}(x)\right|dx$$

Based on Lemma 1.1 and Lemma 2.1, it tends to zero. For the same reasons the third summand in the relation (2.4) tends to zero, too. Finally we get $||g(x)-N_n^{(3)}(x)||\to 0$, where $n\to\infty$. □

Acknowledgments

The author would like to express his warm thanks to the referees for comments and many valuable suggestions.

References

[1] Bary, K. N. *Trigonometric series*, Moscow, (1961)(in Russian.)

[2] Bosanquet, L. S. *Note on convergence and summability factors III*, Proc. London Math. Soc. **50**, 482-496, 1949.

[3] Braha, N. L. *On L_1-convergence of certain cosine sums with third semi-convex coefficients*, Int. J. Open Probl. Comput. Sci. Math. **2** (4), 562-571, 2009.

[4] Kaur, Kulwinder and Bahtia, S. S. *Integrability and L_1- convergence of Rees-Stanojevic sums with generalized semiconvex coefficient*, IJMMS **30** (11), 645-650, 2002.

[5] Moore, C. N. *On the use of Cesaro means in determining criteria for Fourier constants*, Bull. Amer. Math. Soc. **39**, 907-913, 1933.

[6] Zygmund, A. *Trigonometric series*, Vol. 1, Cambridge University Press, Cambridge, 1959.